Linear Algebra and Matrix Theory

Chapter 1 - Linear Systems, Matrices and Determinants

This is a very brief outline of some basic definitions and theorems of linear algebra. We will assume that you know elementary facts such as how to add two matrices, how to multiply a matrix by a number, how to multiply two matrices, what an identity matrix is, and what a solution of a linear system of equations is. Hardly any of the theorems will be proved. More complete treatments may be found in the following references.

1. References

(1) S. Friedberg, A. Insel and L. Spence, *Linear Algebra*, Prentice-Hall.

(2) M. Golubitsky and M. Dellnitz, *Linear Algebra and Differential Equations Using Matlab*, Brooks-Cole.

(3) K. Hoffman and R. Kunze, *Linear Algebra*, Prentice-Hall.

(4) P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, Academic Press.

The solutions, if any, of a linear system of equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
\tag{2.1}
$$

may be found by *Gaussian elimination.* The permitted steps are as follows.

(1) Both sides of any equation may be multiplied by the same nonzero constant.

(2) Any two equations may be interchanged.

(3) Any multiple of one equation may be added to another equation.

Instead of working with the symbols for the variables (the x_i), it is easier to place the coefficients (the a_{ij}) and the forcing terms (the b_i) in a rectangular array called the *augmented matrix* of the system.

$$
\begin{pmatrix}
a_{11} & a_{12} & \dots & a_{1n} & | & b_1 \\
a_{21} & a_{22} & \dots & a_{2n} & | & b_2 \\
\vdots & & & & & \\
a_{m1} & a_{m2} & \dots & a_{mn} & | & b_m
\end{pmatrix}
\tag{2.2}
$$

The steps of Gaussian elimination are carried out by *elementary row operations* applied to the augmented matrix. These are:

(1) Any row of the matrix may be multiplied throughout by any nonzero number.

(2) Any two rows of the matrix may be interchanged.

(3) Any multiple of one row may be added to another row.

When Gaussian elimination steps are applied to a linear system (i.e. when elementary row operations are applied to the augmented matrix), the result is an *equivalent* system, that is, one that has exactly the same set of solutions. The objective of Gaussian elimination is to transform a given linear system through a sequence of elementary row operations to obtain an equivalent linear system whose solutions are easy to find.

Solved Problems:

Problem 1:

$$x + y + z = 1$$

$$x - 2y + 3z = 2$$

$$2x - y + 4z = 3$$

Solution: Let $M(i, c)$ denote the operation of multiplying row i by the constant c. Let $A(c, i, k)$ denote the operation of adding c times row i to row k. Let $S(i, k)$ denote the operation of swapping rows i and k. Then, starting with the augmented matrix of the system,

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 1 & -2 & 3 & | & 2 \\ 2 & -1 & 4 & | & 3 \end{pmatrix} \xrightarrow{A(-1,1,2)} \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -3 & 2 & | & 1 \\ 2 & -1 & 4 & | & 3 \end{pmatrix}$$

$$\xrightarrow{A(-2,1,3)} \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -3 & 2 & | & 1 \\ 0 & -3 & 2 & | & 1 \end{pmatrix} \xrightarrow{A(-1,2,3)} \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -3 & 2 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\xrightarrow{M(2,-1/3)} \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & -\frac{2}{3} & | & \frac{1}{3} \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \xrightarrow{A(-1,2,1)} \begin{pmatrix} 1 & 0 & \frac{5}{3} & | & \frac{2}{3} \\ 0 & 1 & -\frac{2}{3} & | & \frac{1}{3} \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

No further simplification is possible. Reintroducing the variables, the equivalent system is

$$x + \frac{5}{3}z = \frac{2}{3}$$
$$y - \frac{2}{3}z = \frac{1}{3}$$

Thus, the original system has infinitely many solutions obtained by assigning an arbitary value to z and computing x and y from the two equations above. Geometrically, the set of solutions is a straight line in 3-space.

