PART III. FUNCTIONS: LIMITS AND CONTINUITY

III.1. LIMITS OF FUNCTIONS

This chapter is concerned with functions $f : D \to \mathbb{R}$ where D is a nonempty subset of $\mathbb{R}$. That is, we will be considering real-valued functions of a real variable. The set D is called the *domain* of f.

Definition 1. *Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. A number L is the* **limit of f at c** *if to each $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$|f(x) - L| < \epsilon \quad \textit{whenever} \quad x \in D \quad \textit{and} \quad 0 < |x - c| < \delta.$$

This definition can be stated equivalently as follows:

Definition. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. A number L is the **limit of f at c** if to each neighborhood V of L there exists a deleted neighborhood U of c such that $f(U \cap D) \subseteq V$.

Notation $\lim_{x \to c} f(x) = L$.

Examples:

(a) $\lim_{x \to -2} (x^2 - 2x + 4) = 12$.

(b) $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2} = 4$.

(c) $\lim_{x \to 3} \dfrac{x^2 + 3x + 5}{x - 3}$ does not exist.

(d) $\lim_{x \to 1} \dfrac{|x - 1|}{x - 1}$ does not exist.

Example: Let $f(x) = 4x - 5$. Prove that $\lim_{x \to 3} f(x) = 7$.

Proof: Let $\epsilon > 0$.

$$|f(x) - 7| = |(4x - 5) - 7| = |4x - 12| = 4|x - 3|.$$

Choose $\delta = \epsilon/4$. Then

$$|f(x) - 7| = 4|x - 3| < 4\,\frac{\epsilon}{4} = \epsilon \quad \text{whenever} \quad 0 < |x - 3| < \delta.$$

Two Obvious Limits:

(a) For any constant k and any number c, $\lim_{x \to c} k = k$.

(b) For any number c, $\lim_{x \to c} x = c$.

THEOREM 1. *Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Then $\lim_{x \to c} f(x) = L$ if and only if for every sequence $\{s_n\}$ in D such that $s_n \to c$, $s_n \neq c$ for all n, $f(s_n) \to L$.*

Proof: Suppose that $\lim_{x \to c} f(x) = L$. Let $\{s_n\}$ be a sequence in D which converges to c, $s_n \neq c$ for all n. Let $\epsilon > 0$. There exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - c| < \delta \ \ (x \in D).$$

Since $s_n \to c$ there exists a positive integer N such that $|c - s_n| < \delta$ for all $n > N$. Therefore

$$|f(s_n) - L| < \epsilon \quad \text{for all } n > N \quad \text{and} \quad f(s_n) \to L.$$

Now suppose that for every sequence $\{s_n\}$ in D which converges to c, $f(s_n) \to L$. Suppose that $\lim_{x \to c} f(x) \neq L$. Then there exists an $\epsilon > 0$ such that for each $\delta > 0$ there is an $x \in D$ with $0 < |x - c| < \delta$ but $f(x) - L| \geq \epsilon$. In particular, for each positive integer n there is an $s_n \in D$ such that $|c - s_n| < 1/n$ and $|f(s_n) - L| \geq \epsilon$. Now, $s_n \to c$ but $\{f(s_n)\}$ does not converge to L, a contradiction.

Corollary Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. If $\lim_{x \to c} f(x)$ exists, then it is unique. That is, f can have only one limit at c.

THEOREM 2. *Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. If $\lim_{x \to c} f(x)$ does not exist, then there exists a sequence $\{s_n\}$ in D such that $s_n \to c$, but $\{f(s_n)\}$ does not converge.*

Proof: Suppose that $\lim_{x \to c} f(x)$ does not exist. Suppose that for every sequence $\{s_n\}$ in D such that $s_n \to c \ (s_n \neq c)$, $\{f(s_n)\}$ converges. Let $\{s_n\}$ and $\{t_n\}$ be sequences in D which converge to c. Then $\{f(s_n)\}$ and $\{f(t_n)\}$ are convergent sequences. Let $\{u_n\}$ be the sequence $\{s_1, t_1, s_2, t_2, \dots\}$. Then $\{u_n\}\}$ converges to c and $\{f(u_n)\}$ converges to some number L. Since $\{f(s_n)\}$ and $\{f(t_n)\}$ are subsequences of $\{f(u_n)\}$, $f(s_n) \to L$ and $f(t_n) \to$ L. Therefore, for every sequence $\{s_n\}$ in D such that $s_n \to c$, $s_n \neq c$ for all n, $f(s_n) \to L$ and $\lim_{x \to c} f(x) = L$.

Arithmetic of Limits

THEOREM 3. *Let $f,\ g : D \to \mathbb{R}$ and let c be an accumulation point of D. If*

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M,$$

then

1. $\lim_{x\to c}[f(x)+g(x)] = L+M$,

2. $\lim_{x\to c}[f(x)-g(x)] = L-M$,

3. $\lim_{x\to c}[f(x)g(x)] = LM$, $\quad \lim_{x\to c}[k\,f(x)] = kL$, k *constant,*

4. $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ *provided* $M \neq 0$, $g(x) \neq 0$.

