

Chapter 4

Systems of Linear Differential Equations

Introduction to Systems

Up to this point the entries in a vector or matrix have been real numbers. In this section, and in the following sections, we will be dealing with vectors and matrices whose entries are functions. A vector whose components are functions is called a *vector-valued function* or *vector function*. Similarly, a matrix whose entries are functions is called a *matrix function*.

The operations of vector and matrix addition, multiplication by a number and matrix multiplication for vector and matrix functions are exactly as defined in Chapter 5 so there is nothing new in terms of arithmetic. However, there are operations on functions other than arithmetic operations, e.g., limits, differentiation, and integration, that we have to define for vector and matrix functions. These operations from calculus are defined in a natural way.

Let $\mathbf{v}(t) = (f_1(t), f_2(t), \dots, f_n(t))$ be a vector function whose components are defined on an interval I .

Limit: Let $c \in I$. If $\lim_{t \rightarrow c} f_i(t) = \alpha_i$ exists for $i = 1, 2, \dots, n$, then

$$\lim_{t \rightarrow c} \mathbf{v}(t) = \left(\lim_{t \rightarrow c} f_1(t), \lim_{t \rightarrow c} f_2(t), \dots, \lim_{t \rightarrow c} f_n(t) \right) = (\alpha_1, \alpha_2, \dots, \alpha_n).$$

Limits of vector functions are calculated “component-wise.”

Derivative: If f_1, f_2, \dots, f_n are differentiable on I , then \mathbf{v} is

differentiable on I , and

$$\mathbf{v}' = ((f_1'(t), f_2'(t), \dots, f_n'(t)).$$

That is, \mathbf{v}' is the vector function whose components are the derivatives of the components of \mathbf{v} .

Integration: Since differentiation of vector functions is done component-wise, integration must also be component-wise. That is

$$\int \mathbf{v}(t) dt = \left(\int f_1(t) dt, \int f_2(t) dt, \dots, \int f_n(t) dt \right).$$

Limits, differentiation and integration of matrix functions is done in exactly the same way, component-wise.

4.1. Systems of Linear Differential Equations

Consider the third-order linear differential equation

$$y''' + p(t)y'' + q(t)y' + r(t)y = f(t)$$

where p, q, r, f are continuous functions on some interval I . Solving the equation for y''' , we get

$$y''' = -r(t)y - q(t)y' - p(t)y'' + f(t).$$

Introduce new dependent variables x_1, x_2, x_3 , as follows:

$$\begin{aligned} x_1 &= y \\ x_2 &= x_1' (= y') \\ x_3 &= x_2' (= y'') \end{aligned}$$

Then

$$y''' = x_3' = -r(t)x_1 - q(t)x_2 - p(t)x_3 + f(t)$$

and the third-order equation can be written equivalently as a system of three first-order equations:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -r(t)x_1 - q(t)x_2 - p(t)x_3 + f(t) \end{aligned}$$

Example 1. (a) Consider the third-order nonhomogeneous equation

$$y''' - y'' - 8y' + 12y = 2e^t.$$

Solving the equation for y''' , we have

$$y''' = -12y + 8y' + y'' + 2e^t.$$

Let $x_1 = y$, $x'_1 = x_2 (= y')$, $x'_2 = x_3 (= y'')$. Then

$$y''' = x'_3 = -12x_1 + 8x_2 + x_3 + 2e^t$$

and the equation converts to the equivalent system:

$$\begin{aligned}x'_1 &= x_2 \\x'_2 &= x_3 \\x'_3 &= -12x_1 + 8x_2 + x_3 + 2e^t\end{aligned}$$

Note: This system is just a very special case of the “general” system of three, first-order differential equations:

$$\begin{aligned}x'_1 &= a_{11}(t)x_1 + a_{12}(t)x_2 + a_{13}(t)x_3(t) + b_1(t) \\x'_2 &= a_{21}(t)x_1 + a_{22}(t)x_2 + a_{23}(t)x_3(t) + b_2(t) \\x'_3 &= a_{31}(t)x_1 + a_{32}(t)x_2 + a_{33}(t)x_3(t) + b_3(t)\end{aligned}$$

(b) Consider the second-order homogeneous equation

$$t^2y'' - ty' - 3y = 0.$$

Solving this equation for y'' , we get

$$y'' = \frac{3}{t^2}y + \frac{1}{t}y'.$$

To convert this equation to an equivalent system, we let $x_1 = y$, $x'_1 = x_2 (= y')$. Then we have

$$\begin{aligned}x'_1 &= x_2 \\x'_2 &= \frac{3}{t^2}x_1 + \frac{1}{t}x_2\end{aligned}$$

which is just a special case of the general system of two first-order differential equations:

$$\begin{aligned}x'_1 &= a_{11}(t)x_1 + a_{12}(t)x_2 + b_1(t) \\x'_2 &= a_{21}(t)x_1 + a_{22}(t)x_2 + b_2(t)\end{aligned}$$

