

CHAPTER 1

Introduction to Differential Equations

1.1 Basic Terminology

Most of the phenomena studied in the sciences and engineering involve processes that change with time. For example, it is well known that the rate of decay of a radioactive material at time t is proportional to the amount of material present at time t . In mathematical terms this says that

$$\frac{dy}{dt} = ky, \quad k \text{ a negative constant} \quad (1)$$

where $y = y(t)$ is the amount of material present at time t .

If an object, suspended by a spring, is oscillating up and down, then Newton's Second Law of Motion ($F = ma$) combined with Hooke's Law (the restoring force of a spring is proportional to the displacement of the object) results in the equation

$$\frac{d^2y}{dt^2} + k^2y = 0, \quad k \text{ a positive constant} \quad (2)$$

where $y = y(t)$ denotes the position of the object at time t .

The basic equation governing the diffusion of heat in a uniform rod of finite length L is given by

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2} \quad (3)$$

where $u = u(x, t)$ is the temperature of the rod at time t at position x on the rod.

Each of these equations is an example of what is known as a differential equation.

DIFFERENTIAL EQUATION A *differential equation* is an equation which contains an unknown function together with one or more of its derivatives.

Here are some additional examples of differential equations.

Example 1.

- (a) $y' = \frac{x^2y - y}{y + 1}$.
- (b) $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 4x^3$.
- (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace's equation)
- (d) $\frac{d^3y}{dx^3} - 4 \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} = 3e^{-x}$.

TYPE As suggested by these examples, a differential equation can be classified into one of two general categories determined by the type of unknown function appearing in the equation. If the

unknown function depends on a single independent variable, then the equation is an *ordinary differential equation*; if the unknown function depends on more than one independent variable, then the equation is a *partial differential equation*.

The differential equations (1) and (2) are ordinary differential equations, and (3) is a partial differential equation. In Example 1, equations (a), (b) and (d) are ordinary differential equations and equation (c) is a partial differential equation.

ORDER The *order* of a differential equation is the order of the highest derivative of the unknown function appearing in the equation.

Equation (1) is a first order equation, and equations (2) and (3) are second order equations. In Example 1, equation (a) is a first order equation, (b) and (c) are second order equations, and equation (d) is a third order equation.

The obvious question that we want to consider is that of “solving” a given differential equation.

SOLUTION A *solution of a differential equation* is a function defined on some interval I (in the case of an ordinary differential equation) or on some domain D in two or higher dimensional space (in the case of a partial differential equation) with the property that the equation reduces to an identity when the function is substituted into the equation.

Example 2. Given the second-order ordinary differential equation

$$x^2 y'' - 2x y' + 2y = 4x^3. \quad [\text{Example 1 (b)}]$$

Show that:

(a) $y(x) = x^2 + 2x^3$ is a solution.

(b) $z(x) = 2x^2 + 3x$ is not a solution.

SOLUTION (a) The first step is to calculate the first two derivatives of y .

$$\begin{aligned} y &= x^2 + 2x^3, \\ y' &= 2x + 6x^2, \\ y'' &= 2 + 12x. \end{aligned}$$

Next, we substitute y and its derivatives into the differential equation.

$$x^2(2 + 12x) - 2x(2x + 6x^2) + 2(x^2 + 2x^3) \stackrel{?}{=} 4x^3.$$

Simplifying the left-hand side, we get

$$2x^2 + 12x^3 - 4x^2 - 12x^3 + 2x^2 + 4x^3 \stackrel{?}{=} 4x^3 \quad \text{and} \quad 4x^3 = 4x^3.$$

The equation is satisfied; $y = x^2 + 2x^3$ is a solution.

(b) The first two derivatives of z are:

$$\begin{aligned} z &= 2x^2 + 3x, \\ z' &= 4x + 3, \\ z'' &= 4. \end{aligned}$$

Substituting into the differential equation, we have

$$x^2(4) - 2x(4x + 3) + 2(2x^2 + 3x) \stackrel{?}{=} 4x^3.$$

Simplifying the left-hand side, we get

$$4x^2 - 8x^2 - 6x + 4x^2 + 6x = 0 \neq 4x^3.$$

The function $z = 2x^2 + 3x$ is not a solution of the differential equation. ■

Example 3. Show that $u(x, y) = \cos x \sinh y + \sin x \cosh y$ is a solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

SOLUTION The first step is to calculate the indicated partial derivatives.