Examples:

(a) Since $\lim_{x\to c} x = c$, $\quad \lim_{x\to c} x^n = c^n$ for every positive integer n, by (3).

(b) If $p(x) = 2x^3 + 3x^2 - 5x + 4$, then, by (1), (2) and (3),

$$\lim_{x\to -2} p(x) = 2(-2)^3 + 3(-2)^2 - 5(-2) + 4 = 10 = p(-2).$$

(c) If $R(x) = \dfrac{x^3 - 2x^2 + x - 5}{x^2+4}$, then, by (1) – (4),

$$\lim_{x\to 2} R(x) = \frac{2^3 - 2(2)^2 + 2 - 5}{2^2+4} = \frac{-3}{8} = R(2).$$

THEOREM 4. *(***"Pinching Theorem"***) Let* $f,\ g,\ h : D \to \mathbb{R}$ *and let* c *be an accumulation point of* D. *Suppose that* $f(x) \leq g(x) \leq h(x)$ *for all* $x \in D$, $x \neq c$. *If*

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L,$$

then $\lim_{x\to c} g(x) = L$.

Proof: Let $\epsilon > 0$. There exists a positive number δ_1 such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x-c| < \delta_1 \ \ (x \in D).$$

That is

$$-\epsilon < f(x) - L < \epsilon \quad \text{whenever} \quad 0 < |x-c| < \delta_1.$$

Similarly, there exists a positive number δ_2 such that

$$-\epsilon < h(x) - L < \epsilon \quad \text{whenever} \quad 0 < |x-c| < \delta_2.$$

Let $\delta = \min\,\{\delta_1, \delta_2\}$. Then

$$-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon \quad \text{whenever} \quad 0 < |x-c| < \delta.$$

Therefore, $\lim_{x\to c} g(x) = L$.

One-Sided Limits

Definition 2. *Let* $f : D \to \mathbb{R}$ *and let* c *be an accumulation point of* D*. A number* L *is the* **right-hand limit of** f **at** c *if to each* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$|f(x) - L| < \epsilon \quad \textit{whenever} \quad x \in D \quad \textit{and} \quad c < x < c + \delta.$$

Notation: $\lim\limits_{x \to c^+} f(x) = L.$

A number M *is the* **left-hand limit of** f **at** c *if to each* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$|f(x) - L| < \epsilon \quad \textit{whenever} \quad x \in D \quad \textit{and} \quad c - \delta < x < c.$$

Notation: $\lim\limits_{x \to c^-} f(x) = M.$

Examples

(a) $\lim\limits_{x \to 1^-} \dfrac{|x-1|}{x-1} = -1; \quad \lim\limits_{x \to 1^+} \dfrac{|x-1|}{x-1} = 1.$

(b) Let $f(x) = \begin{cases} x^2 - 1, & x \le 2 \\ \dfrac{1}{x-2}, & x > 2 \end{cases}$; $\quad \lim\limits_{x \to 2^-} f(x) = 3, \quad \lim\limits_{x \to 2^+} f(x)$ does not exist.

THEOREM 5. $\lim\limits_{x \to c} f(x) = L$ *if and only if each of the one-sided limits* $\lim\limits_{x \to c^+} f(x)$ *and* $\lim\limits_{x \to c^-} f(x)$ *exists, and*

$$\lim_{x \to c+} f(x) = \lim_{x \to c^-} f(x) = L.$$

Exercises 3.1

1. Evaluate the following limits.

   (a) $\lim\limits_{x \to 2} \dfrac{x^2 - 4x + 3}{x - 1}$ (b) $\lim\limits_{x \to 1} \dfrac{x^2 - 4x + 3}{x - 1}$

   (c) $\lim\limits_{x \to 2} \dfrac{x^2 - x - 6}{x + 2}$ (d) $\lim\limits_{x \to -2} \dfrac{x^2 - x - 6}{x + 2}$

   (e) $\lim\limits_{x \to 2} \dfrac{x^2 - x - 6}{(x + 2)^2}$ (f) $\lim\limits_{x \to 1} \dfrac{\sqrt{x} - 1}{x - 1}$

   (g) $\lim\limits_{x \to 0} \dfrac{x}{\sqrt{4 + x} - 2}$ (h) $\lim\limits_{x \to 1^+} \dfrac{1 - x^2}{|x - 1|}$

2. Given that $f(x) = x^3$, evaluate the following limits.

(a) $\lim_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$

(b) $\lim_{x \to 3} \dfrac{f(x) - f(2)}{x - 3}$

(c) $\lim_{x \to 3} \dfrac{f(x) - f(2)}{x - 2}$

(d) $\lim_{x \to 1} \dfrac{f(x) - f(1)}{x - 1}$

3. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.