General Theory

Let $a_{11}(t), a_{12}(t), \dots, a_{1n}(t), a_{21}(t), \dots, a_{nn}(t), b_1(t), b_2(t), \dots, b_n(t)$ be continuous functions on some interval I . The system of n first-order differential equations

$$\begin{aligned}x'_1 &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n(t) + b_1(t) \\x'_2 &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n(t) + b_2(t) \\&\vdots \\x'_n &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n(t) + b_n(t)\end{aligned}\tag{S}$$

is called a *first-order linear differential system*.

The system (S) is *homogeneous* if

$$b_1(t) \equiv b_2(t) \equiv \cdots \equiv b_n(t) \equiv 0 \quad \text{on } I.$$

(S) is *nonhomogeneous* if the functions $b_i(t)$ are not all identically zero on I ; that is, if there is at least one point $a \in I$ and at least one function $b_i(t)$ such that $b_i(a) \neq 0$.

Let $A(t)$ be the $n \times n$ matrix

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}$$

and let \mathbf{x} and \mathbf{b} be the vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Then (S) can be written in the *vector-matrix form*

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{b}.\tag{S}$$

The matrix $A(t)$ is called the *matrix of coefficients* or the *coefficient matrix*.

Example 2. The vector-matrix form of the system in Example 1(a) is:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 8 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 0 \\ 2e^t \end{pmatrix},$$

a nonhomogeneous system.

The vector-matrix form of the system in Example 1(b) is:

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 3/t^2 & 1/t \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3/t^2 & 1/t \end{pmatrix} \mathbf{x}, \quad \text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

a homogeneous system.

A *solution* of the linear differential system (S) is a differentiable vector function

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

that satisfies (S) on the interval I .

Example 3. Verify that $\mathbf{x}(t) = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}e^t \\ \frac{1}{2}e^t \\ \frac{1}{2}e^t \end{pmatrix}$ is a solution of the nonhomogeneous system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 8 & 1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 0 \\ 2e^t \end{pmatrix}$$

of Example 2.

SOLUTION

$$\begin{aligned}
 \mathbf{x}' &= \left[\begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}e^t \\ \frac{1}{2}e^t \\ \frac{1}{2}e^t \end{pmatrix} \right]' \\
 &= \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \\ 8e^{2t} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}e^t \\ \frac{1}{2}e^t \\ \frac{1}{2}e^t \end{pmatrix} \\
 &\stackrel{?}{=} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 8 & 1 \end{pmatrix} \left[\begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}e^t \\ \frac{1}{2}e^t \\ \frac{1}{2}e^t \end{pmatrix} \right] + \begin{pmatrix} 0 \\ 0 \\ 2e^t \end{pmatrix} \\
 &\stackrel{?}{=} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 8 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 8 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}e^t \\ \frac{1}{2}e^t \\ \frac{1}{2}e^t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2e^t \end{pmatrix} \\
 &= \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \\ 8e^{2t} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}e^t \\ -\frac{1}{2}e^t \\ -\frac{3}{2}e^t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2e^t \end{pmatrix} \\
 &= \begin{pmatrix} 2e^{2t} \\ 4e^{2t} \\ 8e^{2t} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}e^t \\ \frac{1}{2}e^t \\ \frac{1}{2}e^t \end{pmatrix}.
 \end{aligned}$$

\mathbf{x} is a solution.

THEOREM 1. (Existence and Uniqueness Theorem) Let a be any point on the interval I , and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be any n real numbers. Then the initial-value problem

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{b}(t), \quad \mathbf{x}(a) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

has a unique solution.

Exercises 4.1

Convert the differential equation into a system of first-order equations.

- $y'' - ty' + 3y = \sin 2t.$

2. $y'' + y = 2e^{-2t}$.
3. $y''' - y'' + y = e^t$.
4. $my'' + cy' + ky = \cos \lambda t$, m, c, k, λ are constants.

Write the system in vector-matrix form.