Problem 2: Solve

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 1 & -2 & 3 & | & 2 \\ 2 & -1 & 4 & | & -1 \end{pmatrix}$$

Solution:

$$\xrightarrow{A(-1,1,2)} \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -3 & 2 & | & 1 \\ 2 & -1 & 4 & | & -1 \end{pmatrix} \xrightarrow{A(-2,1,3)} \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -3 & 2 & | & 1 \\ 0 & -3 & 2 & | & -3 \end{pmatrix}$$

No further steps are necessary because it is obvious that the last two equations have no solutions. Nevertheless, we shall continue.

$$\xrightarrow{A(-1,2,3)} \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -3 & 2 & | & 1 \\ 0 & 0 & 0 & | & -4 \end{pmatrix} \xrightarrow{M(3,-\frac{1}{4})} \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -3 & 2 & | & 1 \\ 0 & 0 & 0 & | & 1 \end{pmatrix}$$

$$\xrightarrow{M(2,-\frac{1}{3})} \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & -\frac{2}{3} & | & -\frac{1}{3} \\ 0 & 0 & 0 & | & 1 \end{pmatrix} \xrightarrow{A(-1,2,1)} \begin{pmatrix} 1 & 0 & \frac{5}{3} & | & \frac{4}{3} \\ 0 & 1 & -\frac{2}{3} & | & -\frac{1}{3} \\ 0 & 0 & 0 & | & 1 \end{pmatrix}$$

$$\xrightarrow{A(\frac{1}{3},3,2)} \begin{pmatrix} 1 & 0 & \frac{5}{3} & | & \frac{4}{3} \\ 0 & 1 & -\frac{2}{3} & | & 0 \\ 0 & 0 & 0 & | & 1 \end{pmatrix} \xrightarrow{A(-\frac{4}{3},3,1)} \begin{pmatrix} 1 & 0 & \frac{5}{3} & | & 0 \\ 0 & 1 & -\frac{2}{3} & | & 0 \\ 0 & 0 & 0 & | & 1 \end{pmatrix}$$

Again, it is obvious that the system has no solution because the last equation has no solution. The matrix just above is in a special form called *reduced row echelon form* or *Hermite normal form.* We will discuss it further in a later section.

Unsolved Problems:

Find all solutions of the following systems. For systems with two or three variables describe the solution set geometrically if you can.

1.

$$x_1 + 2x_2 - x_3 + 2x_4 = 1$$

$$2x_1 - x_2 + 2x_3 - x_4 = 2$$

$$3x_1 + x_2 + x_3 + x_4 = 3$$

$$-x_1 + 3x_2 - 3x_3 + 3x_4 = -1$$

2.

$$\begin{pmatrix} 1 & 1 & 1 & | & 2 \\ 0 & 1 & 2 & | & 1 \\ 2 & 3 & 4 & | & 5 \\ 1 & 0 & -1 & | & 1 \end{pmatrix}$$

3.

$$x_1 + 2x_2 - x_3 = -1$$

$$2x_1 + 2x_2 + x_3 = 1$$

$$3x_1 + 5x_2 - 2x_3 = -1$$

4.

$$\begin{pmatrix} 1 & -4 & -1 & 1 & | & 3 \\ 2 & -8 & 1 & -4 & | & 9 \\ -1 & 4 & -2 & 5 & | & -6 \end{pmatrix}$$

3. Matrix Notation

The linear system (2.1) is written in matrix form as

$$Ax = b, \tag{3.1}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is the $m \times n$ coefficient matrix,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

is the $n \times 1$ column vector of unknowns, and

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

is the $m \times 1$ column vector of forcing terms. All of these must have entries or components from the same number field or *scalar* field $\mathbb{F}$. Here the scalar field will almost always be either the real numbers $\mathbb{F} = \mathbb{R}$ or the complex numbers $\mathbb{F} = \mathbb{C}$.

The set of all $m \times n$ matrices with entries from the field $\mathbb{F}$ is denoted by $\mathbb{F}^{m\times n}$. The set $\mathbb{F}^{m\times n}$ is a vector space over the field $\mathbb{F}$.