   (a) $\lim_{x\to c} f(x) = L$ if and only if to each $\epsilon > 0$, there is a $\delta > 0$ such that

   $$|f(x) - f(c)| < \epsilon \quad \text{whenever} \quad |x - c| < \delta, \quad x \in D.$$

   (b) $\lim_{x\to c} f(x) = L$ if and only if for each deleted neighborhood U of c there is a neighborhood V of L such that $f(U \cap D) \subseteq V$.

   (c) $\lim_{x\to c} f(x) = L$ if and only if for every sequence $\{s_n\}$ in D that converges to c, $s_n \neq c$ for all n, the sequence $\{f(s_n)\}$ converges to L.

   (d) $\lim_{x\to c} f(x) = L$ if and only if $\lim_{h\to 0} f(c + h) = L$.

   (e) If f does not have a limit at c, then there exists a sequence $\{s_n\}$ in D $s_n \neq c$ for all n, such that $s_n \to c$, but $\{f(s_n)\}$ diverges.

   (f) For any polynomial P and any real number c, $\lim_{x\to c} P(x) = P(c)$.

   (g) For any polynomials P and Q, and any real number c,

   $$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

4. Find a $\delta > 0$ such that $0 < |x - 3| < \delta$ implies $|x^2 - 5x + 6| < \frac{1}{4}$.

5. Find a $\delta > 0$ such that $0 < |x - 2| < \delta$ implies $|x^2 + 2x - 8| < \frac{1}{10}$.

6. Prove that $\lim_{x\to 1} (4x + 3) = 7$.

7. Prove that $\lim_{x\to 3} (x^2 - 2x + 3) = 6$.

8. Determine whether or not the following limits exist:

   (a) $\lim_{x\to 0} \left|\sin \dfrac{1}{x}\right|$.

   (b) $\lim_{x\to 0} x \sin \dfrac{1}{x}$.

9. Let $f : D \to \mathbb{R}$ and let c be an accumulation point of D. Suppose that $\lim_{x\to c} f(x) = L$ and $L > 0$. Prove that there is a number $\delta > 0$ such that $f(x) > 0$ for all $x \in D$ with $0 < |x - c| < \delta$.

10. (a) Suppose that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} [f(x)g(x)] = 1$. Prove that $\lim_{x\to c} g(x)$ does not exist.

(b) Suppose that $\lim_{x \to c} f(x) = L \neq 0$ and $\lim_{x\to c} [f(x)g(x)] = 1$. Does $\lim_{x\to c} g(x)$ exist, and if so, what is it?

III.2 CONTINUOUS FUNCTIONS

Definition 3. *Let* $f : D \to \mathbb{R}$ *and let* $c \in D$. *Then* f **is continuous at** c *if to each* $\epsilon > 0$ *there is a* $\delta > 0$ *such that*

$$|f(x) - f(c)| < \epsilon \quad \textit{whenever} \quad |x - c| < \delta, \quad x \in D.$$

Let $S \subseteq D$. *Then* f *is continuous on* S *if it is continuous at each point* $c \in S$. f *is continuous if* f *is continuous on* D.

THEOREM 6. Characterizations of Continuity *Let* $f : D \to \mathbb{R}$ *and let* $c \in D$. *The following are equivalent:*

1. f *is continuous at* c.
2. *If* $\{x_n\}$ *is a sequence in* D *such that* $x_n \to c$, *then* $f(x_n) \to f(c)$.
3. *To each neighborhood* V *of* $f(c)$, *there is a neighborhood* U *of* c *such that* $f(U \cap D) \subseteq V$.

Proof: See Theorem 1.

Corollary *If* c *is an accumulation point of* D, *then each of the above is equivalent to*

$$\lim_{x\to c} f(x) = f(c).$$

THEOREM 7. *Let* $f : D \to \mathbb{R}$ *and let* $c \in D$. *Then* f *is discontinuous at* c *if and only if there is a sequence* $\{x_n\}$ *in* D *such that* $x_n \to c$ *but* $\{f(x_n)\}$ *does not converge to* $f(c)$.

Continuity of Combinations of Functions

THEOREM 8. Arithmetic: *Let* f, $g : D \to \mathbb{R}$ *and let* $c \in D$. *If* f *and* g *are continuous at* c, *then*

1. $f + g$ *is continuous at* c.
2. $f - g$ *is continuous at* c.
3. fg *is continuous at* c; kf *is continuous at* c *for any constant* k.

4. f/g *is continuous at* c *provided* $g(c) \neq 0$.

THEOREM 9. Composition: *Let* $f : D \to \mathbb{R}$ *and* $g : E \to \mathbb{R}$ *be functions such that* $f(D) \subseteq E$. *If* f *is continuous at* $c \in D$ *and* g *is continuous at* $f(c) \in E$, *then the composition of* g *with* f, $g \circ f : D \to \mathbb{R}$, *is continuous at* c.

