Chapter 3

Second Order Linear Differential Equations

3.1. Introduction; Basic Terminology

Recall that a first order linear differential equation is an equation which can be written in the form

$$y' + p(x)y = q(x)$$

where p and q are continuous functions on some interval I. A second order linear differential equation has an analogous form.

SECOND ORDER LINEAR DIFFERENTIAL EQUATION: A *second order, linear differential equation* is an equation which can be written in the form

$$y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

where p, q, and f are continuous functions on some interval I.

The functions p and q are called the *coefficients* of the equation; the function f on the right-hand side is called the *forcing function* or the *nonhomogeneous term*. The term "forcing function" comes from the applications of second-order equations; an explanation of the alternative term " nonhomogeneous" is given below.

A second order equation which is not linear is said to be *nonlinear* .

Remarks on "Linear." Set $L[y] = y'' + p(x)y' + q(x)y$. If we view L as an "operator" that transforms a twice differentiable function $y = y(x)$ into the continuous function

$$L[y(x)] = y''(x) + p(x)y'(x) + q(x)y(x),$$

then, for any two twice differentiable functions $y_1(x)$ and $y_2(x)$,

$$L[y_1(x) + y_2(x)] = L[y_1(x)] + L[y_2(x)]$$

and, for any constant c,

$$L[cy(x)] = cL[y(x)].$$

As introduced in Section 2.1, L is a linear transformation, specifically, a *linear differential operator:*

$$L : C^2(I) \to C(I)$$

where $C^2(I)$ is the vector space of twice continuously differentiable functions on I and $C(I)$ is the vector space of continuous functions on I. ■

The first thing we need to know is that an initial-value problem has a solution, and that it is unique.

THEOREM 1. (Existence and Uniqueness Theorem:) Given the second order linear equation (1). Let a be any point on the interval I, and let α and β be any two real numbers. Then the initial-value problem

$$y'' + p(x)\,y' + q(x)\,y = f(x), \quad y(a) = \alpha,\; y'(a) = \beta$$

has a unique solution.

A proof of this theorem is beyond the scope of this course.

Remark: We can solve any first order linear differential equation; Chapter 2 gives a method for finding the general solution of *any* first order linear equation. In contrast, *there is no general method for solving second (or higher) order linear differential equations.* There are, however, methods for solving certain special types of second order linear equations and we'll consider these in this chapter. ■

DEFINITION 1. (**Homogeneous/Nonhomogeneous Equations**) The linear differential equation (1) is *homogeneous* [1] if the function f on the right side is 0 for all $x \in I$. In this case, equation (1) becomes

$$y'' + p(x)\,y' + q(x)\,y = 0. \tag{2}$$

Equation (1) is *nonhomogeneous* if f is not the zero function on I, i.e., (1) is nonhomogeneous if $f(x) \neq 0$ for some $x \in I$.

[1] This use of the term "homogeneous" is completely different from its use to categorize the first order equation $y' = f(x, y)$ in Exercises 2.2.

For reasons which will become clear, almost all of our attention is focused on homogeneous equations.

Homogeneous Equations

As defined above, a second order, linear, homogeneous differential equation is an equation that can be written in the form

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{3}$$

where p and q are continuous functions on some interval I.

The Trivial Solution: The first thing to note is that the zero function, $y(x) = 0$ for all $x \in I$, (also denoted by $y \equiv 0$) is a solution of (1). The zero solution is called the *trivial solution* . Obviously our main interest is in finding *nontrivial* solutions. ■

Let $\mathcal{S} = \{y = y(x) \,:\, y \text{ is a solution of (1)}\}$; $\mathcal{S}$ is a subset of $C^2(I)$.

THEOREM 2. Let $y = u(x)$, $y = v(x) \in \mathcal{S}$, and let C be any real number. Then

$$\begin{aligned} y(x) &= u(x) + v(x) \in \mathcal{S} \quad \text{and} \\ y(x) &= Cu(x) \in \mathcal{S}. \end{aligned}$$

That is, $\mathcal{S}$ is a subspace of $C^2(I)$. Indeed, $\mathcal{S}$ is the *null space* of the linear differential operator L. ■

Theorem 1 can be restated as: If $y = y_1(x)$, $y = y_2(x) \in \mathcal{S}$ and C_1, C_2 are real numbers, then

$$C_1\,y_1 + C_2\,y_2 \in \mathcal{S}.$$

The expression

$$C_1\,y_1 + C_2\,y_2$$

is called a *linear combination* of y_1 and y_2.

Note that the equation

$$y(x) = C_1 y_1(x) + C_2 y_2(x) \tag{4}$$

where C_1 and C_2 are arbitrary constants, has the form of the general solution of equation (1). So the question is: If y_1 and y_2 are solutions of (1), is the expression (2) the general solution of (1)? That is, can every solution of (1) be written as a linear combination of y_1 and y_2? It turns out that (2) may or not be the general solution; it depends on the relation between the solutions y_1 and y_2.

Suppose that $y = y_1(x)$ and $y = y_2(x)$ are solutions of equation (1). Under what conditions is (2) the general solution of (1)?

Let $u = u(x)$ be *any* solution of (1) and choose *any* point $a \in I$. Suppose that

$$\alpha = u(a), \quad \beta = u'(a).$$

Then u is a member of the two-parameter family (2) if and only if there are values for C_1 and C_2 such that

$$\begin{aligned} C_1 y_1(a) + C_2 y_2(a) &= \alpha \\ C_1 y_1'(a) + C_2 y_2'(a) &= \beta \end{aligned}$$

If we multiply the first equation by $y_2'(a)$, the second equation by $-y_2(a)$, and add, we get

$$[y_1(a)y_2'(a) - y_2(a)y_1'(a)]C_1 = \alpha y_2'(a) - \beta y_2(a).$$

Similarly, if we multiply the first equation by $-y_1'(a)$, the second equation by $y_1(a)$, and add, we get

$$[y_1(a)y_2'(a) - y_2(a)y_1'(a)]C_2 = -\alpha y_1'(a) + \beta y_1(a).$$

We are guaranteed that this pair of equations has solutions C_1, C_2 if and only if

$$y_1(a)y_2'(a) - y_2(a)y_1'(a) \neq 0$$

in which case

$$C_1 = \frac{\alpha y_2'(a) - \beta y_2(a)}{y_1(a)y_2'(a) - y_2(a)y_1'(a)} \quad \text{and} \quad C_2 = \frac{-\alpha y_1'(a) + \beta y_1(a)}{y_1(a)y_2'(a) - y_2(a)y_1'(a)}.$$

Since a was chosen to be any point on I, we conclude that (2) is the general solution of (1) if and only if

$$y_1(x)y_2'(x) - y_2(x)y_1'(x) \neq 0 \quad \text{for all } x \in I.$$

DEFINITION 2. (Wronskian) Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of (1). The function W defined by

$$W[y_1, y_2](x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

is called the *Wronskian* of y_1, y_2.

We use the notation $W[y_1, y_2](x)$ to emphasize that the Wronskian is a function of x that is determined by two solutions y_1, y_2 of equation (1). When there is no danger of confusion, we'll shorten the notation to $W(x)$.

Remark Note that

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x). \qquad \blacksquare$$

THEOREM 3. Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of equation (1), and let $W(x)$ be their Wronskian. Exactly one of the following holds:

(i) $W(x) = 0$ for all $x \in I$ and y_1 is a constant multiple of y_2.

(ii) $W(x) \neq 0$ for all $x \in I$ and $y = C_1y_1(x) + C_2y_2(x)$ is the general solution of (1)

DEFINITION 3. (Fundamental Set) A pair of solutions $y = y_1(x)$, $y = y_2(x)$ of equation (1) forms a *fundamental set of solutions* if

$$W[y_1, y_2](x) \neq 0 \quad \text{for all } x \in I.$$

Linear Dependence; Linear Independence

By Theorem 3, if y_1 and y_2 are solutions of equation (1) such that $W[y_1, y_2] \equiv 0$, then y_1 is a constant multiple of y_2. The question as to whether or not one function is a multiple of another function and the consequences of this are of fundamental importance in differential equations and in linear algebra.

In this sub-section we are dealing with functions in general, not just solutions of the differential equation (1)

DEFINITION 4. (Linear Dependence; Linear Independence) Given two functions $f = f(x)$, $g = g(x)$ defined on an interval I. The functions f and g are *linearly dependent on* I if and only if there exist two real numbers c_1 and c_2, not both zero, such that

$$c_1f(x) + c_2g(x) \equiv 0 \quad \text{on } I.$$

The functions f and g are *linearly independent on* I if they are not linearly dependent. $\blacksquare$

Linear dependence can be stated equivalently as: f and g are linearly dependent on I if and only if one of the functions is a constant multiple of the other.

The term Wronskian defined above for two solutions of equation (1) can be extended to any two differentiable functions f and g. Let $f = f(x)$ and $g = g(x)$ be differentiable functions on an interval I. The function $W[f, g]$ defined by

$$W[f, g](x) = f(x)g'(x) - g(x)f'(x)$$

is called the *Wronskian* of f, g.

There is a connection between linear dependence/independence and Wronskian.

THEOREM 4. Let $f = f(x)$ and $g = g(x)$ be differentiable functions on an interval I. If f and g are linearly dependent on I, then $W(x) = 0$ for all $x \in I$ ($W \equiv 0$ on I). ■

This theorem can be stated equivalently as: Let $f = f(x)$ and $g = g(x)$ be differentiable functions on an interval I. If $W(x) \neq 0$ for at least one $x \in I$, then f and g are linearly independent on I.

Going back to differential equations, Theorem 4 can be restated as

Theorem 4' Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of equation (1). Exactly one of the following holds:

(i) $W(x) = 0$ for all $x \in I$; y_1 and y_2 are linear dependent.

(ii) $W(x) \neq 0$ for all $x \in I$; y_1 and y_2 are linearly independent and $y = C_1y_1(x) + C_2y_2(x)$ is the general solution of (1).

The statements "$y_1(x)$, $y_2(x)$ form a fundamental set of solutions of (1)" and "$y_1(x)$, $y_2(x)$ are linearly independent solutions of (1)" are synonymous.

The results of this section can be captured in one statement

The set $\mathcal{S}$ of solutions of (1), a subspace of $C^2(I)$, has dimension 2, the order of the equation.

Exercises 3.1

In Exercises 1 – 2, verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions of the equation?

1. $y'' - 4y' + 4y = 0; \quad y_1(x) = e^{2x},\ y_2(x) = xe^{2x}.$

2. $x^2y'' - x(x+2)y' + (x+2)y = 0; \quad y_1(x) = x,\ y_2(x) = xe^x.$

3. Given the differential equation $y'' - 3y' - 4y = 0.$

   (a) Find two values of r such that $y = e^{rx}$ is a solution of the equation.

   (b) Determine a fundamental set of solutions and give the general solution of the equation.

   (c) Find the solution of the equation satisfying the initial conditions $y(0) = 1,\ y'(0) = 0.$

4. Given the differential equation $y'' - \left(\frac{2}{x}\right)y' - \left(\frac{4}{x^2}\right)y = 0.$

   (a) Find two values of r such that $y = x^r$ is a solution of the equation.