We denote the augmented matrix of the linear system (3.1) by $(A|b)$. This is an example of a *partitioned* matrix, whose parts or submatrices are separated by the vertical bar. It should be noted that if $B \in \mathbb{F}^{k\times m}$, then

$$B(A|b) = (BA|Bb).$$

A similar result holds for matrices partitioned into blocks of columns as $(A_1|A_2|\cdots|A_k)$. This follows from the rules for matrix multiplication.

4. Row Equivalence and Rank

Definition 4.1. *Two matrices A and $B \in \mathbb{F}^{m\times n}$ are* row equivalent *if A can be obtained from B by a sequence of elementary row operations.*

This definition is symmetric with respect to A and B because each elementary row operation has an *inverse* operation of the same type, that is, a row operation of the same type which undoes it. The inverse of $A(c,i,k)$ is $A(-c,i,k)$, the inverse of $M(i,c)$ is $M(i,c^{-1})$, and the inverse of $S(i,k)$ is $S(i,k)$.

Elementary Matrices:

Let I_m denote the $m \times m$ identity matrix. An elementary $m \times m$ matrix is one obtained from I_m by an elementary row operation. For example, applying the operation $A(-2, 3, 1)$ to I_3 yields the elementary matrix

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiplying a given matrix on the left by an elementary matrix has the same effect as applying the corresponding elementary operation to the matrix. Therefore, two matrices are row equivalent if one can be obtained from the other by a sequence of left multiplications by elementary matrices.

Reduced Row Echelon Form:

When we refer to a *zero row* of a matrix we mean a row whose entries are all 0. A matrix B is in *reduced row echelon form* if

(1) All the nonzero rows of B precede all the zero rows of B.

(2) The first nonzero entry in each nonzero row is 1.

(3) The leading 1 in each nonzero row is to the right of the leading 1's in all the preceding rows.

(4) The leading 1 in each nonzero row is the only nonzero entry in its column.

Example 4.1.

$$\begin{pmatrix} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Theorem 4.1. *For each matrix $A \in \mathbb{F}^{m\times n}$ there is a unique matrix $B \in \mathbb{F}^{m\times n}$ such that B in in reduced row echelon form and is row equivalent to A.*

We shall refer to B in the statement of the theorem as the *reduced row echelon form* of A. The reduced row echelon form of a matrix may be found by Gaussian elimination.

Definition 4.2. *The* rank *of a matrix is the number of nonzero rows in its reduced row echelon form.*

The idea of the rank of a matrix is connected to the existence of solutions of a linear system through the following theorem.

Theorem 4.2. *A linear system $Ax = b$ has a solution if and only if the rank of the coefficient matrix A is the same as the rank of the augmented matrix $(A|b)$.*

Unsolved Problems: Find the reduced row echelon form and rank of each of the following matrices.

1.

$$\begin{pmatrix} 0 & 1 & 2 & 1 \\ -2 & 0 & 2 & 0 \\ 3 & 2 & 1 & 2 \end{pmatrix}$$

2.

$$\begin{pmatrix} 2 & -2 & 1 \\ 1 & 0 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

5. Invertible Matrices

A square matrix $A \in \mathbb{F}^{n\times n}$ is *invertible* if there is a matrix $A^{-1} \in \mathbb{F}^{n\times n}$ such that $AA^{-1} = A^{-1}A = I_n$. Elementary matrices are invertible since each elementary operation can be undone by an elementary operation of the same type. It is clear that products of invertible matrices are invertible. Indeed, $(AB)^{-1} = B^{-1}A^{-1}$. From this, it follows that a square matrix is invertible if and only if its reduced row echelon form is invertible. However, the only reduced row echelon matrices that are invertible are the identity matrices. Thus, we have the following theorem.