Proof: Let $\epsilon > 0$. Since g is continuous at $f(c) \in E$ there is a positive number δ_1 such that $|g(f(x)) - g(f(c))| < \epsilon$ whenever $|f(x) - f(c)| < \delta_1$, $f(x) \in E$. Since f is continuous at c there is a positive number δ such that $|f(x) - f(c)| < \delta_1$ whenever $|x - c| < \delta$, $x \in D$. It now follows that

$$|g(f(x)) - g(f(c))| < \epsilon \quad \text{whenever} \quad |x - c| < \delta,\ x \in D$$

and $g \circ f$ is continuous at c.

Definition 4. *Let* $f : D \to \mathbb{R}$, *and let* $G \subseteq \mathbb{R}$. *The pre-image of* G, *denoted by* $f^{-1}(G)$ *is the set*

$$f^{-1}(G) = \{x \in D : f(x) \in G\}.$$

THEOREM 10. *A function* $f : D \to \mathbb{R}$ *is continuous on* D *if and only if for each open set* G *in* $\mathbb{R}$ *there is an open set* H *in* $\mathbb{R}$ *such that* $H \cap D = f^{-1}(G)$.

Proof: Suppose f is continuous on D. Let $G \subseteq \mathbb{R}$ be an open set. If $c \in f^{-1}(G)$, then $f(c) \in G$. Since G is open, there exists a neighborhood V of $f(c)$ such that $V \subseteq G$. Therefore, there exists a neighborhood U_c of c such that $f(U_c \cap D) \subseteq V$. Let

$$H = \cup_{c \in f^{-1}(G)} U_c.$$

H is open and $H \cap D = f^{-1}(G)$.

Conversely, choose any $c \in D$, and let V be a neighborhood of $f(c)$. Since V is an open set, there is an open set $H \subseteq \mathbb{R}$ such that $H \cap D = f^{-1}(V)$. Since $f(c) \in V$, $c \in H$. But H is an open set so there is a neighborhood U of c such that $U \subseteq H$. Now

$$f(U \cap D) \subseteq f(H \cap D) = v.$$

It follows that f is continuous on D by Theorem 6.

Corollary A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(G)$ is open in $\mathbb{R}$ whenever G is open in $\mathbb{R}$.

Exercises 3.2

1. Let $f(x) = \dfrac{x^2 + 2x - 15}{x - 3}$. Define f at 3 so that f will be continuous at 3.

2. Each of the following functions is defined everywhere except at $x = 1$. Where possible, define f at 1 so that it becomes continuous at 1.

(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ (b) $f(x) = \dfrac{1}{x - 1}$

(c) $f(x) = \dfrac{x - 1}{|x - 1|}$ (d) $f(x) = \dfrac{(x - 1)^2}{|x - 1|}$

3. In each of the following define f at 5 so that it becomes continuous at 5.

(a) $f(x) = \dfrac{\sqrt{x + 4} - 3}{x - 5}$ (b) $f(x) = \dfrac{\sqrt{x + 4} - 3}{\sqrt{x - 5}}$

(c) $f(x) = \dfrac{\sqrt{2x - 1} - 3}{x - 5}$ (d) $f(x) = \dfrac{\sqrt{x^2 - 7x + 16} - \sqrt{6}}{(x - 5)\sqrt{x + 1}}$

4. Let $f(x) = \begin{cases} A^2x^2, & x < 2 \\ (1 - A)x, & x \geq 2. \end{cases}$ For what values of A is f continuous at 2?

5. Give necessary and sufficient conditions on A and B for the function

$$f(x) = \begin{cases} Ax - B, & x \leq 1 \\ 3x, & 1 < x < 2 \\ Bx^2 - A, & x \geq 2 \end{cases}$$

to be continuous at $x = 1$ but discontinuous at $x = 2$.

6. Let $f : D \to \mathbb{R}$ and let $c \in D$. True – False. Justify your answer by citing a definition or theorem, giving a proof, or giving a counter-example.

(a) f is continuous at c if and only if to each ϵ there is a $\delta > 0$ such that

$$|fx) - f(c)| < \epsilon \quad \text{whenever} \quad |x - c| < \delta \text{ and } x \in D.$$

(b) If $f(D) \subseteq \mathbb{R}$ is bounded, then f is continuous on D.

(c) If c is an isolated point of D, then f is continuous at c.

(d) If f is continuous at c and $\{x_n\}$ is a sequence in D, then $x_n \to c$ whenever $f(x_n) \to f(c)$.

(e) If $\{x_n\}$ is a Cauchy sequence in D, then $\{f(x_n)\}$ is convergent.

7. Prove or give a counterexample.

(a) If f and $f + g$ are continuous on D, then g is continuous on D.