5.

$$\begin{aligned}x_1' &= -2x_1 + x_2 + \sin t \\x_2' &= x_1 - 3x_2 - 2 \cos t\end{aligned}$$

6.

$$\begin{aligned}x_1' &= e^t x_1 - e^{2t} x_2 \\x_2' &= e^{-t} x_1 - 3e^t x_2\end{aligned}$$

7.

$$\begin{aligned}x_1' &= 2x_1 + x_2 + 3x_3 + 3e^{2t} \\x_2' &= x_1 - 3x_2 - 2 \cos t \\x_3' &= 2x_1 - x_2 + 4x_3 + t\end{aligned}$$

8.

$$\begin{aligned}x_1' &= t^2 x_1 + x_2 - t x_3 + 3 \\x_2' &= -3e^t x_2 + 2x_3 - 2e^{-2t} \\x_3' &= 2x_1 + t^2 x_2 + 4x_3\end{aligned}$$

9. Verify that $\mathbf{u}(t) = \begin{pmatrix} t^{-1} \\ -t^{-2} \end{pmatrix}$ is a solution of the system in Example 1 (b).
10. Verify that $\mathbf{u}(t) = \begin{pmatrix} e^{-3t} \\ -3e^{-3t} \\ 9e^{-3t} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} e^t \\ \frac{1}{2} e^t \\ \frac{1}{2} e^t \end{pmatrix}$ is a solution of the system in Example 1 (a).
11. Verify that $\mathbf{w}(t) = \begin{pmatrix} te^{2t} \\ e^{2t} + 2te^{2t} \\ 4e^{2t} + 4te^{2t} \end{pmatrix}$ is a solution of the homogeneous system associated with the system in Example 1 (a).

12. Verify that $\mathbf{x}(t) = \begin{pmatrix} -\sin t \\ -\cos t - 2\sin t \end{pmatrix}$ is a solution of the system

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -3 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 2\sin t \end{pmatrix}.$$

13. Verify that $\mathbf{x}(t) = \begin{pmatrix} -2e^{-2t} \\ 0 \\ 3e^{-2t} \end{pmatrix}$ is a solution of the system

$$\mathbf{x}' = \begin{pmatrix} 1 & -3 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & -2 \end{pmatrix} \mathbf{x}.$$

4.2. Homogeneous Systems

In this section we give the basic theory for linear homogeneous systems. This “theory” is simply a repetition results given in Sections 3.2 and 3.6, phrased this time in terms of the system

$$\begin{aligned} x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n(t) \\ x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n(t) \\ &\vdots \\ x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n(t) \end{aligned} \tag{H}$$

or

$$\mathbf{x}' = A(t)\mathbf{x}. \tag{H}$$

Note first that the zero vector $\mathbf{z}(t) \equiv \mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a solution of (H). As before,

this solution is called the *trivial solution*. Of course, we are interested in finding *nontrivial solutions*.

THEOREM 1. If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are solutions of (H), and if c_1, c_2, \dots, c_k are real numbers, then

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$$

is a solution of (H); any linear combination of solutions of (H) is also a solution of (H).

DEFINITION 1. Let

$$\mathbf{x}_1 = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \\ \vdots \\ x_{n2}(t) \end{pmatrix}, \quad \dots, \quad \mathbf{x}_k = \begin{pmatrix} x_{1k}(t) \\ x_{2k}(t) \\ \vdots \\ x_{nk}(t) \end{pmatrix}$$

be n -component vector functions defined on some interval I . The vectors are *linearly dependent* on I if there exist k real numbers c_1, c_2, \dots, c_k , not all zero, such that

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \dots + c_k\mathbf{x}_k(t) \equiv 0 \quad \text{on } I.$$

Otherwise the vectors are *linearly independent* on I .

THEOREM 2. Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ be n , n -component vector functions defined on an interval I . If the vectors are linearly dependent, then

$$W(t) = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} \equiv 0 \quad \text{on } I.$$

The determinant in Theorem 4 is called the *Wronskian* of the vector functions $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$.

COROLLARY Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ be n , n -component vector functions defined on an interval I . If the Wronskian $W(t) \neq 0$ for at least one $t \in I$, then the vectors are linearly independent on I .

Example 1. The vector functions

$$\mathbf{u} = \begin{pmatrix} t^3 \\ 3t^2 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} t^{-1} \\ -t^{-2} \end{pmatrix}$$

are solutions of the homogeneous system in Example 1(b), Section 4.1. Their Wronskian is:

$$W(t) = \begin{vmatrix} t^3 & t^{-1} \\ 3t^2 & -t^{-2} \end{vmatrix} = -4t.$$

The solutions are linearly independent.

The vector functions

$$\mathbf{x}_1 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} e^{-3t} \\ -3e^{-3t} \\ 9e^{-3t} \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} te^{2t} \\ e^{2t} + 2te^{2t} \\ 4e^{2t} + 4te^{2t} \end{pmatrix}$$

are solutions of the homogeneous system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 8 & 1 \end{pmatrix} \mathbf{x}.$$

Their Wronskian is:

$$W(t) = \begin{vmatrix} e^{2t} & e^{-3t} & te^{2t} \\ 2e^{2t} & -3e^{-3t} & e^{2t} + 2te^{2t} \\ 4e^{2t} & 9e^{-3t} & 4e^{2t} + 4te^{2t} \end{vmatrix} = -25e^t.$$

These solutions are linearly independent.

