Addendum to the Analysis Module

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I think it is important for you to know the following results. Some proofs are included and you should read them, but the important thing is that you be aware of the results themselves.

1. Completeness of $\mathbb{R}^n$

A set E in $\mathbb{R}^n$ is bounded if there is a constant $M > 0$ such that $||x|| \le M$ for all $x \in E$. Equivalently, there is a (different) constant $M > 0$ such that $|x_i| \le M$ for all $x \in E$ and all $i = 1, \cdots, n$. A set of the form $\{x \in \mathbb{R}^n : |x_i - c_i| \le M, i = 1, \cdots, n\}$, where $c = (c_1, \cdots, c_n)$ is a given point, will be called a hypercube with center c and width $2M$ in $\mathbb{R}^n$.

Completeness of $\mathbb{R}^n$ follows from completeness of the real number system and can be expressed in different ways. The following theorem is one way of expressing it.

Theorem 1.1. *A bounded infinite set in $\mathbb{R}^n$ has a limit point.*

In general, the limit points need not be in the set itself. However, if the set is closed it contains all its limit points. This leads to the following important definition.

Definition 1.1. *A nonempty set E in $\mathbb{R}^n$ is* compact *if it is bounded and closed.*

A useful characterization of compact sets is given in the following theorem. In more general topological spaces it is the definition of compactness. A collection $\{O_\alpha\}$ of sets in $\mathbb{R}^n$ *covers* a set E if $E \subseteq \underset{\alpha}{\cup} O_\alpha$.

Theorem 1.2 (Heine-Borel Theorem)**.** *E is compact if and only if for each collection $\{O_\alpha\}$ of open sets that covers E, there is a finite subcollection $\{O_\alpha | \alpha \in F\}$ that covers E.*

Proof. We shall prove the "only if" direction because it is the one we need. The other direction is not terribly difficult and is left as an exercise. Suppose that E is covered by a collection $\{O_\alpha\}$ of open sets and is not covered by any finite subcollection. Since E is bounded, it is contained in a hypercube H_1 of width M, say. Let x^1 be any element of E. Partition H_1 into 2^n congruent hypercubes of width $M/2$. At least one of these, say H_2, has the property that $E \cap H_2$ is not covered by any finite subcollection of $\{O_\alpha\}$. Clearly, $E \cap H_2$ must be an infinite set, so let $x^2 \in E \cap H_2$ be different from x^1. Now subdivide H_2 into 2^n congruent hypercubes of width $M/4$. At least one of these, say H_3, has the property that $E \cap H_3$ is not covered by any finite subcollection of $\{O_\alpha\}$. Clearly, $E \cap H_3$ is infinite, so there is an $x^3 \in E \cap H_3$ different from x^1 and x^2. Continuing in this manner, we may inductively define a sequence of hypercubes $H_1, H_2, , \cdots$ and points $x^1, x^2, \cdots$ such that:

(1) $H_1 \supset H_2 \supset H_3 \supset \cdots$.

(2) The width of H_k is $M/2^{k-1}$.

(3) $x^k \in E \cap H_k$ is distinct from $x^1, \cdots, x^{k-1}$.

(4) $E \cap H_k$ is not covered by a finite subcollection of $\{O_\alpha\}$.

Now consider the infinite set $\{x^k | k = 1, 2, \cdots\}$. It is bounded, so by completeness it has a limit x. Since E is closed, $x \in E$. Indeed, $x \in E \cap H_k$ for each k. Since $\{O_\alpha\}$ covers E, $x \in O_\alpha$ for some α. Therefore, there is an $r > 0$ such that $B(x; r) \subset O_\alpha$. Since $x \in H_k$, if we choose k large enough, we have $H_k \subset B(x; r) \subset O_\alpha$. (Choose k large enough that $\sqrt{n}M/2^{k-1} < r$.) But then $E \cap H_k$ is covered by a finite subcollection of $\{O_\alpha\}$, contradicting its definition. Therefore, the assumption that no finite subcollection of $\{O_\alpha\}$ covers E must be false. $\square$

There are other characterizations and consequences of compactness. Here is one that is especially useful.

Theorem 1.3. *Let $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$ be a nested sequence of (nonempty) compact sets. Then $\bigcap_{k=1}^{\infty} E_k \neq \Phi$.*

2. More on Continuity

Definition 2.1. *Let $E \subseteq \mathbb{R}^n$ and let $f : E \longrightarrow \mathbb{R}$. f is* uniformly continuous *on E if for each $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in E$ and $||x - y|| < \delta$.*

Uniform continuity is stronger than continuity at each point. The following examples illustrate this.

Example 2.1. *On $E = (0,1)$, the functions $f(x) = 1/x$ and $g(x) = sin(1/x)$ are continuous but not uniformly continuous. The functions $sinx$ and $xsin(1/x)$ are uniformly continuous.*

Notice that E is the preceding example is not compact. That makes a big difference, as the following theorem shows.

Theorem 2.1. *Let E be compact and $f : E \longrightarrow \mathbb{R}$ continuous. Then f is uniformly continuous on E.*

Proof. Let $\epsilon > 0$ be given. For each $x \in E$, there is a $\delta_x > 0$ such that if $u \in E \cap B(x; \delta_x)$ then $|f(x) - f(u)| < \epsilon/2$. The collection of open sets $\{B(x; \delta_x/2)|x \in E\}$ covers E. Therefore, there is a finite subcollection $\{B(x; \delta_x/2)|x \in F\}$ which covers E. Let $\delta = min\{\delta_x/2|x \in F\}$. Let $u, v \in E$ with $d(u, v) < \delta$. Then there exists an $x \in F$ such that $d(x, u) < \delta_x/2$. By the triangle inequality it follows that $d(x, v) < \delta_x$. Hence, $|f(u) - f(v)| \leq |f(u) - f(x)| + |f(x) - f(v)| < \epsilon/2 + \epsilon/2 = \epsilon$. □

In the examples above, the functions that are continuous on E but not uniformly continuous cannot be extended continuously so that their domains include the limit points of E. In contrast, the functions that are uniformly continuous can be continuously extended in one and only one way to the closure of E. This is characteristic of uniformly continuous functions defined on non-compact domains.

Possibly the most important connection between compactness and continuity is the following.

Theorem 2.2. *Let $E \subseteq \mathbb{R}^n$ be compact and $f : E \longrightarrow \mathbb{R}$ continuous. Then f has a maximum (and a minimum) value on E.*

Proof. For each positive integer k, let $E_k = \{x \in E | f(x) \geq k\}$. By continuity of f, E_k is closed, and as a subset of E it is clearly bounded, so it is either compact or empty. We will show that for large enough k it is empty. If this were not true, then we would have a nested sequence $E_1 \supseteq E_2 \supseteq \cdots$ of compact sets. By Theorem 1.2 above, $\bigcap_{k=1}^{\infty} E_k \neq \Phi$, which implies that there is an $x \in E$ such that $f(x) \geq k$ for each integer k. Obviously, this cannot happen, so E_k is empty for sufficiently large k. Thus, the values of f have a least upper bound y. Now, for each positive integer k, let $E_k = \{x \in E | f(x) \geq y - 1/k\}$. This time, E_k is not empty by the definition of a least upper bound and so it must be compact. Therefore, there is an $x \in \bigcap_{k=1}^{\infty} E_k$. Since $f(x) \geq y - 1/k$ for all k, $f(x) = y$ and f achieves its maximum value at x. $\square$