(b) If f and fg are continuous on D, then g is continuous on D.

(c) If f and g are not continuous on D, then $f+g$ is not continuous on D.

(d) If f and g are not continuous on D, then fg is not continuous on D.

(e) If f^2 is continuous on D, then f is continuous on D.

(f) If f is continuous on D, then $f(D)$ is a bounded set.

8. Let $f : D \to \mathbb{R}$.

   (a) Prove that if f is continuous at c, then $|f|$ is continuous at c.

   (b) Suppose that $|f|$ is continuous at c. Does it follow that f is continuous at c? Justify your answer.

9. Let $f : D \to \mathbb{R}$ be continuous at $c \in D$. Prove that if $f(c) > 0$, then there is an $\alpha > 0$ and a neighborhood U of c such that $f(x) > \alpha$ for all $x \in U \cap D$.

10. Let $f : D \to \mathbb{R}$ be continuous at $c \in D$. Prove that there exists an $M > 0$ and a neighborhood U of c such that $|f(x)| \le M$ for all $x \in U \cap D$.

III.3. PROPERTIES OF CONTINUOUS FUNCTIONS

Definition 5. *A function $f : D \to \mathbb{R}$ is* **bounded** *if there exists a number M such that $|f(x)| \le M$ for all $x \in D$. That is, f is bounded if $f(D)$ is a bounded subset of $\mathbb{R}$.*

THEOREM 11. *Let $f : D \to \mathbb{R}$ be continuous. If D is compact, then $f(D)$ is compact. (The continuous image of a compact set is compact.)*

Proof: Let $\mathcal{G} = \{\mathcal{G}_\alpha\}$ be an open cover of $f(D)$. Since f is continuous, for each open set G_α in $\mathcal{G}$ there is an open set H_α such that $H_\alpha \cap D = f^{-1}(G_\alpha)$. Also, since $f(D) \subseteq \cup G_\alpha$, it follows that

$$D \subseteq \cup f^{-1}(G_\alpha) \subseteq \cup H_\alpha.$$

Thus, the collection $\{H_\alpha\}$ is an open cover of D. Since D is compact this open cover has a finite subcover H_{α_1}, H_{α_1}, $\ldots$, H_{α_n}. Now,

$$D \subseteq (H_{\alpha_1} \cap D) \cup (H_{\alpha_2} \cap D) \cup \cdots \cup (H_{\alpha_n} \cap D)$$

and

$$f(D) \subseteq G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}.$$

Therefore, the open cover $\mathcal{G}$ has a finite subcover and $f(D)$ is compact.

Definition 6. *Let $f : D \to \mathbb{R}$. $f(x_0)$ is the* **minimum value of** f **on** D *if $f(x_0) \le f(x)$ for all $x \in D$. $f(x_1)$ is the* **maximum value of** f **on** D *if $f(x) \le f(x_1)$ for all $x \in D$.*

COROLLARY 1. If $f : D \to \mathbb{R}$ is continuous and D is compact, then f has a maximum value and a minimum value. That is, there exist points $x_0,\ x_1 \in D$ such that $f(x_0) \le f(x) \le f(x_1)$ for all $x \in D$.

COROLLARY 2. If $f : D \to \mathbb{R}$ is continuous and D is compact, then $f(D)$ is closed and bounded.

THEOREM 12. *Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a)$ and $f(b)$ have opposite sign, then there is at least one point $c \in (a, b)$ such that $f(c) = 0$.*

Proof: Suppose that $f(a) < 0$ and $f(b) > 0$. Since $f(a) < 0$ we know from the continuity of f that there is an interval $[a, \delta)$ such that $f(x) < 0$ on $[a, \delta)$. (See Exercises 3.2, #9) Let

$$c = \sup \{\delta \,:\, f \text{ is negative on } [a, \delta)\}.$$

Clearly $c \le b$.

We cannot have $f(c) > 0$ for then $f(x) > 0$ on some interval to the left of c, and we know that to the left of c, $f(x) < 0$. This also shows that $c < b$.

We cannot have $f(c) < 0$ for then $f(x) < 0$ on some interval $[a, t)$, with $t > c$ which contradicts the definition of c.

It follows that $f(c) = 0$.

THEOREM 13. Intermediate Value Theorem *Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that $f(a) \ne f(b)$. If k is a number between $f(a)$ and $f(b)$, then there is at least one number $c \in (a, b)$ such that $f(c) = k$.*

COROLLARY If $f : D \to \mathbb{R}$ is continuous and $I \subseteq D$ is an interval, then $f(I)$ is an interval.