   (b) Determine a fundamental set of solutions and give the general solution of the equation.

   (c) Find the solution of the equation satisfying the initial conditions $y(1) = 2,\ y'(1) = -1.$

   (d) Find the solution of the equation satisfying the initial conditions $y(2) = y'(2) = 0.$

5. Given the differential equation $(x^2 + 2x - 1)y'' - 2(x+1)y' + 2y = 0.$

   (a) Show that the equation has a linear polynomial and a quadratic polynomial as solutions.

   b Find two linearly independent solutions of the equation and give the general solution.

6. Let $y = y_1(x)$ be a solution of (1): $y'' + p(x)y' + q(x)y = 0$ where p and q are continuous function on an interval I. Let $a \in I$ and assume that $y_1(x) \neq 0$ on I. Set
$$y_2(x) = y_1(x)\int_a^x \frac{e^{-\int_a^t p(u)\,du}}{y_1^2(t)}\,dt.$$
   Show that y_2 is a solution of (1) and that y_1 and y_2 are linearly independent.

   Use Exercise 6 to find a fundamental set of solutions of the given equation starting from the given solution y_1.

7. $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0; \quad y_1(x) = x.$

8. $y'' - \dfrac{2x-1}{x}y' + \dfrac{x-1}{x}y = 0; \quad y_1(x) = e^x.$

9. Let $y = y_1(x)$ and $y = y_2(x)$ be solutions of equation (1):
$$y'' + p(x)y' + q(x)y = 0$$
on an interval I. Let $a \in I$ and suppose that
$$y_1(a) = \alpha,\ y_1'(a) = \beta \qquad \text{and} \qquad y_2(a) = \gamma,\ y_2'(a) = \delta.$$
Under what conditions on $\alpha,\ \beta,\ \gamma, \delta$ will the functions y_1 and y_2 be linearly independent on I?

10. Suppose that $y = y_1(x)$ and $y = y_2(x)$ are solutions of (1). Show that if $y_1(x) \neq 0$ on I and $W[y_1, y_2](x) \equiv 0$ on I, then $y_2(x) = \lambda y_1(x)$ on I.

3.2. Homogenous Equations with Constant Coefficients

We have emphasized that there are no general methods for solving second (or higher) order linear differential equations. However, there are some special cases for which solution methods do exist. In this and the following sections we consider such a case, linear equations with constant coefficients.

A *second order, linear, homogeneous differential equation with constant coefficients* is an equation which can be written in the form
$$y'' + ay' + by = 0 \tag{1}$$
where a and b are real numbers.

You have seen that the function $y = e^{-ax}$ is a solution of the first-order linear equation
$$y' + ay = 0,$$
the equation modeling exponential growth and decay. This suggests that equation (1) may also have an exponential function $y = e^{rx}$ as a solution.

If $y = e^{rx}$, then $y' = r\,e^{rx}$ and $y'' = r^2\,e^{rx}$. Substitution into (1) gives
$$r^2\,e^{rx} + a\,(r\,e^{rx}) + b\,(e^{rx}) = e^{rx}\left(r^2 + ar + b\right) = 0.$$
Since $e^{rx} \neq 0$ for all x, we conclude that $y = e^{rx}$ is a solution of (1) if and only if
$$r^2 + ar + b = 0. \tag{2}$$
Thus, if r is a root of the quadratic equation (2), then $y = e^{rx}$ is a solution of equation (1); we can find solutions of (1) by finding the roots of the quadratic equation (2).

DEFINITION 1. Given the differential equation (1). The corresponding quadratic equation (2)

$$r^2 + ar + b = 0$$

is called the *characteristic equation of* (1); the quadratic polynomial $r^2 + ar + b$ is called the *characteristic polynomial.* The roots of the characteristic equation are called the *characteristic roots* . ■

The nature of the solutions of the differential equation (1) depends on the nature of the roots of its characteristic equation (2). There are three cases to consider:

(1) Equation (2) has two, distinct real roots, $r_1 = \alpha,\ r_2 = \beta$.

(2) Equation (2) has only one real root, $r = \alpha$.

(3) Equation (2) has complex conjugate roots, $r_1 = \alpha + i\,\beta,\ r_2 = \alpha - i\,\beta,\ \beta \neq 0$.

Case I: The characteristic equation has two, distinct real roots, $r_1 = \alpha,\ r_2 = \beta$. In this case,

$$y_1(x) = e^{\alpha x} \qquad \text{and} \qquad y_2(x) = e^{\beta x}$$

are solutions of (1). Since $\alpha \neq \beta$, y_1 and y_2 are not constant multiples of each other, the pair y_1, y_2 forms a fundamental set of solutions of equation (1) and

$$y = C_1\, e^{\alpha x} + C_2\, e^{\beta x}$$

is the general solution.

Note: We can use the Wronskian to verify the independence of y_1 and y_2:

$$W(x) = y_1 y_2' - y_2 y_1' = e^{\alpha x}\left(\beta\, e^{\beta x}\right) - e^{\beta x}\left(\alpha\, e^{\alpha x}\right) = (\alpha - \beta)\, e^{(\alpha+\beta)x} \neq 0. \qquad ■$$

Example 1. Find the general solution of the differential equation

$$y'' + 2y' - 8y = 0.$$

SOLUTION The characteristic equation is

$$\begin{aligned} r^2 + 2r - 8 &= 0 \\ (r+4)(r-2) &= 0 \end{aligned}$$

The characteristic roots are: $r_1 = -4,\ r_2 = 2$. The functions $y_1(x) = e^{-4x},\ y_2(x) = e^{2x}$ form a fundamental set of solutions of the differential equation and

$$y = C_1\, e^{-4x} + C_2\, e^{2x}$$

is the general solution of the equation. ■

Case II: The characteristic equation has only one real root, $r = \alpha$.[2] Then

$$y_1(x) = e^{\alpha x} \qquad \text{and} \qquad y_2(x) = x\, e^{\alpha x}$$

are linearly independent solutions of equation (1) and

$$y = C_1\, e^{\alpha x} + C_2\, x\, e^{\alpha x}$$

is the general solution.

Proof: We know that $y_1(x) = e^{\alpha x}$ is one solution of the differential equation; we need to find another solution which is independent of y_1. Since the characteristic equation has only one real root, α, the equation must be

$$r^2 + ar + b = (r - \alpha)^2 = r^2 - 2\alpha r + \alpha^2 = 0$$

and the differential equation (1) must have the form

$$y'' - 2\alpha\, y' + \alpha^2 y = 0. \tag{*}$$

Now, $z = C\, e^{\alpha x}$, C any constant, is also a solution of (*), but z is not independent of y_1 since it is simply a multiple of y_1. We replace C by a function u which is to be determined (if possible) so that $y = ue^{\alpha x}$ is a solution of (*).[3] Calculating the derivatives of y, we have

$$\begin{aligned} y &= u\, e^{\alpha x} \\ y' &= \alpha\, u\, e^{\alpha x} + u'\, e^{\alpha x} \\ y'' &= \alpha^2 u\, e^{\alpha x} + 2\alpha\, u'\, e^{\alpha x} + u''\, e^{\alpha x} \end{aligned}$$

Substitution into (*) gives

$$\alpha^2 u\, e^{\alpha x} + 2\alpha\, u'\, e^{\alpha x} + u''\, e^{\alpha x} - 2\alpha\left[\alpha\, u\, e^{\alpha x} + u'\, e^{\alpha x}\right] + \alpha^2\, u\, e^{\alpha x} = 0.$$

[2] In this case, α is said to be a *double root* of the characteristic equation.

[3] This is an application of a general method called *variation of parameters.* We will use the method several times in the work that follows.

This reduces to

$$u''\,e^{\alpha x} = 0 \qquad \text{which becomes} \qquad u'' = 0 \qquad \text{since } \; e^{\alpha x} \neq 0.$$

Now, $u'' = 0$ is the simplest second order, linear differential equation with constant coefficients; the general solution is $u = C_1 + C_2 x = C_1 \cdot 1 + C_2 \cdot x$, and $u_1(x) = 1$ and $u_2(x) = x$ form a fundamental set of solutions. Since $y = u\,e^{\alpha x}$, we conclude that

$$y_1(x) = 1 \cdot e^{\alpha x} = e^{\alpha x} \qquad \text{and} \qquad y_2(x) = x\,e^{\alpha x}$$

are solutions of (*). It's easy to see that y_1 and y_2 form a fundamental set of solutions of (*). This can also be checked by using the Wronskian:

$$W(x) = e^{\alpha x}\left[e^{\alpha x} + \alpha\, x\, e^{\alpha x}\right] - \alpha\, x\, e^{\alpha x} = e^{2\alpha x} \neq 0.$$

Finally, the general solution of (*) is

$$y = C_1\, e^{\alpha x} + C_2\, x\, e^{\alpha x} \qquad \blacksquare$$

Example 2. Find the general solution of the differential equation

$$y'' - 6y' + 9y = 0.$$

SOLUTION The characteristic equation is

$$\begin{aligned} r^2 - 6r + 9 &= 0 \\ (r-3)^2 &= 0 \end{aligned}$$

There is only one characteristic root: $r_1 = r_2 = 3$. The functions $y_1(x) = e^{3x}$, $y_2(x) = x\,e^{3x}$ are linearly independent solutions of the differential equation and

$$y = C_1\, e^{3x} + C_2\, x\, e^{3x}$$

is the general solution. $\blacksquare$

Case III: The characteristic equation has complex conjugate roots:

$$r_1 = \alpha + i\,\beta,\; r_2 = \alpha + i\,\beta, \quad \beta \neq 0$$

In this case

$$y_1(x) = e^{\alpha x} \cos\, \beta x \qquad \text{and} \qquad y_2(x) = e^{\alpha x} \sin\, \beta x$$

are linearly independent solutions of equation (1) and

$$y = C_1\, e^{\alpha x} \cos \beta x + C_2\, e^{\alpha x} \sin \beta x = e^{\alpha x} \left[C_1 \cos \beta x + C_2 \sin \beta x\right]$$

is the general solution.