$$\begin{aligned}\frac{\partial u}{\partial x} &= -\sin x \sinh y + \cos x \cosh y, \\ \frac{\partial^2 u}{\partial x^2} &= -\cos x \sinh y - \sin x \cosh y, \\ \frac{\partial u}{\partial y} &= \cos x \cosh y + \sin x \sinh y, \\ \frac{\partial^2 u}{\partial y^2} &= \cos x \sinh y + \sin x \cosh y.\end{aligned}$$

Substituting into the differential equation, we find that

$$(-\cos x \sinh y - \sin x \cosh y) + (\cos x \sinh y + \sin x \cosh y) = 0$$

and the equation is satisfied; $u(x, y) = \cos x \sinh y + \sin x \cosh y$ is a solution of Laplace's equation. ■

You know from your experience in previous mathematics courses that the calculus of functions of several variables (limits, graphing, differentiation, integration and applications) is more complicated than the calculus of functions of a single variable. By extension, therefore, you would expect that the study of partial differential equations would be more complicated than the study of ordinary differential equations. This is indeed the case! Since the intent of this material is to introduce some of the basic theory and methods for differential equations, we shall confine ourselves to ordinary differential equations from this point forward. Hereafter the term *differential equation* shall be interpreted to mean *ordinary differential equation*. ■

We begin by considering the simple first-order differential equation

$$y' = f(x)$$

where f is some given function. In this case we can find y simply by integrating:

$$\begin{aligned}y = y(x) &= \int f(x) dx \\ y &= F(x) + C\end{aligned}$$

where F is an antiderivative of f and C is an arbitrary constant. Not only did we find a solution of the differential equation, we found a whole family of solutions each member of which is determined by assigning a specific value to the constant C . In this context, the arbitrary constant is called a *parameter* and the family of solutions is called a *one-parameter family*.

Remark In calculus you learned that not only is each member of the family $y = F(x) + C$ a solution of the differential equation but this family actually represents the set of *all* solutions of the equation; that is, there are no other solutions outside of this family. ■

Example 4. The differential equation

$$y' = 3x^2 - \sin 2x$$

has the one-parameter family of solutions

$$\begin{aligned} y(x) &= x^3 + \frac{1}{2} \cos 2x + C \\ &= \int (3x^2 - \sin 2x) dx = x^3 + \frac{1}{2} \cos 2x + C. \quad \blacksquare \end{aligned}$$

In a similar manner, if we are given a second order equation of the form

$$y'' = f(x)$$

then we can find y by integrating twice with each integration step producing an arbitrary constant of integration.

Example 5. If

$$y'' = 6x + 4e^{2x},$$

then

$$y' = \int (6x + 4e^{2x}) dx = 3x^2 + 2e^{2x} + C_1$$

and

$$y = \int (3x^2 + 2e^{2x} + C_1) dx = x^3 + e^{2x} + C_1x + C_2, \quad C_1, C_2 \text{ arbitrary constants.}$$

The set of functions

$$y = x^3 + e^{2x} + C_1x + C_2$$

is a *two-parameter family* of solutions of the differential equation

$$y'' = 6x + 4e^{2x}. \quad \blacksquare$$

n -PARAMETER FAMILY OF SOLUTIONS The examples given above are very special cases. In general, to find a set of solutions of an n -th order differential equation we would expect, intuitively, to "integrate" n times with each integration step producing an arbitrary constant of integration. As a result, we *expect* an n -th order differential equation to have an *n -parameter family of solutions*.

SOLVING A DIFFERENTIAL EQUATION To *solve* an n -th order differential equation means to find an n -parameter family of solutions. It is important to understand that the two n 's here are the same. For example, to solve a fourth-order differential equation we need to find a four-parameter family of solutions.

Example 6. Show that $y = C_1x^2 + C_2x + 2x^3$ is a two-parameter family of solutions of

$$x^2y'' - 2xy' + 2y = 4x^3.$$

SOLUTION We calculate the first two derivatives of y and then substitute into the differential equation:

$$y = C_1x^2 + c_2x + 2x^3$$

$$y' = 2C_1x + C_2 + 6x^2,$$

$$y'' = 2C_1 + 12x;$$

$$x^2(2C_1 + 12x) - 2x(2C_1x + C_2 + 6x^2) + 2(C_1x^2 + C_2x + 2x^3) \stackrel{?}{=} 4x^3.$$

Simplifying the left-hand side and re-arranging the terms, we get

$$C_1(2x^2 - 4x^2 + 2x^2) + C_2(-2x + 2x) + 12x^3 - 12x^3 + 4x^3 \stackrel{?}{=} 4x^3$$

$$C_1(0) + C_2(0) + 4x^3 \stackrel{?}{=} 4x^3$$

$$4x^3 = 4x^3$$

Thus, for any two constants C_1, C_2 , the function $y = C_1x^2 + C_2x + 2x^3$, is a solution of the differential equation. The set of functions $y = C_1x^2 + C_2x + 2x^3$ is a two-parameter family of solutions of the equation. ■

GENERAL SOLUTION For most of the equations that we will study in this course, an n -parameter family of solutions of a given n -th order equation will represent the set of *all* solutions of the equation. In such cases, the term *general solution* is often used in place of n -parameter family of solutions.