THEOREM 14. *Suppose that $f : D \to \mathbb{R}$ is continuous. If $I \subseteq D$ is a compact interval, then $f(I)$ is a compact interval.*

Exercises 3.3

1. Show that the equation $x^3 - 4x + 2 = 0$ has three distinct roots in $[-3, 3]$ and locate the roots between consecutive integers.
2. Prove that $\sin x + 2\cos x = x^2$ for some $x \in [0, \pi/2]$.
3. Prove that there exists a positive number c such that $c^2 = 2$.

4. True – False. Justify your answer by citing a theorem, giving a proof, or giving a counter-example.

   (a) Suppose that $f : D \to \mathbb{R}$ is continuous. Then there exists a point $x_1 \in D$ such that $f(x) \le f(x_1)$ for all $x \in D$.
   (b) If $D \subseteq \mathbb{R}$ is bounded and $f : D \to \mathbb{R}$ is continuous, then $f(D)$ is bounded.
   (c) Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose that $f(a) \le k \le f(b)$. Then there exists a point $c \in [a, b]$ such that $f(c) = k$.
   (d) Let $f : (a, b) \to \mathbb{R}$ be continuous. Then there is a point $x_1 \in (a, b)$ such that $f(x) \le f(x_1)$ for all $x \in (a, b)$.
   (e) If $f : D \to \mathbb{R}$ is continuous and bounded on D, then f has a maximum value and a minimum value on D.

5. Let $f : D \to \mathbb{R}$ be continuous. For each of the following, prove or give a counterexample.

   (a) If D is open, then $f(D)$ is open.
   (b) If D is closed, then $f(D)$ is closed.
   (c) If D is not open, then $f(D)$ is not open.
   (d) If D is not closed, then $f(D)$ is not closed.
   (e) If D is not compact, then $f(D)$ is not compact.
   (f) If D is not bounded, then $f(D)$ is not bounded.
   (g) If D is an interval, then $f(D)$ is an interval.
   (h) If D is an interval and $f(D) \subseteq \mathcal{Q}$ (the rational numbers), then f is constant.

6. Prove that every polynomial of odd degree has at least one real root.

7. Prove Theorem 13.

8. Prove Theorem 14.

9. Suppose that $f : [a, b] \to [a, b]$ is continuous. Prove that there is at least one point $c \in [a, b]$ such that $f(c) = c$. (Such a point is called a *fixed point of* f.)

10. Suppose that $f,\ g : [a, b] \to \mathbb{R}$ are continuous, and suppose that $f(a) \le g(a)$, $f(b) \ge g(b)$. Prove that there is at least one point $c \in [a, b]$ such that $f(c) = g(c)$.

III.4. THE DERIVATIVE

DEFINITION 1. *Let* I *be an interval, let* $f : I \to \mathbb{R}$, *and let* $c \in I$. f *is* **differentiable at** c *if*

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = m$$

exists. f *is* **differentiable on** I *if it is differentiable at each point of* I.

Notation: If f is differentiable at c, then the limit m is called the **derivative of** f **at** c and is denoted by $f'(c)$.

An equivalent definition of differentiability is:

Definition. f is **differentiable at** c if

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = m$$

exists.

Two basic derivatives

(a) Let $f(x) \equiv k$, $x \in \mathbb{R}$, k constant. For any $c \in \mathbb{R}$, $f'(c) = 0$

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} \frac{k - k}{x - c} = \lim_{x\to c} 0 = 0.$$

(b) Let $f(x) = x$, $x \in \mathbb{R}$. For any $c \in \mathbb{R}$, $f'(c) = 1$

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} \frac{x - c}{x - c} = \lim_{x\to c} 1 = 1.$$

Examples:

(a) Let $f(x) = x^2 + 3x - 1$ on $\mathbb{R}$. Then for any $c \in \mathbb{R}$, we have

$$\begin{aligned}
\lim_{x\to c} \frac{f(x) - f(c)}{x - c} &= \lim_{x\to c} \frac{x^2 + 3x - 1 - (c^2 + 3c - 1)}{x - c} \\
&= \lim_{x\to c} \frac{x^2 - c^2 + 3(x - c)}{x - c} = \lim_{x\to c} \frac{(x - c)(x + c + 3}{x - c} \\
&= \lim_{x\to c} (x + c + 3) = 2c + 3
\end{aligned}$$

Thus $f'(c) = 2c + 3$.