Theorem 5.1. *Let $A \in \mathbb{F}^{n\times n}$. The following are equivalent.*

(1) *A is invertible.*

(2) *The rank of A is n.*

(3) *The reduced row echelon form of A is I_n.*

(4) *A is a product of elementary matrices.*

The inverse of a matrix $A \in \mathbb{F}^{n\times n}$ may be found by Gaussian elimination. Perform elementary row operations on the $n \times 2n$ partitioned matrix $(A|I_n)$ until A is in reduced row echelon form. If the rank of A is less than n that will be apparent. If the rank of A is n, the result of the calculations will be $(I_n|A^{-1})$.

Other criteria for invertibility are given in the following theorem.

Theorem 5.2. *$A \in \mathbb{F}^{n\times n}$ is invertible if and only if for each $b \in \mathbb{F}^{n\times 1}$ the linear system $Ax = b$ has a unique solution x. $A \in \mathbb{F}^{n\times n}$ is invertible if and only if the homogeneous linear system $Ax = 0$ has only $x = 0$ as a solution.*

Solved Problems:

1. Use Gaussian elimination to find the inverse.

$$\begin{pmatrix} 2 & -2 & 1 \\ 1 & 0 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

Solution: Adjoin a 3×3 identity matrix to the right of the given matrix and then use gaussian elimination steps. Two or more steps may be combined.

$$\begin{pmatrix} 2 & -2 & 1 & | & 1 & 0 & 0 \\ 1 & 0 & 0 & | & 0 & 1 & 0 \\ 2 & -1 & 1 & | & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -2 & 1 & | & 1 & -2 & 0 \\ 1 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & -1 & 1 & | & 0 & -2 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 1 & -1 & | & 0 & 2 & -1 \\ 0 & -2 & 1 & | & 1 & -2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 1 & -1 & | & 0 & 2 & -1 \\ 0 & 0 & 1 & | & -1 & -2 & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 1 & 0 & | & -1 & 0 & 1 \\ 0 & 0 & 1 & | & -1 & -2 & 2 \end{pmatrix}$$

Thus the inverse of the given matrix is

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & -2 & 2 \end{pmatrix}$$

This should be verified by multiplying the two matrices together.

2. Show that if $A, B \in \mathbb{F}^{n\times n}$ and $AB = I_n$, then $BA = I_n$.

Solution: Let $E_1, \cdots, E_k$ be a sequence of elementary matrices such that $E_1 \cdots E_k A = H$ is in reduced row echelon form. Since $AB = I$,

$$HB = E_1 \cdots E_k$$

If the last row of H were zero, the last row of the right hand side above would be zero also. Below you are asked to show that no product of elementary matrices has a zero row. Therefore, $H = I$, $B = E_1 \cdots E_k$, and $BA = I$.

Unsolved Problems: Determine whether the matrices below are invertible or not. If they are, find their inverses.

1.

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 1 & -3 & 1 \end{pmatrix}$$

2.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

3. Show that if $A \in \mathbb{F}^{m \times n}$ has a zero row its rank is less than m. Then show that a product of elementary matrices does not have a zero row. This completes the solution of the second solved problem above.

4. Show that if E is an elementary matrix, then its transpose E^t is an elementary matrix of the same type. If $\mathbb{F} = \mathbb{C}$, then the conjugate transpose E^* is elementary. matrix.

6. Determinants

A permutation of the set $\mathbb{N}_n = \{1, 2, \cdots, n\}$ is a one-to-one function from this set onto itself. A permutation σ has a sign, either +1 or -1 depending on whether the number of pairs $(i, j$ such that $i < j$ and $\sigma(i) > \sigma(j)$ is even or odd. The sign of a permutation σ is denoted by $sgn(\sigma)$.

Definition 6.1. *Let $B = (b_{i,j}) \in \mathbb{F}^{n \times n}$. The* determinant *of B is*

$$det(B) = \sum_{\sigma} sgn(\sigma) \prod_{i=1}^{n} b_{i,\sigma(i)}.$$

The sum is taken over all permuations σ of $\mathbb{N}_n$.