Solutions which are not included in the n -parameter family of solutions are called *singular solutions*.

Example 7. Consider the differential equation

$$y' = 4x(y - 1)^{1/2}.$$

$y = (x^2 + C)^2 + 1$ is a one-parameter family of solutions (verify this). (In Section 2.2 you will learn how to solve this equation.) Also, it is easy to see that the constant function $y \equiv 1$ is a solution of the equation:

$$y \equiv 1 \quad \text{implies} \quad y' \equiv 0.$$

and

$$0 = 4x(1 - 1)^{1/2} = 0;$$

the equation is satisfied. This solution is not included in the general solution because there is no value that you can assign to C that will produce the solution $y \equiv 1$; $y \equiv 1$ is a singular solution. ■

PARTICULAR SOLUTION If specific values are assigned to the arbitrary constants in the general solution of a differential equation, then the resulting solution is called a *particular solution* of the equation.

Example 8. $y = C_1x^2 + C_2x + 2x^3$ is the general solution of the second order differential equation

$$x^2y'' - 2xy' + 2y = 4x^3.$$

Setting $C_1 = 2$ and $C_2 = 3$, we get the particular solution $y = 2x^2 + 3x + 2x^3$. ■

THE DIFFERENTIAL EQUATION OF AN n -PARAMETER FAMILY: If we are given an n -parameter family of curves, then we can regard the family as the general solution of an n^{th} -order differential equation and attempt to find the equation. The equation that we search for, called the *differential equation of the family*, should be free of the parameters (arbitrary constants), and its order should equal the number of parameters. The strategy for finding the differential equation of a given n -parameter family is to differentiate the equation n times. This produces a system of $n + 1$ equations which can be used to eliminate the arbitrary constants.

Example 9. Given the one-parameter family $y^2 = Cx^3 + 3$. Find the differential equation of the family.

SOLUTION Since we have a one-parameter family, we are looking for a first order equation. Differentiating the given equation, we obtain

$$2yy' = 3Cx^2$$

We can solve this equation for C to get $C = \frac{2yy'}{3x^2}$. Substituting this into the given equation gives

$$y^2 = \left(\frac{2yy'}{3x^2}\right)x^2 + 3 \quad \text{which simplifies to} \quad y' = \frac{3y^2 - 9}{2xy}.$$

This is the differential equation of the given family. ■

Example 10. Find the differential equation of the two-parameter family

$$y = C_1x + C_2x^2.$$

SOLUTION We are looking for a second order differential equation. Differentiating twice, we obtain the equations

$$y' = C_1 + 2C_2x \quad \text{and} \quad y'' = 2C_2.$$

From the second equation, we get $C_2 = y''/2$. Substituting this into the first equation gives

$$y' = C_1 + 2\left(\frac{y''}{2}\right)x \quad \text{so that} \quad C_1 = y' - xy''.$$

Substituting these values into the original equation, we have

$$y = (y' - xy'')x + \frac{y''}{2}x^2 \quad \text{which simplifies to} \quad x^2y'' - xy' - 2y = 0. \quad \blacksquare$$

INITIAL-VALUE PROBLEMS As noted above, we can obtain a particular solution of an n th order differential equation simply by assigning specific values to the n constants in the general solution. However, in typical applications of differential equations you will be asked to find a solution of a given equation that satisfies certain *preassigned* conditions.

Example 11. Find a solution of

$$y' = 3x^2 - 2x$$

that passes through the point $(1, 3)$.

SOLUTION In this case, we can find the general solution by integrating:

$$y = \int (3x^2 - 2x) dx = x^3 - x^2 + C.$$

The general solution is $y = x^3 - x^2 + C$.

To find a solution that passes through the point $(2, 6)$, we set $x = 2$ and $y = 6$ in the general solution and solve for C :

$$6 = 2^3 - 2^2 + C = 8 - 4 + C \quad \text{which implies} \quad C = 2.$$

Thus, $y = x^3 - x^2 + 2$ is a solution of the differential equation that satisfies the given condition. In fact, it is the only solution that satisfies the condition since the general solution represented all solutions of the equation and the constant C was uniquely determined. ■

Example 12. Find a solution of

$$x^2y'' - 2xy' + 2y = 4x^3$$

which passes through the point $(1, 4)$ with slope 2.