(b) Same function using the alternative definition.

$$\begin{aligned}
\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} &= \lim_{h\to 0}\frac{(c+h)^2+3(c+h)-1-(c^2+3c-1)}{x-c}\\
&= \lim_{h\to 0}\frac{c^2+2ch+h^2+3c+3h-c^2-3c}{h} = \lim_{h\to 0}\frac{2ch+h^2+3h}{h}\\
&= \lim_{h\to 0}(2c+h+3) = 2c+3
\end{aligned}$$

Note: The alternative definition is usually easier to use when calculating the derivative of a given function because it's usually easier to expand an expression than it is to factor. For example:

(c) Let $f(x)=\sin x$ on $\mathbb{R}$, and let $c\in\mathbb{R}$. Then

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c}\frac{\sin x-\sin c}{x-c} = ????$$

On the other hand

$$\begin{aligned}
\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} &= \lim_{h\to 0}\frac{\sin(c+h)-\sin c}{h}\\
&= \lim_{h\to 0}\frac{\sin c\cos h+\cos c\sin h-\sin c}{h}\\
&= \lim_{h\to 0}\frac{\sin c[\cos h-1]+\cos c\sin h}{h}\\
&= \lim_{h\to 0}\sin c\left[\frac{\cos h-1}{h}\right]+\lim_{h\to 0}\cos c\left[\frac{\sin h}{h}\right] = \cos c.
\end{aligned}$$

Therefore $f'(c)=\cos c$. Here we used the important trigonometric limits:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \quad\text{and}\quad \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0.$$

(c) Let $f(x)=\sqrt{x}$, $x\geq 0$ and let $c>0$.

$$\begin{aligned}
\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} &= \lim_{h\to 0}\frac{\sqrt{c+h}-\sqrt{c}}{h} = \lim_{h\to 0}\frac{\sqrt{c+h}-\sqrt{c}}{h}\,\frac{\sqrt{c+h}+\sqrt{c}}{\sqrt{c+h}+\sqrt{c}}\\
&= \lim_{h\to 0}\frac{h}{h\left(\sqrt{c+h}+\sqrt{c}\right)} = \lim_{h\to 0}\frac{1}{\left(\sqrt{c+h}+\sqrt{c}\right)} = \frac{1}{2\sqrt{c}}
\end{aligned}$$

Thus, $f'(c)=\dfrac{1}{2\sqrt{c}}$.

NOTE: In each of the examples we started with a function f and "derived" a new function f' which is called the **derivative** of f. If we start with a function of x, then it is standard to denote the derivative as a function of x. For example, if $f(x) = x^2 + 3x - 1$, then $f'(x) = 2x + 3$; if $f(x) = \sin x$, then $f'(x) = \cos x$; if $f(x) = \sqrt{x}$, then $f'(x) = 1/2\sqrt{x}$

Example: A function that fails to be differentiable at a point c.

Set

$$f(x) = \begin{cases} x^2 + 1, & x \leq 1 \\ 3 - x, & x > 1 \end{cases}$$

You can verify that f is continuous for all x; in particular, f continuous at $x = 1$. We show that f is not differentiable at 1.

For $h < 0$,

$$\frac{f(1+h) - f(1)}{h} = \frac{(1+h)^2 + 1 - (2)}{h} = \frac{2h + h^2}{h} = 2 + h$$

and

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} (2 + h) = 2.$$

For $h > 0$,

$$\frac{f(1+h) - f(1)}{h} = \frac{3 - (1+h) - (2)}{h} = \frac{-h}{h} = -1$$

and

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} (-1) = -1.$$

Therefore,

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

does not exist.

THEOREM 15. *If $f : I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at c.*

Proof: For $x \in I$, $x \neq c$, we have

$$f(x) = (x - c)\,\frac{f(x) - f(c)}{x - c} + f(c).$$

Since f is differentiable at c,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

exists. Therefore,

$$\lim_{x \to c} f(x) = \left[\lim_{x \to c} (x - c)\right] \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c}\right] + \lim_{x \to c} f(c) = 0 \cdot f'(c) + f(c) = f(c)$$

By the Corollary to Theorem 6, f is continuous at c.

Differentiability of Combinations of Functions

THEOREM 16. Arithmetic: *Let $f,\ g : I \to \mathbb{R}$ and let $c \in I$. If f and g are differentiable at c, then*

(a) $f+g$ is differentiable at c and

$$(f+g)'(c) = f'(c) + g'(c).$$

(b) $f-g$ is differentiable at c and

$$(f-g)'(c) = f'(c) - g'(c).$$

(c) fg is differentiable at c and

$$(fg)'(c) = f(c)g'(c) + g(c)f'(c).$$

For any constant k, kf is differentiable at c and $(kf)'(c) = kf'(c)$.

(d) If $g(c) \neq 0$, then f/g is differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g^2(c)}.$$

Proof: (c)

$$\begin{aligned}\frac{f(x)g(x) - f(c)g(c)}{x-c} &= \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x-c} \\ &= f(x)\frac{g(x)-g(c)}{x-c} + g(c)\frac{f(x)-f(c)}{x-c}.\end{aligned}$$

Since f is continuous at c, $\lim_{x\to c} f(x) = f(c)$. Therefore, since f and g are continuous at c,

$$\lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x-c} = f(c)g'(c) + g(c)f'(c).$$