THEOREM 3. Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ be n solutions of (H). Exactly one of the following holds:

1. $W(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)(t) \equiv 0$ on I and the solutions are linearly dependent.
2. $W(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)(t) \neq 0$ for all $t \in I$ and the solutions are linearly independent.

It is easy to construct sets of n linearly independent solutions of (H). Simply pick any point $a \in I$ and any nonsingular $n \times n$ matrix A . Let α_1 be the first column of A , α_2 the second column of A , and so on. Then let \mathbf{x}_1 be the solution of (H) such that $\mathbf{x}_1(a) = \alpha_1$, let \mathbf{x}_2 be the solution of (H) such that $\mathbf{x}_2(a) = \alpha_2, \dots$, and let \mathbf{x}_n be the solution of (H) such that $\mathbf{x}_n(a) = \alpha_n$. The existence and uniqueness theorem guarantees the existence of these solutions. Now

$$W(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)(a) = \det A \neq 0.$$

Therefore, $W(t) \neq 0$ for all $t \in I$ and the solutions are linearly independent.

A particularly nice set of n linearly independent solutions is obtained by choosing $A = I_n$, the identity matrix.

THEOREM 4. Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ be n linearly independent solutions of (H). Let \mathbf{u} be *any* solution of (H). Then there exists a unique set of constants c_1, c_2, \dots, c_n such that

$$\mathbf{u} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n.$$

That is, every solution of (H) can be written as a unique linear combination of $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$.

DEFINITION 2. A set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ of n linearly independent solutions of (H) is called a *fundamental set of solutions*. A fundamental set of solutions is also called a *solutions basis* for (H). If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ is a fundamental set of solutions of (H), then the $n \times n$ matrix

$$X(t) = \begin{pmatrix} x_{11}(t) & x_{12}(t) & \cdots & x_{1n}(t) \\ x_{21}(t) & x_{22}(t) & \cdots & x_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}(t) & x_{n2}(t) & \cdots & x_{nn}(t) \end{pmatrix}$$

(the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are the columns of X) is called a *fundamental matrix* for (H).

DEFINITION 3. Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ be a fundamental set of solutions of (H). Then

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n,$$

where c_1, c_2, \dots, c_n are arbitrary constants, is the *general solution* of (H).

Exercises 4.2

Determine whether or not the vector functions are linearly dependent.

1. $\mathbf{x}_1 = \begin{pmatrix} 2t - 1 \\ -t \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -t + 1 \\ 2t \end{pmatrix}$

2. $\mathbf{x}_1 = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

3. $\mathbf{x}_1 = \begin{pmatrix} 2 - t \\ t \\ -2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} t \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 2 + t \\ t - 2 \\ 2 \end{pmatrix}.$

4. $\mathbf{x}_1 = \begin{pmatrix} e^t \\ -e^t \\ e^t \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -e^t \\ 2e^t \\ -e^t \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ e^t \\ 0 \end{pmatrix}.$

5. $\mathbf{x}_1 = \begin{pmatrix} e^t \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ e^t \end{pmatrix}$

6. Given the linear differential system

$$\mathbf{x}' = \begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix} \mathbf{x}.$$

Let

$$\mathbf{x}_1 = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} 3e^{3t} \\ 2e^{3t} \end{pmatrix}.$$

(a) Show that $\mathbf{x}_1, \mathbf{x}_2$ are a fundamental set of solutions of the system.

(b) Let X be the corresponding fundamental matrix. Show that

$$X' = AX.$$

(c) Give the general solution of the system.

(d) Find the solution of the system that satisfies $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

7. Let X be the matrix function

$$X(t) = \begin{pmatrix} 0 & 4te^{-t} & e^{-t} \\ 1 & e^{-t} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(a) Verify that X is a fundamental matrix for the system

$$\mathbf{x}' = \begin{pmatrix} -1 & 4 & -4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x}.$$

(b) Find the solution of the system that satisfies $\mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

4.3. Homogeneous Systems with Constant Coefficients

A homogeneous system with constant coefficients is a linear differential system having the form

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned}$$

where $a_{11}, a_{12}, \dots, a_{nn}$ are constants. The system in vector-matrix form is

$$\begin{array}{l} x_1' \\ x_2' \\ - \\ x_n' \end{array} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ - \\ x_n \end{pmatrix} \quad \text{or} \quad \vec{x}' = A\vec{x}.$$

Example 1. Consider the 3rd order linear homogeneous differential equation

$$y''' - y'' - 8y' + 12y = 0.$$

The characteristic equation is:

$$r^3 - r^2 - 8r + 12 = (r - 2)^2(r + 3) = 0$$

and $\{e^{2t}, te^{2t}, e^{-3t}\}$ is a solution basis for the equation.