Proof: It is true that the functions $z_1(x) = e^{(\alpha + i\beta)x}$ and $z_2(x) = e^{(\alpha - i\beta)x}$ are linearly independent solutions of (1), but these are complex-valued functions and want real-valued solutions of (1). The characteristic equation in this case is

$$r^2 + ar + b = (r - [\alpha + i\,\beta])(r - [\alpha - i\,\beta]) = r^2 - 2\alpha\, r + \alpha^2 + \beta^2 = 0$$

and the differential equation (1) has the form

$$y'' - 2\alpha\, y' + \left(\alpha^2 + \beta^2\right) y = 0. \tag{*}$$

We'll proceed in a manner similar to Case II. Set $y = u\, e^{\alpha x}$ where u is to be determined (if possible) so that y is a solution of (*). Calculating the derivatives of y, we have

$$\begin{aligned} y &= u\, e^{\alpha x} \\ y' &= \alpha\, u\, e^{\alpha x} + u'\, e^{\alpha x} \\ y'' &= \alpha^2 u\, e^{\alpha x} + 2\alpha\, u'\, e^{\alpha x} + u''\, e^{\alpha x} \end{aligned}$$

Substitution into (*) gives

$$\alpha^2 u\, e^{\alpha x} + 2\alpha\, u'\, e^{\alpha x} + u''\, e^{\alpha x} - 2\alpha \left[\alpha\, u\, e^{\alpha x} + u'\, e^{\alpha x}\right] + \left(\alpha^2 + \beta^2\right)\, u\, e^{\alpha x} = 0.$$

This reduces to

$u''\, e^{\alpha x} + \beta^2\, u\, e^{\alpha x} = 0$ which becomes $u'' + \beta^2\, u = 0$ since $e^{\alpha x} \neq 0$.

Now,

$$u'' + \beta^2\, u = 0$$

is the equation of *simple harmonic motion* (for example, it models the oscillatory motion of a weight suspended on a spring). The functions $u_1(x) = \cos \beta x$ and $u_2(x) = \sin \beta x$ form a fundamental set of solutions. (Verify this.)

Since $y = u\, e^{\alpha x}$, we conclude that

$$y_1(x) = e^{\alpha x} \cos \beta x \qquad \text{and} \qquad y_2(x) = e^{\alpha x} \sin \beta x$$

are solutions of (*). It's easy to see that y_1 and y_2 form a fundamental set of solutions. This can also be checked by using the Wronskian

Finally, we conclude that the general solution of equation (1) is:

$$y = C_1\, e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x = e^{\alpha x}\left[C_1 \cos \beta x + C_2 \sin \beta x\right]. \qquad \blacksquare$$

Example 3. Find the general solution of the differential equation

$$y'' - 4y' + 13y = 0.$$

SOLUTION The characteristic equation is: $r^2 - 4r + 13 = 0$. By the quadratic formula, the roots are

$$r_1,\ r_2 = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2} = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6\,i}{2} = 2 \pm 3\,i.$$

The characteristic roots are the complex numbers: $r_1 = 2 + 3\,i$, $r_2 = 2 - 3\,i$. The functions $y_1(x) = e^{2x} \cos 3x$, $y_2(x) = e^{2x} \sin 3x$ are linearly independent solutions of the differential equation and

$$y = C_1\, e^{2x} \cos 3x + C_2\, e^{2x} \sin 3x = e^{2x}\left[C_1 \cos 3x + C_2 \sin 3x\right]$$

is the general solution. $\blacksquare$

Example 4. (Important Special Case) Find the general solution of the differential equation

$$y'' + \beta^2 y = 0.$$

SOLUTION The characteristic equation is: $r^2 + \beta^2 = 0$. The characteristic roots are the complex numbers

$$r_1,\ r_2 = 0 \pm \beta\, i$$

The functions $y_1(x) = e^{0x} \cos \beta x = \cos \beta x$, $y_2(x) = e^{0} \sin \beta 3x = \sin \beta x$ are linearly independent solutions of the differential equation and

$$y = C_1 \cos \beta x + C_2 \sin \beta x$$

is the general solution. $\blacksquare$

Recovering a Differential Equation from Solutions

You can also work backwards using the results above. That is, we can determine a second order, linear, homogeneous differential equation with constant coefficients that has given functions u and v as solutions. Here are some examples.

Example 5. Find a second order, linear, homogeneous differential equation with constant coefficients that has the functions $u(x) = e^{2x}$, $v(x) = e^{-3x}$ as solutions.

SOLUTION Since e^{2x} is a solution, 2 must be a root of the characteristic equation and $r-2$ must be a factor of the characteristic polynomial. Similarly, e^{-3x} a solution means that -3 is a root and $r-(-3) = r+3$ is a factor of the characteristic polynomial. Thus the characteristic equation must be

$$(r-2)(r+3) = 0 \qquad \text{which expands to} \qquad r^2 + r - 6 = 0.$$

Therefore, the differential equation is

$$y'' + y' - 6y = 0. \qquad \blacksquare$$

Example 6. Find a second order, linear, homogeneous differential equation with constant coefficients that has $y(x) = e^x \cos 2x$ as a solution.

SOLUTION Since $e^x \cos 2x$ is a solution, the characteristic equation must have the complex numbers $1+2i$ and $1-2i$ as roots. (Although we didn't state it explicitly, $e^x \sin 2x$ must also be a solution.) The characteristic equation must be

$$(r - [1+2i])(r - [1-2i]) = 0 \qquad \text{which expands to} \qquad r^2 - 2r + 5 = 0$$

and the differential equation is

$$y'' - 2y' + 5y = 0. \qquad \blacksquare$$

Exercises 3.2

Find the general solution of the given differential equation.

1. $y'' + 2y' - 8y = 0.$
2. $y'' - 13y' + 42y = 0.$
3. $y'' - 10y' + 25y = 0.$
4. $y'' + 2y' + 5y = 0.$
5. $y'' + 4y' + 13y = 0.$
6. $y'' = 0.$
7. $y'' + 2y' = 0.$

8. $2y'' + 5y' - 3y = 0$.

9. $y'' - 9y = 0$.

10. $y'' + 16y = 0$.

11. $y'' - 2y' + 2y = 0$.

12. $y'' - y' - 30y = 0$.

Find the solution of the initial-value problem.

13. $y'' - 5y' + 6y = 0; \quad y(0) = 1,\ y'(0) = 1$.

14. $y'' + 4y' + 3y = 0; \quad y(0) = y'(0) = 0$.

15. $y'' + 2y' + y = 0; \quad y(0) = -3,\ y'(0) = 1$.

16. $y'' + 4y = 0; \quad y(0) = 1,\ y'(0) = -2$.

Find a differential equation $y'' + ay' + by = 0$ that is satisfied by the given function(s).

17. $y_1(x) = e^{2x},\ y_2(x) = e^{-5x}$.

18. $y(x) = 2xe^{3x}$.

19. $y(x) = \cos 2x$.

20. $y_1(x) = 3e^{2x},\ y_2(x) = -4e^{-6x}$.

21. $y(x) = e^{-2x} \sin 4x$.

Find a differential equation $y'' + ay' + by = 0$ whose general solution is the given expression.

22. $y = C_1 e^{x/2} + C_2 e^{2x}$.

23. $y = C_1 e^{3x} + C_2 e^{-4x}$.

24. $y = C_1 e^{-x} \cos 3x + C_2 e^{-x} \sin 3x$.

25. $y = C_1 e^{2x} + C_2 x e^{2x}$.

26. $y = C_1 \cos 4x + C_2 \sin 4x$.

27. Find the solution $y = y(x)$ of the initial-value problem $y'' - y' - 2y = 0;\ y(0) = \alpha,\ y'(0) = 2$. Then find α such that $y(x) \to 0$ as $x \to \infty$.

28. Find the solution $y = y(x)$ of the initial-value problem $4y'' - y = 0;\ y(0) = 2,\ y'(0) = \beta$. Then find β such that $y(x) \to 0$ as $x \to \infty$.

Euler Equations: A second order linear homogeneous equation of the form

$$x^2\frac{d^2y}{dx^2} + \alpha x\frac{dy}{dx} + \beta y = 0 \tag{E}$$

where α and β are constants, is called an *Euler equation* .

29. Prove that the Euler equation (E) can be transformed into the second order equation with constant coefficients

$$\frac{d^2y}{dz^2} + a\frac{dy}{dz} + by = 0$$

where a and b are constants, by means of the change of independent variable $z = \ln\, x$.

Find the general solution of the Euler equations.

30. $x^2y'' - xy' - 8y = 0$.

31. $x^2y'' - 3xy' + 4y = 0$.

32. $x^2y'' - xy' + 5y = 0$.

3.3. Nonhomogeneous Equations

In this section we consider the general second order, linear, nonhomogeneous equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{1}$$

where $p,\, q,\, f$ are continuous functions on an interval I.

The objectives of this section are to determine the "structure" of the set of solutions of (1).

As we shall see, there is a close connection between equation (1) and

$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

In this context, equation (2) is called the *reduced equation* of equation (1).

General Results

THEOREM 1. If $z = z_1(x)$ and $z = z_2(x)$ are solutions of equation (1), then

$$y(x) = z_1(x) - z_2(x)$$

is a solution of equation (2). $\blacksquare$

Thus *the difference of any two solutions of the nonhomogeneous equation (1) is a solution of its reduced equation* (2).

Our next theorem gives the "structure" of the set of solutions of (1).

THEOREM 2. Let $y = y_1(x)$ and $y = y_2(x)$ be linearly independent solutions of the reduced equation (2) and let $z = z(x)$ be a particular solution of (1). If $u = u(x)$ is *any* solution of (1), then there exist constants C_1 and C_2 such that

$$u(x) = C_1 y_1(x) + C_2 y_2(x) + z(x). \quad \blacksquare$$

According to Theorem 2, if $y = y_1(x)$ and $y = y_2(x)$ are linearly independent solutions of the reduced equation (2) and $z = z(x)$ is a particular solution of (1), then

$$y = C_1 y_1(x) + C_2 y_2(x) + z(x) \tag{3}$$

represents the set of all solutions of (1). That is, (3) is the general solution of (1). Another way to look at (3) is: The general solution of (1) consists of the general solution of the reduced equation (2) *plus* a particular solution of (1):

$$\underbrace{y}_{\text{general solution of (1)}} = \underbrace{C_1 y_1(x) + C_2 y_2(x)}_{\text{general solution of (2)}} + \underbrace{z(x).}_{\text{particular solution of (1)}}$$

The next result is sometimes useful in finding particular solutions of nonhomogeneous equations. It is known as the *superposition principle.*

THEOREM 3. If $z = z_1(x)$ and $z = z_2(x)$ are particular solutions of

$$y'' + p(x)y' + q(x)y = f(x) \quad \text{and} \quad y'' + p(x)y' + q(x)y = g(x),$$

respectively, then $z(x) = z_1(x) + z_2(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) + g(x). \quad \blacksquare$$

This result can be extended to nonhomogeneous equations whose right-hand side is the sum of an arbitrary number of functions.

COROLLARY If $z = z_1(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x),$$

$z = z_2(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x),$$

and so on

$z = z_n(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f_n(x),$$

then $z(x) = z_1(x) + z_2(x) + \cdots + z_n(x)$ is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x) + \cdots + f_n(x). \qquad \blacksquare$$

The importance of Theorem 7 and its Corollary is that we need only consider nonhomogeneous equations in which the function on the right-hand side consists of one term only.

Variation of Parameters

By our work above, to find the general solution of (1) we need to find:

(i) a linearly independent pair of solutions y_1, y_2 of the reduced equation (2), and

(ii) a particular solution z of (1).

The *method of variation of parameters* uses a pair of linearly independent solutions of the reduced equation to construct a particular solution of (1).

Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the reduced equation

$$y'' + p(x)y' + q(x)y = 0.$$

Then

$$y = C_1 y_1(x) + C_2 y_2(x)$$

is the general solution. We replace the arbitrary constants C_1 and C_2 by functions $u = u(x)$ and $v = v(x)$, which are to be determined so that

$$z(x) = u(x)y_1(x) + v(x)y_2(x)$$

is a particular solution of the nonhomogeneous equation (1). The replacement of the parameters C_1 and C_2 by the "variables" u and v is the basis for the term "variation of parameters." Since there are two unknowns u and v to be determined we shall impose two conditions on these unknowns. One condition is that z should solve the differential equation (1). The second condition is at our disposal and we shall choose it in a manner that will simplify our calculations.

Differentiating z we get

$$z' = u\,y_1' + y_1\,u' + v\,y_2' + y_2\,v'.$$

For our second condition on u and v, we set

$$y_1\,u' + y_2\,v' = 0. \tag{a}$$

This condition is chosen because it simplifies the first derivative z' and because it will lead to a simple pair of equations in the unknowns u and v. With this condition the equation for z' becomes

$$z' = u\,y_1' + v\,y_2' \tag{b}$$

and

$$z'' = u\,y_1'' + y_1'\,u' + v\,y_2'' + y_2'\,v'.$$

Now substitute z, z' (given by (b)), and z'' into the left side of equation (1). This gives

$$\begin{aligned} z'' + pz' + qz &= (u\,y_1'' + y_1'\,u' + v\,y_2'' + y_2'\,v') + p(u\,y_1' + v\,y_2') + q(u\,y_1 + v\,y_2) \\ &= u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + y_1'\,u' + y_2'\,v'. \end{aligned}$$

Since y_1 and y_2 are solutions of (2),

$$y_1'' + py_1' + qy_1 = 0 \quad \text{and} \quad y_2'' + py_2' + qy_2 = 0$$

and so

$$z'' + pz' + qz = y_1'\,u' + y_2'\,v'.$$

The condition that z should satisfy (1) is

$$y_1'\,u' + y_2'\,v' = f(x). \tag{c}$$

Equations (a) and (c) constitute a system of two equations in the two unknowns u and v:

$$\begin{aligned} y_1\, u' + y_2\, v' &= 0 \\ y_1'\, u' + y_2'\, v' &= f(x) \end{aligned}$$

Obviously this system involves u' and v' not u and v, but if we can solve for u' and v', then we can integrate to find u and v. Solving for u' and v', we find that

$$u' = \frac{-y_2\, f}{y_1\, y_2' - y_2\, y_1'} \quad \text{and} \quad v' = \frac{y_1\, f}{y_1\, y_2' - y_2\, y_1'}$$

We know that the denominators here are non-zero because the expression

$$y_1(x)y_2'(x) - y_2(x)y_1'(x) = W(x)$$

is the Wronskian of y_1 and y_2, and y_1, y_2 are linearly independent solutions of the reduced equation.

We can now get u and v by integrating:

$$u = \int \frac{-y_2(x)f(x)}{W(x)}\, dx \quad \text{and} \quad v = \int \frac{y_1(x)f(x)}{W(x)}\, dx.$$

Finally

$$z(x) = y_1(x) \int \frac{-y_2(x)f(x)}{W(x)}\, dx + y_2(x) \int \frac{y_1(x)f(x)}{W(x)}\, dx \tag{4}$$

is a particular solution of the nonhomogeneous equation (1).

Remark This result illustrates why the emphasis is on linear homogeneous equations. To find the general solution of the nonhomogeneous equation (1) we need a fundamental set of solutions of the reduced equation (2) and one particular solution of (1). But, as we have just shown, if we have a fundamental set of solutions of (2), then we can use them to construct a particular solution of (1). Thus, all we really need to solve (1) is a fundamental set of solutions of its reduced equation (2). ■

Example 1. Find a particular solution of the nonhomogeneous equation

$$y'' - 5\, y' + 6\, y = 4e^{2x}. \tag{*}$$

SOLUTION The functions $y_1(x) = e^{2x}$, $y_2(x) = e^{3x}$ are linearly independent solutions of the reduced equation. The Wronskian of y_1, y_2 is

$$W(x) = y_1\, y_2' - y_2\, y_1' = e^{5x}.$$

By the method of variation of parameters, a particular solution of the nonhomogeneous equation is

$$z(x) = u(x)\, e^{2x} + v(x)\, e^{3x}$$

where, from (4),

$$u(x) = \int \frac{-e^{3x}(4e^{2x})}{e^{5x}}\, dx = \int -4\, dx = -4x$$

and

$$v(x) = \int \frac{e^{2x}(4e2x)}{e^{5x}}\, dx = \int 4e^{-x}\, dx = -4e^{-x}.$$

(NOTE: Since we are seeking only one function u and one function v we have not included arbitrary constants in the integration steps.)

Now

$$z(x) = -4x\, e^{2x} - 4e^{-x}\, e^{3x} = -4x\, e^{2x} - 4e^{2x}$$

is a particular solution of the nonhomogeneous equation (*) and

$$y = C_1\, e^{2x} + C_2\, e3x - 4x\, e2x - 4e^{2x} = C_1\, e^{2x} + C_2\, e^{3x} - 4x\, e2x$$

is the general solution (we "absorbed" $-4e^{2x}$ in the $C_1\, e^{2x}$ term). As you can check $-4xe^{2x}$ is a solution of the nonhomogeneous equation. ■

Exercises 3.3

Verify that the given functions y_1 and y_2 form a fundamental set of solutions of the reduced equation of the given nonhomogeneous equation; then find a particular solution of the nonhomogeneous equation and give the general solution of the equation.

1. $y'' - \dfrac{2}{x^2}\, y = 3 - x^{-2}; \quad y_1(x) = x^2,\ y_2(x) = x^{-1}.$
2. $y'' - \dfrac{1}{x} y' + \dfrac{1}{x^2} y = \dfrac{2}{x}; \quad y_1(x) = x,\ y_2(x) = x \ln x.$
3. $(x-1)y'' - xy' + y = (x-1)^2; \quad y_1(x) = x,\ y_2(x) = e^x.$
4. $x^2y'' - xy' + y = 4x \ln x.$

Find the general solution of the given nonhomogeneous differential equation.

5. $y'' - 4y' + 4y = \frac{1}{3}\, x^{-1} e^{2x}.$

6. $y'' + 4y' + 4y = \dfrac{e^{-2x}}{x^2}$.

7. $y'' + 2y' + y = e^{-x}\ln x$.

8. The function $y_1(x) = x$ is a solution of $x^2y'' + xy' - y = 0$. Find the general solution of the differential equation

$$x^2y'' + xy' - y = 2x.$$

9. The functions $y_1(x) = x^2 + x\ln x$, $y_2(x) = x + x^2$ and $y_3(x) = x^2$ are solutions of a second order, linear, nonhomogeneous equation. What is the general solution of the equation?

3.4. Undetermined Coefficients

Solving a linear nonhomgeneous equation depends, in part, on finding a particular solution of the equation. We have seen one method for finding a particular solution, the method of variation of parameters. In this section we present another method, the method of *undetermined coefficients.*

Remark: Limitations of the method. In contrast to variation of parameters, which can be applied to any nonhomogeneous equation, the method of undetermined coefficients can be applied only to nonhomogeneous equations of the form

$$y'' + ay' + by = f(x) \tag{1}$$

where a and b are constants and the nonhomogeneous term f is a polynomial, an exponential function, a sine, a cosine, or a combination of such functions. ■

To motivate the method of undetermined coefficients, consider the linear operator on the left side of (1):

$$y'' + ay' + by. \tag{2}$$

If we calculate (2) for an exponential function $z = Ae^{rx}$, A a constant, we have

$$z = Ae^{rx}, \qquad z' = Are^{rx}, \qquad z'' = Ar^2e^{rx}$$

and

$$\begin{aligned} y'' + ay' + by &= Ar^2e^{rx} + a(Are^{rx}) + b(Ae^{rx} = \left(Ar^2 + aAr + bA\right)e^{rx} \\ &= K\,e^{rx} \qquad \text{where } K = Ar^2 + aAr + bA. \end{aligned}$$

That is, the operator (2) "transforms" Ae^{rx} into a constant multiple of e^{rx}. We can use this result to determine a particular solution of a nonhomogeneous equation of the form

$$y'' + ay' + by = ce^{rx}.$$

Here is a specific example.

Example 1. Find a particular solution of the nonhomogeneous equation

$$y'' - 2y' + 5y = 6e^{3x}.$$

SOLUTION As we saw above, if we "apply" $y'' - 2y' + 5y$ to $z(x) = Ae^{3x}$ we will get an expression of the form Ke^{3x}. We want to determine A so that $K = 6$. The constant A is called an *undetermined coefficient.* We have

$$z = Ae^{3x}, \quad z' = 3Ae^{3x}, \quad z'' = 9Ae^{3x}.$$

Substituting z and its derivatives into the left side of the differential equation, we get

$$9Ae^{3x} - 2\left(3Ae^{3x}\right) + 5\left(Ae^{3x}\right) = (9A - 6A + 5A)e^{3x} = 8Ae^{3x}.$$

We want

$$z'' - 2z' + 5z = 6e^{3x},$$

so we set

$$8Ae^{3x} = 6e^{3x} \quad \text{which gives} \quad 8A = 6 \quad \text{and} \quad A = \tfrac{3}{4}.$$

Thus, $z(x) = \frac{3}{4}\,e^{3x}$ is a particular solution of $y'' - 2y' + 5y = 6e^{3x}$. (Verify this.)

You can also verify that

$$y = e^{x}\left(C_1 \cos\, 2x + C_2\, \sin\, 2x\right) + \tfrac{3}{4}\,e^{3x}$$

is the general solution of the equation. ■

If we set $z(x) = A\,\cos\beta x$ and calculate z' and z'', we get

$$z = A\,cos\,\beta x, \qquad z' = -\beta A\,\sin\,\beta x, \qquad z'' = -\beta^2 A\,\cos\,\beta x.$$

Therefore, $y'' + ay' + by$ applied to z gives

$$\begin{aligned} z'' + az' + bz &= -\beta^2 A\,\cos\,\beta x + a\left(-\beta A\,\sin\,\beta x\right) + b(A\,cos\,\beta x) \\ &= (-\beta^2 A + bA)\cos\,\beta x + (-a\beta A)\sin\,\beta x. \end{aligned}$$

That is, $y'' + ay' + by$ "transforms" $z = A\cos\beta x$ into an expression of the form

$$K\cos\beta x + M\sin\beta x$$

where K and M are constants which depend on a, b, β and A. We will get exactly the same type of result if we apply $y'' + ay' + by$ to $z = B\sin\beta x$. Combining these two results, it follows that $y'' + ay' + by$ applied to

$$z = A\cos\beta x + B\sin\beta x$$

will produce the expression

$$K\cos\beta x + M\sin\beta x$$

where K and M are constants which depend on $a, b, \beta, A,$ and B.