SOLUTION The general solution of the differential equation is

$$y = C_1x^2 + C_2x + 2x^3.$$

Setting $x = 1$ and $y = 4$ in the general solution yields the equation

$$C_1 + C_2 + 2 = 4 \quad \text{which implies} \quad C_1 + C_2 = 2.$$

The second condition, slope 2 at $x = 1$, is a condition on y' . We want $y'(1) = 2$. We calculate y' :

$$y' = 2C_1x + C_2 + 6x^2,$$

and then set $x = 1$ and $y' = 2$. This yields the equation

$$2C_1 + C_2 + 6 = 2 \quad \text{which implies} \quad 2C_1 + C_2 = -4.$$

Now we solve the two equations simultaneously:

$$\begin{aligned} C_1 + C_2 &= 2 \\ 2C_1 + C_2 &= -4 \end{aligned}$$

We get: $C_1 = -6$, $C_2 = 8$. A solution of the differential equation satisfying the given conditions is $y(x) = -6x^2 + 8x + 2x^3$. Moreover, this is the only solution that satisfies the conditions because, again, the values of the C 's were uniquely determined. ■

INITIAL CONDITIONS Conditions such as those imposed on the solutions in Examples 11 and 12 are called *initial conditions*. This term originated with applications where processes are usually observed over time, starting with some initial state at time $t = 0$.

Example 13. The position $y(t)$ of a weight suspended on a spring and oscillating up and down is governed by the differential equation

$$y'' + 9y = 0.$$

(a) Show that the general solution of the differential equation is: $y(t) = C_1 \sin 3t + C_2 \cos 3t$.

(b) Find a solution that satisfies the initial conditions $y(0) = 1$, $y'(0) = -2$.

SOLUTION

(a)

$$y = C_1 \sin 3t + C_2 \cos 3t$$

$$y' = 3C_1 \cos 3t - 3C_2 \sin 3t$$

$$y'' = -9C_1 \sin 3t - 9C_2 \cos 3t$$

Substituting into the differential equation, we get

$$y'' + 9y = (-9C_1 \sin 3t - 9C_2 \cos 3t) + 9(C_1 \sin 3t + C_2 \cos 3t) = 0.$$

Thus $y(t) = C_1 \sin 3t + C_2 \cos 3t$ is the general solution.

(b) Applying the initial conditions, we obtain the pair of equations

$$y(0) = 1 = C_1 \sin 0 + C_2 \cos 0 = C_2 \quad \text{which implies} \quad C_2 = 1,$$

$$y'(0) = -2 = 3C_1 \cos 0 - 3C_2 \sin 0 \quad \text{which implies} \quad C_1 = -\frac{2}{3}.$$

A solution which satisfies the initial conditions is: $y(t) = -\frac{2}{3} \sin 3t + \cos 3t$. ■

Any n -th order differential equation with independent variable x and unknown function y can be written in the form

$$F(x, y, y', y'', \dots, y^{(n-1)}, y^{(n)}) = 0. \tag{4}$$

by moving all the non-zero terms to the left-hand side. Since we are talking about an n -th order equation, $y^{(n)}$ must appear explicitly in the expression F . Each of the other arguments may or may not appear explicitly.

n -th ORDER INITIAL-VALUE PROBLEM An n -th order initial-value problem consists of an n -th order differential equation

$$F(x, y, y', y'', \dots, y^{(n)}) = 0$$

together with n conditions of the form

$$y(c) = k_0, y'(c) = k_1, y''(c) = k_2, \dots, y^{(n-1)}(c) = k_{n-1}.$$

It is important to understand that to be an n -th order initial-value problem there must be n conditions (same n) of exactly the form indicated in the definition. For example, the problem:

EXISTENCE AND UNIQUENESS The fundamental questions in any course on differential equations are:

- (1) Does a given initial-value problem *have* a solution? That is, do solutions to the problem *exist*?
- (2) If a solution does exist, is it *unique*? That is, is there exactly one solution to the problem or is there more than one solution.

The initial-value problems in Examples 1, 2, and 3 each had a unique solution; values for the arbitrary constants in the general solution were uniquely determined.