The corresponding linear homogeneous system is

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & 8 & 1 \end{pmatrix} \mathbf{x}$$

and

$$\mathbf{x}_1(t) = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

is a solution vector. Similarly,

$$\mathbf{x}_2(t) = e^{-3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$$

is a solution vector.

The example suggests that homogeneous systems with constant coefficients might have solution vectors of the form $\mathbf{x}(t) = e^{\lambda t} \mathbf{v}$, for some number λ and some constant vector \mathbf{v} .

If $\mathbf{x}(t) = e^{\lambda t} \mathbf{v}$ is a solution vector of (H), then

$$\mathbf{x}' = A\mathbf{x} \quad \text{which implies} \quad \lambda e^{\lambda t} \mathbf{v} = A e^{\lambda t} \mathbf{v} \quad \text{and so} \quad A\mathbf{v} = \lambda \mathbf{v}.$$

The latter equation is an eigenvalue-eigenvector equation for A . Thus, we look for solutions of the form $\mathbf{x}(t) = e^{\lambda t} \mathbf{c}$ where λ is an eigenvalue of A and \mathbf{c} is a corresponding eigenvector.

Example 2. Find a fundamental set of solution vectors of

$$\mathbf{x}' = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix} \mathbf{x}$$

and give the general solution of the system.

SOLUTION First we find the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 5 \\ 3 & 3 - \lambda \end{vmatrix} = (\lambda - 6)(\lambda + 2).$$

The eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = -2$.

Next, we find corresponding eigenvectors. For $\lambda_1 = 6$ we have:

$$(A - 6I)\mathbf{x} = \begin{pmatrix} -5 & 5 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{which implies } x_1 = x_2, x_2 \text{ arbitrary.}$$

Setting $x_2 = 1$, we get the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Repeating the process for $\lambda_2 = -2$, we get the eigenvector $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$.

Thus $\mathbf{x}_1 = e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{x}_2 = e^{-2t} \begin{pmatrix} 5 \\ -3 \end{pmatrix}$ are solution vectors of the system.

The Wronskian of \mathbf{x}_1 and \mathbf{x}_2 is:

$$W(t) = \begin{vmatrix} e^{6t} & 5e^{-2t} \\ e^{6t} & -3e^{-2t} \end{vmatrix} = -8e^{4t} \neq 0.$$

Thus \mathbf{x}_1 and \mathbf{x}_2 are linearly independent; they form a fundamental set of solutions.

The general solution of the system is

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1e^{6t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{-2t} \begin{pmatrix} 5 \\ -3 \end{pmatrix}.$$

Example 3. Find a fundamental set of solution vectors of

$$\mathbf{x}' = \begin{pmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{pmatrix} \mathbf{x}$$

and find the solution that satisfies the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

SOLUTION

$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -1 & -1 \\ -12 & -\lambda & 5 \\ 4 & -2 & -1 - \lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 + \lambda - 2.$$

Now

$$\det(A - \lambda I) = 0 \quad \text{implies} \quad \lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda - 2)(\lambda - 1)(\lambda + 1) = 0.$$

The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = -1$.

As you can check, corresponding eigenvectors are:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

A fundamental set of solution vectors is:

$$\mathbf{x}_1 = e^{2t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = e^t \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix}, \quad \mathbf{x}_3 = e^{-t} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

since distinct exponential vector-functions are linearly independent (calculate the Wronskian to verify.)

To find the solution vector satisfying the initial condition, solve

$$c_1 \mathbf{x}_1(0) + c_2 \mathbf{x}_2(0) + c_3 \mathbf{x}_3(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

which is:

$$c_1 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 3 & 1 \\ -1 & -1 & 2 \\ 2 & 7 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Note: The matrix of coefficients is the fundamental matrix evaluated at $t = 0$

Using the solution method of your choice (row reduction, inverse, Cramer's rule), the solution is: $c_1 = 3$, $c_2 = -1$, $c_3 = 1$. The solution of the initial-value problem is

$$\mathbf{x} = 3e^{2t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - e^t \begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix} + e^{-t} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

Two Difficulties

There are two difficulties that can arise:

1. A has complex eigenvalues.

If $\lambda = a + bi$ is a complex eigenvalue with corresponding (complex) eigenvector $\mathbf{u} + i\mathbf{x}$, then $\bar{\lambda} = a - bi$ (the complex conjugate of λ) is also an eigenvalue of A and $\mathbf{u} - i\mathbf{x}$ is a corresponding eigenvector. The corresponding linearly independent complex solutions of $\mathbf{x}' = A\mathbf{x}$ are:

$$\begin{aligned} \mathbf{w}_1 &= e^{(a+bi)t}(\mathbf{u} + i\mathbf{x}) = e^{at}(\cos bt + i \sin bt)(\mathbf{u} + i\mathbf{x}) \\ &= e^{at}[(\cos bt \mathbf{u} - \sin bt \mathbf{x}) + i(\cos bt \mathbf{x} + \sin bt \mathbf{u})] \end{aligned}$$

$$\begin{aligned} \mathbf{w}_2 &= e^{(a-bi)t}(\mathbf{u} - i\mathbf{x}) = e^{at}(\cos bt - i \sin bt)(\mathbf{u} - i\mathbf{x}) \\ &= e^{at}[(\cos bt \mathbf{u} - \sin bt \mathbf{x}) - i(\cos bt \mathbf{x} + \sin bt \mathbf{u})] \end{aligned}$$

Now

$$\mathbf{x}_1(t) = \frac{1}{2} [\mathbf{w}_1(t) + \mathbf{w}_2(t)] = e^{at}(\cos bt \mathbf{u} - \sin bt \mathbf{x})$$

and

$$\mathbf{x}_2(t) = \frac{1}{2i} [\mathbf{w}_1(t) - \mathbf{w}_2(t)] = e^{at}(\cos bt \mathbf{x} + \sin bt \mathbf{u})$$

are linearly independent solutions of the system, and they are real-valued vector functions. It is worth noting that \mathbf{x}_1 and \mathbf{x}_2 are simply the real and imaginary parts of \mathbf{w}_1 (or of \mathbf{w}_2).

Example 4. Determine a fundamental set of solution vectors of

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 & -1 \\ 3 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} \mathbf{x}.$$

SOLUTION

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & -4 & -1 \\ 3 & 2 - \lambda & 3 \\ 1 & 1 & 3 - \lambda \end{vmatrix} = -\lambda^3 + 6\lambda^2 - 21\lambda + 26 = -(\lambda - 2)(\lambda^2 - 4\lambda + 13).$$

The eigenvalues are: $\lambda_1 = 2$, $\lambda_2 = 2 + 3i$, $\lambda_3 = 2 - 3i$. The corresponding eigenvectors are:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -5 + 3i \\ 3 + 3i \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$$
$$\mathbf{v}_3 = \begin{pmatrix} -5 - 3i \\ 3 - 3i \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}.$$

Now

$$e^{(2+3i)t} \left[\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \right] =$$
$$e^{2t}(\cos 3t + i \sin 3t) \left[\begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} + i \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \right] =$$
$$e^{2t} \left[\cos 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - \sin 3t \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \right] + i e^{2t} \left[\cos 3t \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} \right].$$

A fundamental set of solution vectors for the system is:

$$\mathbf{x}_1 = e^{2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{x}_2 = e^{2t} \left[\cos 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} - \sin 3t \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \right],$$
$$\mathbf{x}_3 = e^{2t} \left[\cos 3t \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \sin 3t \begin{pmatrix} -5 \\ 3 \\ 2 \end{pmatrix} \right].$$

2. A has an eigenvalue of multiplicity greater than 1

We'll look first at the case where A has an eigenvalue of multiplicity 2.

Example 5. Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & -3 & 3 \\ 3 & -5 - \lambda & 3 \\ 6 & -6 & 4 - \lambda \end{vmatrix} = -\lambda^3 + 12\lambda - 16 = -(\lambda - 4)(\lambda + 2)^2.$$

The eigenvalues are: $\lambda_1 = 4$, $\lambda_2 = \lambda_3 = -2$.

As you can check, an eigenvector corresponding to $\lambda_1 = 4$ is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

We'll carry out the details involved in finding an eigenvector corresponding to the "double" eigenvalue -2 .

$$[A - (-2)I]\mathbf{v} = \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix for this system of equations is

$$\left(\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{array} \right) \text{ which row reduces to } \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

The solutions of this system are: $v_1 = v_2 - v_3$, v_2, v_3 arbitrary. We can assign values to v_2 and v_3 independently and obtain two linearly independent eigenvectors. For

example, setting $v_2 = 1$, $v_3 = 0$, we get the eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Reversing

the roles, we set $v_2 = 0$, $v_3 = -1$ to get the eigenvector $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Clearly

\mathbf{v}_2 and \mathbf{v}_3 are linearly independent. You should understand that there is nothing magic about our two choices for v_2, v_3 ; any choice which produces two independent vectors will do.

The important thing to note here is that this eigenvalue of multiplicity 2 produced two independent eigenvectors.