Now suppose we have a nonhomogeneous equation of the form

$$y'' + ay' + by = c\,\cos\beta x \quad \text{or} \quad y'' + ay' + by = d\,\sin\beta x,$$

or even

$$y'' + ay' + by = c\,\cos\beta x + d\,\sin\beta x.$$

Then we will look for a solution of the form $z(x) = A\cos\beta x + B\sin\beta x$.

Continuing with these ideas, if $y'' + ay' + by$ is applied to $z = Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$, then the result will have the form

$$Ke^{\alpha x}\cos\beta x + Ke^{\alpha x}\sin\beta x$$

where K and M are constants which depend on $a, b, \alpha, \beta, A, B$. Therefore, we expect that a nonhomogeneous equation of the form

$$y'' + ay' + by = ce^{\alpha x}\cos\beta x + de^{\alpha x}\sin\beta x$$

will have a particular solution of the form $z = Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$.

The following table summarizes our discussion to this point.

A particular solution of $y'' + ay' + by = f(x)$

If $f(x) =$	try $z(x) =$
ce^{rx}	Ae^{rx}
$c\,\cos\beta x + d\,\sin\beta x$	$z(x) = A\,\cos\beta x + B\,\sin\beta x$
$ce^{\alpha x}\cos\beta x + de^{\alpha x}\sin\beta x$	$z(x) = Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$

Note: The first line includes the case $r = 0$;
if $f(x) = ce^{0x} = c$, then $z = Ae^{0x} = A$.

Unfortunately, the situation is not quite as simple as it appears; there is a difficulty.

Example 2. Find a particular solution of the nonhomogeneous equation

$$y'' - 5y' + 6y = 4e^{2x}. \tag{*}$$

SOLUTION According to the table, we should set $z(x) = Ae^{2x}$. Calculating the derivatives of z, we have

$$z = Ae^{2x}, \quad z' = 2Ae^{2x}, \quad z'' = 4Ae^{2x}.$$

Substituting z and its derivatives into the left side of (*), we get

$$z'' - 5z' + 6z = 4Ae^{2x} - 5(2Ae^{2x}) + 6(Ae^{2x}) = 0Ae^{2x}.$$

Clearly the equation

$$0Ae^{2x} = 4e^{2x} \quad \text{which is equivalent to} \quad 0A = 4$$

does not have a solution. Therefore equation (*) does not have a solution of the form $z = Ae^{2x}$.

The problem here is $z = Ae^{2x}$ is a solution of the reduced equation

$$y'' - 5y' + 6y = 0.$$

(The characteristic equation is $r^2 - 5r + 6 = 0$; the roots are $r = 2, 3$; and $y_1 = e^{2x}$, $y_2 = e^{3x}$ are linearly independent solutions.)

In Example 6 of the preceding section we saw that $z(x) = -4xe^{2x}$ is a particular solution of (*). So, in the context here, since our trial solution $z = Ae^{2x}$ solves the reduced equation, we'll try $z = Axe^{2x}$. The derivatives of this z are:

$$z = Axe^{2x}, \quad z' = 2Axe^{2x} + Ae^{2x}, \quad z'' = 4Axe^{2x} + 4Ae^{2x}.$$

Substituting into the left side of (*), we get

$$\begin{aligned} z'' - 5z' + 6z &= 4Axe^{2x} + 4Ae^{2x} - 5(2Axe^{2x} + Ae^{2x}) + 6(Axe^{2x}) \\ &= -Ae^{2x}. \end{aligned}$$

Setting $z'' - 5z' + 6z = 4e^{2x}$ gives

$$-Ae^{2x} = 4e^{2x} \quad \text{which implies} \quad A = -4.$$

Thus, $z(x) = -4xe^{2x}$ is a particular solution of (*) (as we already know). ■

We learn from this example that we have to make an adjustment if our trial solution z (from the table) satisfies the reduced equation. Here's another example.

Example 3. Find a particular solution of

$$y'' + 6y' + 9y = 5e^{-3x}. \tag{**}$$

SOLUTION The reduced equation, $y'' + 6y' + 9y = 0$ has characteristic equation

$$r^2 + 6r + 9 = (r+3)^2 = 0.$$

Thus, $r = -3$ is a double root and $y_1(x) = e^{-3x}$, $y_2(x) = xe^{-3x}$ form a fundamental set of solutions.

According to our table, to find a particular solution of (**) we should try $z = Ae^{-3x}$. But this won't work, z is a solution of the reduced equation. Based on the result of the preceding example, we should try $z = Axe^{-3x}$, but this won't work either; $z = Axe^{-3x}$ is also a solution of the reduced equation. So we'll try $z = Ax^2e^{-3x}$. You can verify that

$$z(x) = \frac{5}{2}\,x^2e^{-3x}$$

is a particular solution of (**).

The general solution of (**) is: $y = C_1e^{-3x} + C_2xe^{-3x} + \frac{5}{2}\,x^2e^{-3x}$. ■

Based on these examples we amend our table to read:

Table 1

A particular solution of $y'' + ay' + by = f(x)$

If $f(x) =$	try $z(x) =$*
ce^{rx}	Ae^{rx}
$c\cos\beta x + d\sin\beta x$	$z(x) = A\cos\beta x + B\sin\beta x$
$ce^{\alpha x}\cos\beta x + de^{\alpha x}\sin\beta x$	$z(x) = Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x$

*Note: If z satisfies the reduced equation, try xz;
if xz also satisfies the reduced equation, then
x^2z will give a particular solution

Remark In practice it is a good idea to solve the homogeneous equation before selecting the trial solution z of the nonhomogeneous equation. That way you will not waste your time selecting a z that satisfies the reduced equation. ■

Summary The method of variation of parameters can be applied to *any* linear nonhomogeneous equations but it has the limitation of requiring a fundamental set of solutions of the reduced equation.

The method of undetermined coefficients is limited to linear nonhomogeneous equations with constant coefficients and with restrictions on the nonhomogeneous term f.

In cases where both methods are applicable, the method of undetermined coefficients is usually simpler and, hence, the preferable method. ■

Exercises 3.4

Find the general solution.

1. $y'' + 2y' + 2y = 10e^x$.
2. $y'' + 6y' + 9y = 9e^{3x}$.
3. $y'' + 4y' + 4y = e^{-2x}$.
4. $y'' + 5y' + 6y = e^{2x} + 4$.
5. $y'' - 2y' - 3y = 3e^{2x}$.
6. $y'' + 2y' = 4\sin 2x$.
7. $2y'' + 3y' + y = x^2 + 3\sin x$.
8. $y'' - 6y' + 9y = e^{-3x}$.
9. $y'' + 5y' + 6y = 3x + 4$.
10. $y'' + 6y' + 8y = 3e^{-2x}$.
11. $y'' + y' - 6y = 2e^{-3x} + e^{2x}$.

Find the solution of the given initial-value problem.

12. $y'' + y' - 2y = 2e^{2x}; \quad y(0) = 0,\ y'(0) = 1$.

13. $y'' + 4y = 3e^x; \quad y(0) = 0, \; y'(0) = 2.$

14. $y'' - y' - 2y = \sin 2x; \quad y(0) = 1, \; y'(0) = -1.$

15. $y'' - 2y' + y = e^{3x} + 4; \quad y(0) = 1, \; y'(0) = 1.$

Determine a suitable form for a particular solution $z = z(x)$ of the given equation.

16. $y'' - 2y' - 3y = 6 - 3e^{-x} + 4\cos 3x.$

17. $y'' + y = 3\cos x - 2\sin x.$

18. $y'' - 5y' + 6y = 2e^{2x}\cos x - 3e^{3x} + 5.$

19. $y'' - 4y' + 4y = 2e^{2x} + 2\cos 2x.$

20. $y'' + 2y' = 2 + 5e^{-3x} + \sin 2x.$

3.5. Vibrating Mechanical Systems

Undamped Vibrations A spring of length l_0 units is suspended from a support. When an object of mass m is attached to the spring, the spring stretches to a length l_1 units. If the object is then pulled down (or pushed up) an additional y_0 units at time $t = 0$ and then released, what is the resulting motion of the object? That is, what is the position $y(t)$ of the object at time $t > 0$? Assume that time is measured in seconds

We begin by analyzing the forces acting on the object at time $t > 0$. First, there is the weight of the object (gravity):

$$F_1 = mg.$$

This is a downward force. We choose our coordinate system so that the positive direction is down. Next, there is the restoring force of the spring. By Hooke's Law, this force is proportional to the total displacement $l_1 + y(t)$ and acts in the direction opposite to the displacement:

$$F_2 = -k[l_1 + y(t)] \qquad \text{with } k > 0.$$

The constant of proportionality k is called the *spring constant.* If we assume that the spring is frictionless and that there is no resistance due to the surrounding medium (for example, air resistance), then these are the only forces acting on the object. Under these conditions, the total force is

$$F = F_1 + F_2 = mg - k[l_1 + y(t)] = (mg - kl_1) - ky(t).$$

Before the object was displaced, the system was in equilibrium, so the force of gravity, mg plus the force of the spring, $-kl_1$, must have been 0:

$$mg - kl_1 = 0.$$

Therefore, the total force F reduces to

$$F = -ky(t).$$

By Newton's Second Law of Motion, $F = ma$ (force = mass $\times$ acceleration), we have

$$ma = -ky(t) \quad \text{and} \quad a = -\frac{k}{m}y(t).$$

Therefore, at any time t we have

$$a = y''(t) = -\frac{k}{m}y(t) \quad \text{or} \quad y''(t) + \frac{k}{m}y(t) = 0.$$

When the acceleration is a constant negative multiple of the displacement, the object is said to be in *simple harmonic motion.*

Since $k/m > 0$, we can set $\omega = \sqrt{k/m}$ and write this equation as

$$y''(t) + \omega^2 y(t) = 0, \tag{3}$$

a second order, linear homogeneous equation with constant coefficients. The characteristic equation is

$$r^2 + \omega^2 = 0$$

and the characteristic roots are $\pm\omega i$. The general solution of (1) is

$$y = C_1 \cos \omega t + C_2 \sin \omega t.$$

In the Exercises you are asked to show that the general solution can be written as

$$y = A \sin(\omega t + \phi_0), \tag{4}$$

where A and ϕ_0 are constants with $A > 0$ and $\phi_0 \in [0, 2\pi)$. For our purposes here, this is the preferred form. The motion is *periodic* with *period* T given by

$$T = \frac{2\pi}{\omega},$$

a complete oscillation takes $2\pi/\omega$ seconds. The reciprocal of the period gives the number of oscillations per second. This is called the *frequency*, denoted by f:

$$f = \frac{\omega}{2\pi}.$$

Since $\sin(\omega t + \phi_0)$ oscillates between -1 and 1,

$$y(t) = A \sin(\omega t + \phi_0)$$

oscillates between $-A$ and A. The number A is called the *amplitude* of the motion. The number ϕ_0 is called the *phase constant* or the *phase shift*. The figure gives a typical graph of (2).