Example 14. The function $y = x^2$ is a solution of the differential equation $y' = 2\sqrt{y}$ and $y(0) = 0$. Thus the initial-value problem

$$y' = 2\sqrt{y}; \quad y(0) = 0.$$

has a solution. However, $y \equiv 0$ also satisfies the differential equation and $y(0) = 0$. Thus, the initial-value problem does not have a *unique* solution. In fact, for any positive number a , the function

$$y(x) = \begin{cases} 0, & x \leq a \\ (x - a)^2, & x > a \end{cases}$$

is a solution of the initial-value problem. ■

Example 15. The one-parameter family of functions $y = Cx$ is the general solution of

$$y' = \frac{y}{x}.$$

There is no solution that satisfies $y(0) = 1$; the initial-value problem

$$y' = \frac{y}{x}, \quad y(0) = 1$$

does not have a solution. ■

The questions of *existence and uniqueness* of solutions will be addressed in the specific cases of interest to us. A general treatment of existence and uniqueness of solutions of initial-value problems is beyond the scope of this survey.

Exercises

1. Classify the following differential equations with respect to type (i.e., ordinary or partial) and order.

(a) $(y')^2 + xy y' = \sin x$.

(b) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$.

(c) $\left(\frac{d^2 y}{dx^2}\right)^3 + xy^2 = x$.

(d) $\frac{d^2 y}{dx^2} - 2y\frac{dy}{dx} + xy^2 = \frac{d^3}{dx^3}[e^{-2x}]$.

For each differential equation determine whether or not the given functions are solutions.

2. $\frac{d^3 y}{dx^3} + \frac{dy}{dx} = e^x$; $y(x) = 1 + \sin x + \frac{1}{2}e^x$, $z(x) = 2 \cos x + \frac{1}{2}e^x$.

3. $xy'' + y' = 0$; $y_1(x) = \ln(1/x)$, $y_2(x) = x^2$.

4. $(x+1)y'' + xy' - y = (x+1)^2$; $y(x) = e^{-x} + x^2 + 1$, $z(x) = x^2 + 1$.

5. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$; $u_1(x, y) = \ln \sqrt{x^2 + y^2}$, $u_2(x, y) = x^3 - 3xy^2$.

6. $y'' - y = 2 - x$; $y(x) = e^{-x} + x - 2$, $z(x) = \sinh x + x - 2$.

Determine values of r , if possible, so that the given differential equation has a solution of the form $y = e^{rx}$.

7. $y'' + 2y' - 8y = 0$.

8. $y'' - 6y' + 9y = 0$.

9. $y''' - 4y'' + 5y' - 2y = 0$.

Determine values of r , if possible, so that the given differential equation has a solution of the form $y = x^r$.

10. $x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0$.

11. $x^2 y'' - 3xy' + 4y = 0$.

12. (a) Show that each member of the two-parameter family of functions

$$y = C_1 e^{2x} + C_2 e^{-x}$$

is a solution of the differential equation $y'' - y' - 2y = 0$.

- (b) Find a solution of the initial value problem $y'' - y' - 2y = 0$; $y(0) = 2$, $y'(0) = 1$.

13. (a) Show that each member of the two-parameter family of functions

$$y = C_1x^2 + C_2x^2 \ln x$$

is a solution of the differential equation $x^2y'' - 3xy' + 4y = 0$.

- (b) Find a solution of the initial value problem $x^2y'' - 3xy' + 4y = 0$; $y(1) = 0$, $y'(1) = 1$.
- (c) Is there a member of the two-parameter family which satisfies the initial condition $y(0) = y'(0) = 0$?
- (d) Is there a member of the two-parameter family which satisfies the initial condition $y(0) = 0$, $y'(0) = 1$? If not, why not?

14. Each member of the two-parameter family of functions

$$y = C_1 \sin x + C_2 \cos x$$

is a solution of the differential equation $y'' + y = 0$.

- (a) Determine whether there are one or more members of this family that satisfy the conditions

$$y(0) = 0, \quad y(\pi) = 0.$$

- (b) Show that the zero function, $y \equiv 0$, is the only member of the family that satisfies the conditions

$$y(0) = 0, \quad y(\pi/2) = 0.$$

15. Given the differential equation $x(y')^2 - 2yy' + 4x = 0$.

- (a) Show that the one-parameter family $y = \frac{x^2 + C^2}{C}$ is the general solution of the equation
- (b) Show that each of $y = 2x$ and $y = -2x$ is a solution of the equation. Note that these functions are not included in the general solution of the equation; they are singular solutions of the equation.

Find the differential equation of the given family.

16. $y^3 = Cx^2 + 3$.
17. $y = Ce^{2x} + e^{-2x}$
18. $y = C_1e^x + C_2e^{-2x}$.
19. $y = C_1x + C_2x^2$.
20. $y = C_1 + C_2x + C_3x^2$.