Based on our work above, a fundamental set of solutions for the differential system

$$\mathbf{x}' = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} \mathbf{x}$$

is

$$\mathbf{x}_1 = e^{4t} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = e^{-2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Example 6. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & 8 & -1 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 12 & 8 & -1 - \lambda \end{vmatrix} = -\lambda^3 - \lambda^2 + 8\lambda - 12 = -(\lambda - 3)(\lambda + 2)^2.$$

The eigenvalues are: $\lambda_1 = 3$, $\lambda_2 = \lambda_3 = -2$.

As you can check, an eigenvector corresponding to $\lambda_1 = 3$ is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$.

We'll carry out the details involved in finding an eigenvector corresponding to the "double" eigenvalue -2 .

$$[A - (-2)I]\mathbf{v} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 12 & 8 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The augmented matrix for this system of equations is

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 12 & 8 & 1 & 0 \end{array} \right) \quad \text{which row reduces to} \quad \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

The solutions of this system are $v_1 = \frac{1}{4}v_3$, $v_2 = -\frac{1}{2}v_3$, v_3 arbitrary. Here there is only one parameter and so we'll get only one eigenvector. Setting $v_3 = 4$ we get the

eigenvector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$.

In contrast to the preceding example, the "double" eigenvalue here has only one (independent) eigenvector.

Suppose that we were asked to find a fundamental set of solutions of the linear differential system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 12 & 8 & -1 \end{pmatrix} \mathbf{x}.$$

By our work above, we have two independent solutions

$$\mathbf{x}_1 = e^{3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}.$$

We need a third solution which is independent of these two.

Our system has a special form; it is equivalent to the third order equation

$$y''' + y'' - 8y' - 12y = 0.$$

The characteristic equation is

$$r^3 + r^2 - 8r - 12 = (r - 3)(r + 2)^2 = 0$$

(compare with $\det(A - \lambda I)$.) The roots are: $r_1 = 3$, $r_2 = r_3 = -2$ and a fundamental set of solutions is $\{y_1 = e^{3t}, y_2 = e^{-2t}, y_3 = te^{-2t}\}$. The correspondence between these solutions and the solution vectors we found above should be clear:

$$e^{3t} \longrightarrow e^{3t} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}, \quad e^{-2t} \longrightarrow e^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}.$$

The solution $y_3 = te^{-2t}$ of the equation corresponds to the solution vector

$$\mathbf{x}_3 = \begin{pmatrix} y_3 \\ y_3' \\ y_3'' \end{pmatrix} = \begin{pmatrix} te^{-2t} \\ e^{-2t} - 2te^{-2t} \\ -4e^{-2t} - 4te^{-2t} \end{pmatrix} = e^{-2t} \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + te^{-2t} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}.$$

The appearance of the $te^{-2t}\mathbf{v}_2$ term should not be unexpected since we know that a characteristic root of multiplicity 2 produces a solution of the form te^{rt} .

You can check that \mathbf{x}_3 is independent of \mathbf{x}_1 and \mathbf{x}_2 . Therefore, the solution vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are a fundamental set of solutions of the system.

The question is: What is the significance of the vector $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}$? How is it related to the eigenvalue -2 which generated it, and to the corresponding eigenvector?

Let's look at $[A - (-2)I]\mathbf{w} = [A + 2I]\mathbf{w}$:

$$[A + 2I]\mathbf{w} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 12 & 8 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = \mathbf{v}_2;$$

$A - (-2)I$ “maps” \mathbf{w} onto the eigenvector \mathbf{v}_2 . The corresponding solution of the system has the form

$$\mathbf{x}_3 = e^{-2t}\mathbf{w} + te^{-2t}\mathbf{v}_2$$

where \mathbf{v}_2 is the eigenvector corresponding to -2 and \mathbf{w} satisfies

$$[A - (-2)I]\mathbf{w} = \mathbf{v}_2.$$

General Result

Given the linear differential system $\mathbf{x}' = A\mathbf{x}$. Suppose that A has an eigenvalue λ of multiplicity 2. Then exactly one of the following holds:

1. λ has two linearly independent eigenvectors, \mathbf{v}_1 and \mathbf{v}_2 . Corresponding linearly independent solution vectors of the differential system are $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda t}\mathbf{v}_2$.
2. λ has only one (independent) eigenvector \mathbf{v} . Then a linearly independent pair of solution vectors corresponding to λ are:

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v} \quad \text{and} \quad \mathbf{x}_2(t) = e^{\lambda t}\mathbf{w} + te^{\lambda t}\mathbf{v}$$

where \mathbf{w} is a vector that satisfies $(A - \lambda I)\mathbf{w} = \mathbf{v}$. The vector \mathbf{w} is called a *generalized eigenvector* corresponding to the eigenvalue λ .