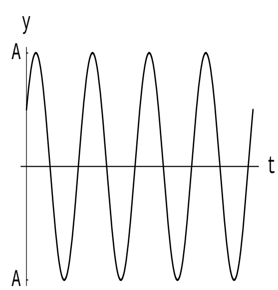


Figure 1

Damped Vibrations

If the spring is not frictionless or if there the surrounding medium resists the motion of the object (for example, air resistance), then the resistance tends to dampen the oscillations. Experiments show that such a resistant force R is approximately proportional to the velocity $v = y'$ and acts in a direction opposite to the motion:

$$R = -cy' \quad \text{with } c > 0.$$

Taking this force into account, the force equation reads

$$F = -ky(t) - cy'(t).$$

Newton's Second Law $F = ma = my''$ then gives

$$my''(t) = -ky(t) - cy'(t)$$

which can be written as

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0. \qquad (c,\ k,\ m \text{ all constant}) \tag{5}$$

This is the equation of motion in the presence of a *damping factor.*

The characteristic equation

$$r^2 + \frac{c}{m}r + \frac{k}{m} = 0$$

has roots

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}.$$

There are three cases to consider:

$$c^2 - 4km < 0, \qquad c^2 - 4km > 0, \qquad c^2 - 4km = 0.$$

Case 1: $c^2 - 4km < 0$. In this case the characteristic equation has complex roots:

$$r_1 = -\frac{c}{2m} + i\omega, \quad r_2 = -\frac{c}{2m} - i\omega \quad \text{where } \omega = \frac{\sqrt{4km - c^2}}{2m}.$$

The general solution is

$$y = e^{(-c/2m)t}\left(C_1 \cos \omega t + C_2 \sin \omega t\right)$$

which can also be written as

$$y(t) = A\, e^{(-c/2m)t} \sin\left(\omega t + \phi_0\right) \tag{6}$$

where, as before, A and ϕ_0 are constants, $A > 0$, $\phi_0 \in [0, 2\pi)$. This is called the *underdamped case.* The motion is similar to simple harmonic motion except that the damping factor $e^{(-c/2m)t}$ causes $y(t) \to 0$ as $t \to \infty$. The oscillations continue indefinitely with constant frequency $f = \omega/2\pi$ but diminishing amplitude $Ae^{(-c/2m)t}$.

The figure below illustrates this motion. ■

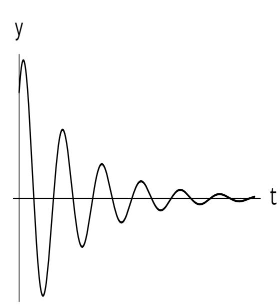


Figure 2

Case 2: $c^2 - 4km > 0$. In this case the characteristic equation has two distinct real roots:

$$r_1 = \frac{-c + \sqrt{c^2 - 4km}}{2m}, \qquad r_2 = \frac{-c - \sqrt{c^2 - 4km}}{2m}.$$

The general solution is

$$y(t) = y = C_1 e^{r_1 t} + C_2 e^{r_2 t}. \tag{7}$$

This is called the *overdamped case.* The motion is nonoscillatory. Since

$$\sqrt{c^2 - 4km} < \sqrt{c^2} = c,$$

r_1 and r_2 are both negative and $y(t) \to 0$ as $t \to \infty$. ■

Case 3: $c^2 - 4km = 0$. In this case the characteristic equation has only one real root:

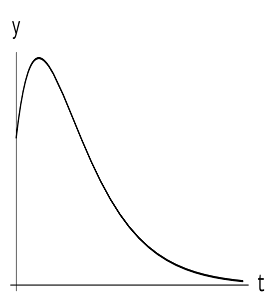
$$r_1 = \frac{-c}{2m},$$

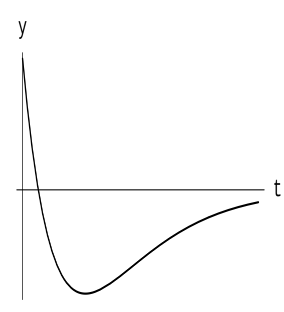
and the general solution is

$$y(t) = y = C_1 e^{-(c/2m)\,t} + C_2 t\, e^{-(c/2m)\,t}. \tag{8}$$

This is called the *critically damped case.* Once again, the motion is nonoscillatory and $y(t) \to 0$ as $t \to \infty$. ■

In both the overdamped and critically damped cases, the object moves back to the equilibrium position $(y(t) \to 0$ as $t \to \infty)$. The object may move through the equilibrium position once, but only once. Two typical examples of the motion are shown below.





Forced Vibrations

The vibrations that we have considered thus far result from the interplay of three forces: gravity, the restoring force of the spring, and the retarding force of friction or the surrounding medium. Such vibrations are called *free vibrations* .

The application of an external force to a freely vibrating system modifies the vibrations and produces what are called *forced vibrations* . As an example we'll investigate the effect of a periodic external force $F_0 \cos \gamma t$ where F_0 and γ are positive constants.

In an undamped system the force equation is

$$F = -kx + F_0 \cos \gamma t$$

and the equation of motion takes the form

$$y'' + \frac{k}{m} y = \frac{F_0}{m} \cos \gamma t.$$

We set $\omega = \sqrt{k/m}$ and write the equation of motion as

$$y'' + \omega^2 y = \frac{F_0}{m} \cos \gamma t. \tag{9}$$

As we'll see, the nature of the motion depends on the relation between the *applied frequency* , $\gamma/2\pi$, and the *natural frequency* of the system, $\omega/2\pi$.

Case 1: $\gamma \neq \omega$. In this case the method of undetermined coefficients gives the particular solution

$$z(t) = \frac{F_0/m}{\omega^2 - \gamma^2} \cos \gamma t$$

and the general equation of motion is

$$y = A \sin(\omega t + \phi_0) + \frac{F_0/m}{\omega^2 - \gamma^2} \cos \gamma t. \tag{10}$$

If ω/γ is rational, the vibrations are periodic. If ω/γ is not rational, then the vibrations are not periodic and can be highly irregular. In either case, the vibrations are bounded by

$$|A| + \left| \frac{F_0/m}{\omega^2 - \gamma^2} \right|. \qquad \blacksquare$$

Case 2: $\gamma = \omega$. In this case the method of undetermined coefficients gives

$$z(t) = \frac{F_0}{2\omega m} t \sin \omega t$$

and the general solution has the form

$$y = A \sin(\omega t + \phi_0) + \frac{F_0}{2\omega m} t \sin \omega t. \tag{11}$$

The system is said to be in *resonance* . The motion is oscillatory but, because of the t factor in the second term, it is not periodic. As $t \to \infty$, the amplitude of the vibrations increases without bound.

A typical illustration of the motion is given in the figure below. ■

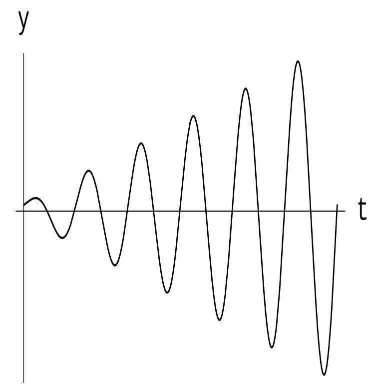


Figure 4

Exercises 3.5

1. Show that simple harmonic motion $y(t) = y = C_1 \cos \omega t + C_2 \sin \omega t$ can be written as: (a) $A \sin(\omega t + \phi_0)$ (b) $y(t) = A \cos(\omega t + \phi_1)$.

2. What is the effect of an increase in the resistance constant c on the amplitude and frequency of the vibrations given by (4)?

3. Show that the motion given by (5) can pass through the equilibrium point at most once. How many times can the motion change directions?

4. Show that if $\gamma \neq \omega$, then the method of undetermined coefficients applied to (7) gives
$$z = \frac{F_0/m}{\omega^2 - \gamma^2} \cos \gamma t.$$

5. Show that if $\gamma = \omega$, then the method of undetermined coefficients applied to (7) gives
$$z = \frac{F_0}{2\omega m} t \sin \omega t.$$

3.6 Higher-Order Linear Differential Equations

This section is a continuation Sections 3.1 - 3.4. All of the "theory" that we developed for second-order linear differential equations carries over, essentially verbatim, to linear differential equations of order greater than two.

Recall that a first order, linear differential equation is an equation which can be written in the form

$$y' + p(x)y = q(x)$$

where p and q are continuous functions on some interval I. A second order, linear differential equation has an analogous form.

$$y'' + p(x)y' + q(x)y = f(x)$$

where p, q, and f are continuous functions on some interval I.

In general, an n^{th}*-order linear differential equation* is an equation that can be written in the form

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \qquad \text{(L)}$$

where $p_0,\ p_1,\ \ldots,\ p_{n-1},$ and f are continuous functions on some interval I. As before, the functions $p_0,\ p_1,\ \ldots,\ p^{n-1}$ are called the *coefficients*, and f is called the *forcing function* or the *nonhomogeneous term*.

Equation (L) is *homogeneous* if the function f on the right side is 0 for all $x \in I$. In this case, equation (L) becomes

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = 0 \qquad \text{(H)}$$

Equation (L) is *nonhomogeneous* if f is not the zero function on I, i.e., (L) is nonhomogeneous if $f(x) \neq 0$ for some $x \in I$. As in the case of second order linear equations, almost all of our attention will be focused on homogeneous equations.

THEOREM 1. (Existence and Uniqueness Theorem) Given the n^{th}- order linear equation (L). Let a be any point on the interval I, and let $\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_{n-1}$ be any n real numbers. Then the initial-value problem

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x);$$

$$y(a) = \alpha_0,\ y'(a) = \alpha_1,\ \ldots,\ y^{(n-1)}(a) = \alpha_{n-1}$$

has a unique solution.

Remark: We can solve any first order linear differential equation, see Section 2.1. In contrast, *there is no general method for solving second or higher order linear differential equations.* However, as we saw in our study of second order equations, there are methods for solving certain special types of higher order linear equations and we shall look at these later in this section. ■

Homogeneous Equations

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = 0. \qquad \text{(H)}$$

Note first that the zero function, $y(x) = 0$ for all $x \in I$, (also denoted by $y \equiv 0$) is a solution of (H). As before, this solution is called the *trivial solution* . Obviously, our main interest is in finding *nontrivial* solutions.

The essential facts about homogeneous equations are as follows. The proofs are identical to those given in Section 3.2

THEOREM 2. If $y = y_1(x),\ y = y_2(x),\ \ldots,\ y = y_k(x)$ are solutions of (H), and if $c_1,\ c_2,\ \ldots,\ c_k$ are any k real numbers, then

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$$

is also a solution of (H).

Any linear combination of solutions of (H) is also a solution of (H).

Note that if $k = n$ in the linear combination above, then the equation

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) \qquad (1)$$

has the form of a general solution of equation (H). So the question is: If $y_1,\ y_2,\ \ldots,\ y_n$ are solutions of (H), is the expression (1) the general solution of (H)? That is, can every solution of (H) be written as a linear combination of $y_1,\ y_2,\ \ldots,\ y_n$? It turns out that (1) may or not be the general solution; it depends on the relation between the solutions $y_1,\ y_2,\ \ldots, y_n$.