Determinants are seldom evaluated from the definition. However, several useful results are easily proved. In the following theorem, we use the same notation for an elementary matrix as for the elementary row operation it corresponds to. Also, a matrix $B = (b_{i,j})$ is upper triangular if $b_{i,j} = 0$ for $i > j$, lower triangular if $b_{i,j} = 0$ for $i < j$, and triangular if it is either upper triangular or lower triangular.

Theorem 6.1. *Properties of determinants:*

(1) *If B is triangular, $det(B) = \prod_i b_{i,i}$.*

(2) *$det(B^t) = det(B)$.*

(3) *$det(A(c,i,k)) = 1$ for $i \neq k$.*

(4) *$det(M(i,c)) = c$.*

(5) *$det(S(i,k)) = -1$ for $i \neq k$.*

(6) *The determinant is multiplicative, i.e., if $B, C \in \mathbb{F}^{n\times n}$, $det(BC) = det(B)det(C)$.*

For a square matrix B we may write $B = E_1 \cdots E_k H$, where each E_i is elementary and H is the reduced row echelon form of B. $det(B) \neq 0$ if and only if $det(H) \neq 0$. Since H is upper triangular, $det(H) \neq 0$ if and only if for each i, $h_{i,i} \neq 0$. This occurs if and only if $H = I$, i.e., if and only if B is invertible.

Theorem 6.2. *Let $B \in \mathbb{F}^{n\times n}$. B is invertible if and only if $det(B) \neq 0$.*

Geometric Interpretation of Determinants:

Let $B \in \mathbb{R}^{2\times 2}$ and partition B by columns as $B = (b_{\bullet,1}|b_{\bullet,2})$. By comparing $det(B)$ to the cross product of the plane vectors $b_{\bullet,1}$ and $b_{\bullet,2}$, it can be seen that the absolute value of $det(B)$ is the area enclosed by the parallogram with adjacent sides $b_{\bullet,1}$ and $b_{\bullet,2}$. Similarly, if B is a real 3×3 matrix, the absolute value of $det(B)$ is the volume of the parallepiped in 3-space whose adjacent sides are the column vectors of B. Generalization to spaces of higher dimension is straightforward and important in multivariable calculus.

Solved Problems:

1. Find the determinant of

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

Solution: Row reduce the given matrix to upper triangular form.

$$\xrightarrow{A(-1,1,2)} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \xrightarrow{A(-3,1,3)} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -2 & -2 \end{pmatrix} \xrightarrow{A(2,2,3)} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The last matrix is triangular, so its determinant is the product of its diagonal entries, -2. This is equal to the product of the determinant of the given matrix and the determinants of the elementary matrices used in the reduction. Each of the elementary matrices was of type "A", which have determinant 1. Therefore, -2 is the answer.

2.

$$\begin{pmatrix} i & 2+i & 0 \\ -1 & 3 & 2i \\ 0 & -1 & 1-i \end{pmatrix}$$

Solution: Instead of writing each row operation in gaussian elimination steps, we will simply keep track of their determinants. Operations of type "A" have determinant 1 and thus do not have to be indicated.

$$\xrightarrow{-i} \begin{pmatrix} 1 & 1-2i & 0 \\ -1 & 3 & 2i \\ 0 & -1 & 1-i \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1-2i & 0 \\ 0 & 4-2i & 2i \\ 0 & -1 & 1-i \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1-2i & 0 \\ 0 & 0 & 2-4i \\ 0 & -1 & 1-i \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 & 1-2i & 0 \\ 0 & -1 & 1-i \\ 0 & 0 & 2-4i \end{pmatrix}$$

Thus, the determinant of the given matrix is $(-2+4i)/((-1)(-i)) = 4+2i$.

Unsolved Problems:

1. Find the determinant of

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & -5 \\ 6 & -4 & 3 \end{pmatrix}$$

2. Find the determinant of

$$\begin{pmatrix} -1 & 2+i & 3 \\ 1-i & i & 1 \\ 3i & 2 & -1+i \end{pmatrix}$$

3. Let $B \in \mathbb{F}^{n\times n}$ and let $c \in \mathbb{F}$. Show that $det(cB) = c^n det(B)$.