Example 7. Find a fundamental set of solution vectors of $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\mathbf{x}$.

SOLUTION

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & -1 \\ 1 & 3 - \lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$

Characteristic values: $\lambda_1 = \lambda_2 = 2$.

Characteristic vectors:

$$(A - 2I)\mathbf{v} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

$$\left(\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array} \right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right).$$

The solutions are: $v_1 = -v_2$, v_2 arbitrary; there is only one eigenvector. Setting $v_2 = -1$, we get $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The vector $\mathbf{x}_1 = e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a solution of the system.

A second solution, independent of \mathbf{x}_1 is $\mathbf{x}_2 = e^{2t}\mathbf{w} + te^{2t}\mathbf{v}$ where \mathbf{w} is a solution of $(A - 2I)\mathbf{z} = \mathbf{v}$:

$$(A - 2I)\mathbf{z} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix};$$

$$\left(\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array} \right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right).$$

The solutions of this system are $z_1 = -1 - z_2$, z_2 arbitrary. If we choose $z_2 = 0$ (*any* choice for z_2 will do), we get $z_1 = -1$ and $\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. Thus

$$\mathbf{x}_2(t) = e^{2t} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + te^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is a solution of the system independent of \mathbf{x}_1 . The solutions

$$\mathbf{x}_1(t) = e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathbf{x}_2(t) = e^{2t} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + te^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are a fundamental set of solutions of the system.

Eigenvalues of Multiplicity 3.

Given the differential system $\mathbf{x}' = A\mathbf{x}$. Suppose that λ is an eigenvalue of A of multiplicity 3. Then exactly one of the following holds:

1. λ has three linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Then three linearly independent solution vectors of the system corresponding to λ are:

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1, \quad \mathbf{x}_2(t) = e^{\lambda t}\mathbf{v}_2, \quad \mathbf{x}_3(t) = e^{\lambda t}\mathbf{v}_3.$$

2. λ has two linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. Then two linearly independent solutions of the system corresponding to λ are:

$$\mathbf{x}_1(t) = e^{\lambda t} \mathbf{v}_1, \quad \mathbf{x}_2(t) = e^{\lambda t} \mathbf{v}_2$$

A third solution, independent of \mathbf{x}_1 and \mathbf{x}_2 has the form

$$\mathbf{x}_3(t) = e^{\lambda t} \mathbf{w} + t e^{\lambda t} \mathbf{x}$$

where \mathbf{x} is an eigenvector corresponding to λ and $(A - \lambda I)\mathbf{w} = \mathbf{x}$.

3. λ has only one (independent) eigenvector \mathbf{v} . Then three linearly independent solutions of the system have the form:

$$\mathbf{x}_1 = e^{\lambda t} \mathbf{v}, \quad \mathbf{x}_2 = e^{\lambda t} \mathbf{w} + t e^{\lambda t} \mathbf{v},$$

$$\mathbf{x}_3(t) = e^{\lambda t} \mathbf{z} + t e^{\lambda t} \mathbf{w} + t^2 e^{\lambda t} \mathbf{v}$$

where $(A - \lambda I)\mathbf{w} = \mathbf{v}$ and $(A - \lambda I)\mathbf{z} = \mathbf{w}$.

Exercises 4.3

Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$ where A is the given matrix. If an initial condition is given, also find the solution that satisfies the condition.

1. $\begin{pmatrix} -2 & 4 \\ 1 & 1 \end{pmatrix}$.

2. $\begin{pmatrix} -1 & 1 \\ 4 & 2 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

3. $\begin{pmatrix} -2 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & -2 & -1 \end{pmatrix}$. Hint: -3 is an eigenvalue.

4. $\begin{pmatrix} 3 & 0 & -1 \\ -2 & 2 & 1 \\ 8 & 0 & -3 \end{pmatrix}$, $\mathbf{x}(0) = \begin{pmatrix} -1 \\ 2 \\ -8 \end{pmatrix}$. Hint: 2 is an eigenvalue.

5. $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$.

6. $\begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix}$.

7. $\begin{pmatrix} 3 & 2 \\ -8 & -5 \end{pmatrix}$.

8. $\begin{pmatrix} -3 & 0 & -3 \\ 1 & -2 & 3 \\ 1 & 0 & 1 \end{pmatrix}$. Hint: -2 is an eigenvalue.

9. $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 4 \end{pmatrix}$. Hint: 3 is an eigenvalue

10. $\begin{pmatrix} -2 & 1 & -1 \\ 3 & -3 & 4 \\ 3 & -1 & 2 \end{pmatrix}$. Hint: 1 is an eigenvalue.

11. $\begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$.