Let $y = y_1(x),\ y = y_2(x),\ \ldots,\ y = y_n(x)$ be n solutions of (H). The $n \times n$ determinant

$$\begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ y_1''(x) & y_2''(x) & \dots & y_n''(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix} \tag{2}$$

is called the *Wronskian* of the solutions $y_1,\ y_2,\ \dots,\ y_n$.

THEOREM 3. Let $y = y_1(x),\ y = y_2(x),\ \dots,\ y = y_n(x)$ be solutions of equation (H), and let $W(x)$ be their Wronskian. Exactly one of the following holds:

(i) $W(x) = 0$ for all $x \in I$ and $y_1,\ y_2,\ \dots,\ y_n$ are linearly dependent.

(ii) $W(x) \neq 0$ for all $x \in I$ which implies that $y_1,\ y_2,\ \dots,\ y_n$ are linearly independent and

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

is the general solution of (H).

DEFINITION 1. (Fundamental Set) A set of n linearly independent solutions $y = y_1(x),\ y = y_2(x),\ \dots,\ y = y_n(x)$ of (H) is called a *fundamental set of solutions*.

A set of solutions $y_1,\ y_2,\ \dots,\ y_n$ of (H) is a fundamental set if and only if

$$W[y_1, y_2, \dots, y_n](x) \neq 0 \quad \text{for all } x \in I.$$

Homogeneous Equations with Constant Coefficients

An *n^{th}-order linear homogeneous differential equation with constant coefficients* is an equation which can be written in the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_1 y' + a_0 y = 0 \tag{3}$$

where $a_0,\ a_1,\ \dots,\ a^{n-1}$ are real numbers.

We have seen that first- and second-order equations with constant coefficients have solutions of the form $y = e^{rx}$. Thus, we'll look for solutions of (3) of this form

If $y = e^{rx}$, then

$$y' = r\,e^{rx},\ y'' = r^2 e^{rx},\ \dots,\ y^{(n-1)} = r^{n-1}r^{rx},\ y^{(n)} = r^n e^{rx}.$$

Substituting y and its derivatives into (3) gives

$$r^n e^{rx} + a_{n-1}r^{n-1} e^{rx} + \cdots + a_1 r\, e^{rx} + a_0\, e^{rx} = 0$$

or

$$e^{rx}\left(r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0\right) = 0.$$

Since $e^{rx} \neq 0$ for all x, we conclude that $y = e^{rx}$ is a solution of (3) if and only if

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0. \tag{4}$$

DEFINITION 2. Given the differential equation (3). The corresponding polynomial equation

$$p(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

is called the *characteristic equation* of (3); the n^{th}-degree polynomial $p(r)$ is called the *characteristic polynomial*. The roots of the characteristic equation are called the *characteristic roots*.

Thus, we can find solutions of the equation if we can find the roots of the corresponding characteristic polynomial. Appendix 1 gives the basic facts about polynomials with real coefficients.

In Chapter 3 we proved that if $r_1 \neq r_2$, then $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are linearly independent. We also showed that $y_3(x) = e^{rx}$ and $y_4(x) = xe^{rx}$ are linearly independent. Here is the general result.

THEOREM 4.

1. If $r_1,\ r_2,\ \ldots,\ r_k$ are distinct numbers (real or complex), then the distinct exponential functions $y_1 = e^{r_1 x},\ y_2 = e^{r_2 x},\ \ldots,\ y_k = e^{r_k x}$ are linearly independent.
2. For any real number α the functions $y_1(x) = e^{\alpha x},\ y_2(x) = xe^{\alpha x},\ \ldots,\ y_k(x) = x^{k-1}e^{\alpha x}$ are linearly independent.

Proof: In each case, the Wronskian $W[y_1, y_2, \ldots, y_k](x) \neq 0$.

Since all of the ground work for solving linear equations with constant coefficients was established in Section 3.2, we'll simply give some examples here. Theorem 4 will be useful in showing that our sets of solutions are linearly independent.

Example 1. Find the general solution of

$$y''' + 3y'' - y' - 3y = 0$$

given that $r = 1$ is a root of the characteristic polynomial.

SOLUTION The characteristic equation is

$$\begin{aligned} r^3 + 3r^2 - r - 3 &= 0 \\ (r-1)(r^2 + 4r + 3) &= 0 \\ (r-1)(r+1)(r+3) &= 0 \end{aligned}$$

The characteristic roots are: $r_1 = 1,\ r_2 = -1,\ r_3 = -3$. The functions $y_1(x) = e^x,\ y_2(x) = e^{-x},\ y_3(x) = e^{-3x}$ are solutions. Since these are distinct exponential functions, the solutions form a fundamental set and

$$y = C_1\, e^{4x} + C_2\, e^{-x} + C_3 e^{-3x}$$

is the general solution of the equation. ■

Example 2. Find the general solution of

$$y^{(4)} - 4y''' + 3y'' + 4y' - 4y = 0$$

given that $r = 2$ is a root of multiplicity 2 of the characteristic polynomial.

SOLUTION The characteristic equation is

$$\begin{aligned} r^4 - 4r^3 + 3r^2 + 4r - 4 &= 0 \\ (r-2)^2(r^2 - 1) &= 0 \\ (r-2)^2(r-1)(r+1) &= 0 \end{aligned}$$

The characteristic roots are: $r_1 = 1,\ r_2 = -1,\ r_3 = r_4 = 2$. The functions $y_1(x) = e^x,\ y_2(x) = e^{-x},\ y_3(x) = e^{2x}$ are solutions. Based on our work in Chapter 3, we conjecture that $y_4 = xe^{2x}$ is also a solution since $r = 2$ is a "double" root. You can verify that this is the case. Since y_4 is distinct from $y_1,\ y_2,$ and is independent of y_3, these solutions form a fundamental set and

$$y = C_1\, e^x + C_2\, e^{-x} + C_3 e^{2x} + C_4 x e^{2x}$$

is the general solution of the equation. ■

Example 3. Find the general solution of

$$y^{(4)} - 2y''' + y'' + 8y' - 20y = 0$$

given that $r = 1 + 2i$ is a root of the characteristic polynomial.

SOLUTION The characteristic equation is

$$p(r) = r^4 - 2r^3 + r^2 + 8r - 20 = 0.$$

Since $1 + 2i$ is a root of $p(r)$, $1 - 2i$ is also a root, and $r^2 - 2r + 5$ is a factor of $p(r)$. Therefore

$$\begin{aligned} r^4 - 2r^3 + r^2 + 8r - 20 &= 0 \\ (r^2 - 2r + 5)(r^2 - 4) &= 0 \\ (r^2 - 2r + 5)(r - 2)(r + 2) &= 0 \end{aligned}$$

The characteristic roots are: $r_1 = 1 + 2i,\ r_2 = 1 - 2i,\ r_3 = 2,\ r_4 = -2$. Since these roots are distinct, the corresponding exponential functions are linearly independent. Again based on our work in Chapter 3, we convert the complex exponentials

$u_1 = e^{(1+2i)x}$ and $u_2(x) = e^{(1-2i)x}$ into $y_1 = e^x \cos 2x$ and $y_2 = e^x \sin 2x$.

Then, $y_1,\ y_2,\ y_3 = e^{2x},\ y_4 = e^{-2x}$ form a fundamental set and

$$y = C_1\, e^x \cos 2x + C_2\, e^x \sin 2x + C_3 e^{2x} + C_4 e^{-2x}$$

is the general solution of the equation. ■

Recovering a Homogeneous Differential Equation from Its Solutions

Once you understand the relationship between the homogeneous equation, the characteristic equation, the roots of the characteristic equation and the solutions of the differential equation, it is easy to go from the differential equation to the solutions and from the solutions to the differential equation. Here are some examples.

Example 4. Find a fourth order, linear, homogeneous differential equation with constant coefficients that has the functions $y_1(x) = e^{2x}$, $y_2(x) = e^{-3x}$ and $y_3(x) = e^{2x} \cos x$ as solutions.

SOLUTION Since e^{2x} is a solution, 2 must be a root of the characteristic equation and $r-2$ must be a factor of the characteristic polynomial; similarly, e^{-3x} a solution means that -3 is a root and $r - (-3) = r + 3$ is a factor of the characteristic polynomial. The solution $e^{2x} \cos x$ indicates that $2+i$ is a root of the characteristic equation. So $2-i$ must also be a root (and $y_4(x) = e^{2x} \sin x$ must also be a solution). Thus the characteristic equation must be

$$(r-2)(r+3)(r-[2+i])(r-[2-i]) = (r^2+r-6)(r^2-4r+5) = r^4-3r^3-5r^2+29r-30 = 0.$$

Therefore, the differential equation is

$$y^{(4)} - 3y''' - 5y'' + 29y' - 30y = 0. \quad ■$$

Example 5. Find a third order, linear, homogeneous differential equation with constant coefficients that has

$$y = C_1\, e^{-4x} + C_2\, x\, e^{-4x} + C_3 e^{2x}$$

as its general solution.

SOLUTION Since e^{-4x} and xe^{-4x} are solutions, -4 must be a double root of the characteristic equation; since e^{2x} is a solution, 2 is a root of the characteristic equation. Therefore, the characteristic equation is

$$(r+4)^2(r-2) = 0 \qquad \text{which expands to} \qquad r^3 + 6r^2 - 32 = 0$$

and the differential equation is

$$y''' + 6y'' - 32y = 0. \qquad \blacksquare$$

Nonhomogeneous Equations

Now we'll consider linear nonhomogeneous equations:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = f(x) \qquad \text{(N)}$$

where $p_0,\ p_1,\ \ldots,\ p_{n-1},\ f$ are continuous functions on an interval I.

Continuing the analogy with second order linear equations, the corresponding homogeneous equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_1(x)y' + p_0(x)y = 0. \qquad \text{(H)}$$

is called the *reduced equation* of equation (N).

The following theorems are exactly the same as Theorems 1 and 2 in Section 3.3, and exactly the same proofs can be used.

THEOREM 5. If $z = z_1(x)$ and $z = z_2(x)$ are solutions of (N), then

$$y(x) = z_1(x) - z_2(x)$$

is a solution of equation (H).

the difference of any two solutions of the nonhomogeneous equation (N) is a solution of its reduced equation (H).

The next theorem gives the "structure" of the set of solutions of (N).

THEOREM 6. Let $y = y_1(x),\ y_2(x),\ \dots,\ y_n(x)$ be a fundamental set of solutions of the reduced equation (H) and let $z = z(x)$ be a particular solution of (N). If $u = u(x)$ is *any* solution of (N), then there exist constants $c_1,\ c_2,\ \dots,\ c_n$ such that

$$u(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + z(x)$$

According to Theorem68, if $\{y_1(x),\ y_2(x),\ \dots,\ y_n(x)\}$ is a fundamental set of solutions of the reduced equation (H) and if $z = z(x)$ is a particular solution of (N), then

$$y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x) + z(x) \tag{5}$$

represents the set of all solutions of (N). That is, (5) is the general solution of (N). Another way to look at (5) is: The general solution of (N) consists of the general solution of the reduced equation (H) *plus* a particular solution of (N):

$$\underbrace{y}_{\text{general solution of (N)}} = \underbrace{C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)}_{\text{general solution of (H)}} + \underbrace{z(x).}_{\text{particular solution of (N)}}$$

Finding a Particular Solution

The method of variation of parameters can be extended to higher-order linear nonhomogeneous equations but the calculations become quite involved. Instead we'll look at the special equations for which the method of undetermined coefficients can be used.