(d) We show first that

$$\left[\frac{1}{g(c)}\right]' = \frac{-g'(c)}{g^2(c)}.$$

Since g is continuous at c and $g(c) \neq 0$, there is an interval I containing c such that $g(c) \neq 0$ on I. Now

$$\frac{\frac{1}{g(x)} - \frac{1}{g(c)}}{x-c} = \frac{1}{g(x)g(c)}\,\frac{g(c)-g(x)}{x-c} = -\frac{1}{g(x)g(c)}\,\frac{g(x)-g(c)}{x-c}.$$

Since g is continuous at c, $\lim_{x\ toc} g(x) = g(c)$. Therefore

$$\lim_{x\to c} \frac{\frac{1}{g(x)} - \frac{1}{g(c)}}{x-c} = -\frac{1}{g^2(c)}\,g'(c) = \frac{-g'(c)}{g^2(c)}$$

(d) now follows by differentiating the product $f(x)\,\dfrac{1}{g(x)}$ using (c).

Example: If $f(x) = x^n$, n an integer, then $f'(x) = nx^{n-1}$.

Proof: Assume first that n is a positive integer, and use induction. Let S be the set of positive integers for which the statement holds. Then $1 \in S$ since if $f(x) = x$, then $f'(x) = 1 = 1\,x^0$. Now assume that the positive integer $k \in S$ and set $f(x) = x^{k+1}$. Since

$$x^{k+1} = x^k\,x$$

we have, by the product rule,

$$f'(x) = x^k\,1 + x\,k\,x^{k-1} = (k+1)x^k$$

and so $k+1 \in S$ and the statement holds for all positive integers n.

If n is a negative integer, then, for $x \neq 0$,

$$f(x) = x^n = \frac{1}{x^{-n}}$$

where $-n$ is a positive integer. By the quotient rule,

$$f'(x) = \frac{x^{-n}(0) - (-n)x{-}n - 1}{(x^{-n})^2} = \frac{n\,x^{-(n+1)}}{x^{-2n}} = nx^{n-1}.$$

Finally, if $f(x) = x^0 \equiv 1$, then $f'(x) = 0 = 0\dfrac{1}{x}$. There is slight difficulty with $x = 0$ in this case; 0^0 is a so-called *indeterminate form.*

THEOREM 17. (The Chain Rule) *Suppose that* $f : I \to \mathbb{R}$ *and* $g : J \to \mathbb{R}$*, and suppose that* $g(J) \subset I$*. If* g *is differentiable at* $c \in I$ *and* f *is differentiable at* $g(c)\ inJ$*, then* $f(g)$ *is differentiable at* c *and*

$$(f[g(c)])' = f'[g(c)]\,g'(c).$$

Pseudo-proof:

$$\frac{f[g(x)] - f(g(c)]}{x - c} = \frac{f[g(x)] - f[g(c)]}{g(x) - g(c)}\,\frac{f[g(x)] - f(g(c)]}{x - c}.$$

Set $u = g(x)$ and $a = g(c)$. Then, as $x \to x$, $u \to a$ since g is continuous at c. Thus

$$\lim_{x\to c}\frac{f[g(x)] - f(g(c)]}{x - c} = \lim_{u\to a}\frac{f(u) - f(a)}{u - a}\,\lim_{x\to c}\frac{f[g(x)] - f(g(c)]}{x - c} = f'(a)g'(c) = f'[g(c)]g'(c).$$

The problem with this proof is that while we know $x - c \neq 0$, we don't know that $u - a \neq 0$; that is, we don't know that $g(x) \neq g(c)$. This proof can be modified to take care of that contingency.

Exercises 3.4

1. Use either of the definitions of the derivative to find the derivative of each of the following functions.

   (a) $f(x) = \dfrac{1}{x}$.

   (b) $f(x) = \sqrt{x}$.

   (c) $f(x) = \dfrac{1}{\sqrt{x}}$.

   (d) $f(x) = x^{1/3}$.

   (e) $f(x) = \cos\, x$.

2. Determine the values of x for which the given function is differentiable and find the derivative.

   (a) $f(x) = |x - 3|$.

   (b) $f(x) = |x^2 - 1|$.

   (c) $f(x) = x\,|x|$.

3. Set $f(x) = \begin{cases} x^2, & if\ x \geq 0 \\ 0, & x < 0 \end{cases}$

   (a) Sketch the graph of f and show that f is differentiable at 0.

   (b) Find f' and sketch the graph of f'.

   (c) Is f' differentiable at 0?

4. Set $f(x) = \begin{cases} x\ \sin(1/x), & \text{if } x \neq 0 \\ 0, & x = 0 \end{cases}$ Determine whether or not f is differentiable at 0.

5. Set $g(x) = \begin{cases} x^2\ \sin(1/x), & if\ x \neq 0 \\ 0, & x = 0 \end{cases}$

   (a) Calculate the derivative of g at any number $c \neq 0$.

   (b) Use the definition to show that g is differentiable at 0 and find $g'(0)$.

   (c) Is g' continuous at 0?