As we saw in Section 3.4, the method of undetermined coefficients can be applied only to nonhomogeneous equations of the form

$$y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y' + a_0(x) y = f(x),$$

where $a_0,\ a_1,\ \dots,\ a_{n-1}$ are constants and the nonhomogeneous term f is a polynomial, an exponential function, a sine, a cosine, or a combination of such functions.

Here is the basic table from Section 3.4, modified to apply to equations of order greater than 2:

Table 1

A particular solution of $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$

If $f(x) =$	try $z(x) =$*
ce^{rx}	Ae^{rx}
$c \cos \beta x + d \sin \beta x$	$z(x) = A \cos \beta x + B \sin \beta x$
$ce^{\alpha x} \cos \beta x + de^{\alpha x} \sin \beta x$	$z(x) = Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x$

*Note: If z satisfies the reduced equation, then $x^k z$, where k is the least integer such that $x^k z$ does not satisfy the reduced equation, will give a particular solution

The method of undetermined coefficients is applied in exactly the same manner as in Section 3.4.

Example 6. Find the general solution of

$$y''' - 2y'' - 5y' + 6y = 4 - 2e^{2x}. \tag{*}$$

SOLUTION First we solve the reduced equation

$$y''' - 2y'' - 5y' + 6y = 0.$$

The characteristic equation is

$$r^3 - 2r^2 - 5r + 6 = (r - 1)(r + 2)(r - 3) = 0.$$

The roots are $r_1 = 1,\ r_2 = -2,\ r_3 = 3$ and the corresponding solutions of the reduced equation are $y_1 = e^x,\ y_2 = e^{-2x},\ y_3 = e^{3x}$. Since these are distinct exponential functions, they are linearly independent and

$$y = C_1 e^x + C_2 e^{-2x} + C_3 e^{3x}$$

is the general solution of the reduced equation.

Next we find a particular solution of the nonhomogeneous equation. The table indicates that we should look for a solution of the form

$$z = A + Be^{2x}.$$

The derivatives of z are:

$$z = A + Be^{2x}, \quad z' = 2Be^{2x}, \quad z'' = 4Be^{2x}, \quad z''' = 8Be^{2x}.$$

Substituting into the left side of (*), we get

$$\begin{aligned} z''' - 2z'' - 5z' + 6z &= 8Be^{2x} - 2\left(4Be^{2x}\right) - 5\left(2Be^{2x}\right) + 6\left(A + Be^{2x}\right) \\ &= 6A - 4Be^{2x}. \end{aligned}$$

Setting $z'' + 6z' + 9z = 4 - 2e^{2x}$ gives

$$6A = 4 \quad \text{and} \quad -4B = -2 \quad \text{which implies} \quad A = \tfrac{2}{3} \quad \text{and} \quad B = \tfrac{1}{2}.$$

Thus, $z(x) = \frac{2}{3} + \frac{1}{2}\,e^{2x}$ is a particular solution of (*).

The general solution of (*) is

$$y = C_1e^{x} + C_2e^{-2x} + C_3e^{3x} + \tfrac{2}{3} + \tfrac{1}{2}\,e^{2x}. \qquad \blacksquare$$

Example 7. Find the general solution of

$$y^{(4)} + y''' - 3y'' - 5y' - 2y = 6e^{-x} \tag{**}$$

SOLUTION First we solve the reduced equation

$$y^{(4)} + y''' - 3y'' - 5y' - 2y = 0.$$

The characteristic equation is

$$r^4 + r^3 - 3r^2 - 5r - 2 = (r+1)^3(r-2) = 0.$$

The roots are $r_1 = r_2 = r_3 = -1, r_4 = 2$ and the corresponding solutions of the reduced equation are $y_1 = e^{-x},\ y_2 = xe^{-x},\ y_3 = x^2e^{-x},\ y_4 = e^{2x}$. Since distinct powers of x are linearly independent, it follows that $y_1,\ y_2,\ y_3$ are linearly independent; and since e^{2x} and e^{-x} are independent, we can conclude that $y_1,\ y_2,\ y_3,\ y_4$ are linearly independent. Thus, the general solution of the reduced equation is

$$y = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x} + C_4e^{2x}.$$

Next we find a particular solution of the nonhomogeneous equation. The table indicates that we should look for a solution of the form

$$z = Ax^3e^{-x}.$$

The derivatives of z are:

$$\begin{aligned} z &= Ax^3e^{-x} \\ z' &= 3Ax^2e^{-x} - Ax^3e^{-x} \\ z'' &= 6Axe^{-x} - 6Ax^2e^{-x} + Ax^3e^{-x} \\ z''' &= 6Ae^{-x} - 18Axe^{-x} + 9Ax^2e^{-x} - Ax^3e^{-x} \\ z^{(4)} &= -24Ae^{-x} + 36Axe^{-x} - 12Ax^2e^{-x} + Ax^3e^{-x} \end{aligned}$$

Substituting z and its derivatives into the left side of (**), we get

$$z^{(4)} + z''' - 3z'' - 5z' - 2z = -18Ae^{-x}.$$

Thus, we have $-18Ae^{-x} = 6e^{-x}$ which implies $A = -\frac{1}{3}$ and $z = -\frac{1}{3}\,x^2e^{-x}$ is a particular solution of (**).

The general solution of (**) is

$$y = C_1e^{-x} + C_2xe^{-x} + C_3x^2e^{-x} + C_4e^{2x} - \tfrac{1}{3}\,x^3e^{-x}. \qquad \blacksquare$$

Example 8. Give the form of a particular solution of

$$y''' - 3y'' + 3y' - y = 4e^x - 3\cos\, 2x.$$

SOLUTION To get the proper form for a particular solution of the equation we need to find the solutions of the reduced equation:

$$y''' - 3y'' + 3y' - y = 0.$$

The characteristic equation is

$$r^3 - 3r^3 + 3r - 1 = (r-1)^3 = 0.$$

Thus, the roots are $r_1 = r_2 = r_3 = 1$, and the corresponding solutions are $y_1 = e^x$, $y_2 = xe^x$, $y_3 = x^2e^x$. The table indicates that the form of a particular solution z of the nonhomogeneous equation is

$$z = Ax^3e^x + B\cos\, 2x + C\sin\, 2x. \qquad \blacksquare$$

Example 9. Give the form of a particular solution of

$$y^{(4)} - 16y = 4e^{2x} - 2e^{3x} + 5\sin\, 2x + 2\cos\, 2x.$$

SOLUTION To get the proper form for a particular solution of the equation we need to find the solutions of the reduced equation:

$$y^{(4)} - 16y = 0.$$

The characteristic equation is

$$r^4 - 16 = (r^2-4)(r^2+4) = (r-2)(r+2)(r^2+4) = 0.$$

Thus, the roots are $r_1 = 2, r_2 = -2, r_3 = 2i$, $r_4 = -2i$, and the corresponding solutions are $y_1 = e^{2x}$, $y_2 = e^{-2}x$, $y_3 = \cos\, 2x$, $y_4 = \sin\, 2x$. The table indicates that the form of a particular solution z of the nonhomogeneous equation is

$$z = Axe^{2x} + Be^{3x} + Cx\cos\, 2x + Dx\sin\, 2x. \qquad \blacksquare$$

Exercises 3.6

Find the general solution of the homogeneous equation

1. $y''' - 6y'' + 11y' - 6y = 0, \quad r_1 = 1$ is a root of the characteristic equation.
2. $y''' + y' + 10y = 0, \quad r_1 = -2$ is a root of the characteristic equation.
3. $y^{(4)} - 2y''' + y'' + 8y' - 20y = 0, \quad r_1 = 1 + 2i$ is a root of the characteristic equation.
4. $y^{(4)} - 3y'' - 4y = 0, \quad r_1 = i$ is a root of the characteristic equation.
5. $y^{(4)} - 4y''' + 14y'' - 4y' + 13y = 0, \quad r_1 = i$ is a root of the characteristic equation.
6. $y''' + y'' - 4y' - 4y = 0, \quad r_1 = -1$ is a root of the characteristic equation.
7. $y^{(6)} - y'' = 0.$
8. $y^{(5)} - 3y^{(4)} + 3y''' - 3y'' + 2y' = 0.$

Find the solution of the initial-value problem.

9. $y^{(4)} - 4y''' + 4y'' = 0; \quad y(0) = -1,\ y'(0) = 2,\ y''(0) = 0,\ y'''(0) = 0.$
10. $y''' + y' = 0; \quad y(0) = 0,\ y'(0) = 1,\ y''(0) = 2.$
11. $y''' - y'' + 9y' - 9y = 0; \quad y(0) = y'(0) = 0,\ y''(0) = 2.$
12. $2y^{(4)} - y''' - 9y'' + 4y' + 4y = 0; \quad y(0) = 0,\ y'(1) = 2,\ y''(0) = 2,\ y'''(0) = 0.$

Find the homogeneous equation with constant coefficients that has the given general solution.

13. $y = C_1 e^{-3x} + C_2 x e^{-3x} + C_3 e^x \cos 3x + C_4 e^x \sin 3x.$
14. $y = C_1 e^{4x} + C_2 x + C_3 + C_4 e^x \cos 2x + C_5 e^x \sin 2x.$
15. $y = C_1 e^{3x} + C_2 e^{-x} + C_3 \cos x + C_4 \sin x + C_5.$
16. $y = C_1 e^{2x} + C_2 x e^{2x} + C_3 x^2 e^{2x} + C_4.$

Find the homogeneous equation with constant coefficients of least order that has the given function as a solution.

17. $y = 2e^{2x} + 3\sin x - x.$

18. $y = 3xe^{-x} + e^{-x}\cos\, 2x + 1.$

19. $y = 2e^{x} - 3e^{-x} + 2x.$

20. $y = 3e^{3x} - 2\cos\, 2x + 4\sin\, x - 3.$

Find the general solution of the nonhomogeneous equation.

21. $y''' + y'' + y' + y = e^{x} + 4.$

22. $y^{(4)} - y = 2e^{x} + \cos\, 2x.$

23. $y^{(4)} + 2y'' + y = 6 + \cos\, 2x.$

24. $y''' - y'' - y' + y = 2e^{-x} + 4e^{2x}.$

Find the solution of the initial-value problem.

25. $y''' - 8y = e^{2x}; \quad y(0) = y'(0) = y''(0) = 0.$

26. $y''' - 2y'' - 5y' + 6y = 2e^{x}; \quad y(0) = 2,\ y'(0) = 0,\ y''(0) = -1.$