Linear Algebra and Matrix Theory

Part 3 - Linear Transformations

1. References

(1) S. Friedberg, A. Insel and L. Spence, *Linear Algebra*, Prentice-Hall.

(2) M. Golubitsky and M. Dellnitz, *Linear Algebra and Differential Equations Using Matlab*, Brooks-Cole.

(3) K. Hoffman and R. Kunze, *Linear Algebra*, Prentice-Hall.

(4) P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, Academic Press.

(5) G. Strang, *Linear Algebra and Its Applications*, Academic Press.

2. Solved Problems

1. Let $\mathcal{V}_n$ be the real vector space of all polynomials of degree $\leq n$ with real coefficients. For each space $\mathcal{V}_n$, let $\mathcal{B}_n = \{v_1, \cdots, v_{n+1}\}$ be the usual basis of polynomial functions $v_i(t) = t^{i-1}$. Let $D : \mathcal{V}_3 \longrightarrow \mathcal{V}_3$ be the differentiation operator. Find the matrix of D with respect to the basis $\mathcal{B}_3$.

Solution: We have $Dv_1 = 0$ and $Dv_i = (i-1)v_{i-1}$ for $i \geq 2$. Let M be the matrix of D and let $[u]$ denote the coordinate vector of u with respect to the given basis $\mathcal{B}_3$. The rule relating D to M is $[Du] = M[u]$. Hence, $[Dv_i] = M[v_i]$ for $i = 1, \cdots, 4$. If $\{e_1, \cdots, e_4\}$ is the standard basis for $\mathbb{R}^{4\times 1}$, $[v_i] = e_i$. From $Dv_1 = 0$ we get $Me_1 = 0$, from $Dv_2 = v_1$ we get $Me_2 = e_1$. Similarly, $Me_3 = 2e_2$ and $Me_4 = 3e_3$. Partitioned by columns, the matrix

M is $(Me_1|Me_2|Me_3|Me_4)$. Thus,

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. Let $\mathcal{V}$ be the same space as in problem no. 1. Instead of the usual basis $\mathcal{B}_3 = \{v_i\}_{i=1}^4$ consider the basis $\hat{\mathcal{B}}_3 = \{\hat{v}_i\}_{i=1}^4$, where $\hat{v}_i(t) = v_i(t) - 1$. Find the transition matrix from $\mathcal{B}_3$ to $\hat{\mathcal{B}}_3$ and use it to find the matrix $\hat{M}$ of D with respect to $\hat{\mathcal{B}}_3$.

Solution: The same technique used in problem no. 1 may also be used here, with the result that

$$\hat{M} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

You should carry out the details of this derivation. The transition matrix $Q = (q_{i,j})$ is defined by the equations

$$\hat{v}_j = \sum_{i=1}^{4} q_{i,j} v_i.$$

One can see immediately that $\hat{v}_1 = v_1$ and $\hat{v}_j = v_j - v_1$ for $i \geq 2$. From this, the transition matrix is

$$Q = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Its inverse is

$$Q^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The relationship between the two matrix representations of D is $\hat{M} = Q^{-1}MQ$. You should carry out the matrix multiplication to verify the answer given previously.

3. Show that the dimension of the column space of a matrix is its rank.

Solution: Recall that the rank of a matrix was defined as the number of nonzero rows in its reduced row echelon form. If $v_1, \cdots, v_k$ are linearly independent vectors in $\mathbb{F}^{m\times 1}$ and $Q \in \mathbb{F}^{m\times m}$ is nonsingular, then $Qv_1, \cdots, Qv_k$ are linearly independent. This follows almost immediately from the definition of linear independence. Thus, although row equivalent matrices do not necessarily have the same column space, the dimensions of their column spaces are the same. Therefore, the dimension of the column space of a matrix M is the same as that of its reduced row echelon form. In the reduced row echelon form, the columns with the leading 1's are linearly independent and each column without a leading 1 is a linear combination of the preceding columns. Hence, the dimension of the column space is the number of leading 1's, i.e., the number of nonzero rows. This shows that the dimension of the row space of a matrix and the dimension of its column space are the same. It also shows why we use the term "rank" for the dimension of the range space of a linear transformation.

4. Verify the theorem on rank and nullity for the linear transformation D in problem no. 1.

Solution: Let $\mathcal{V}$ be a vector space of dimension n over a field $\mathbb{F}$ and let $\mathcal{W}$ be another vector space over $\mathbb{F}$. Let $T : \mathcal{V} \longrightarrow \mathcal{W}$ be a linear transformation. The nullity of T is defined as the dimension of its null space (or kernel)

$$ker(T) = \{v \in \mathcal{V} | Tv = 0\}$$

and its rank is defined as the dimension of its range space

$$T(\mathcal{V}) = \{w \in \mathcal{W} | w = Tv \text{ for some } \ v \in \mathcal{V}\}.$$

The theorem on rank and nullity asserts that the nullity of T plus the rank of T is equal to $dim(\mathcal{V})$. In problem 1, $dim(\mathcal{V}) = 4$. The range of D is the subspace of all polynomials of degree ≤ 2, so it has dimension 3. In other words, the rank of D is 3. The null space of D is the subspace of all polynomials v such that $Dv = 0$, i.e., all constant polynomials. This subspace is spanned by the first basis vector $v_1(t) \equiv 1$, so the nullity of D is 1.

3. Unsolved Problems

1. Define $T : \mathbb{R}^{3\times 1} \longrightarrow \mathbb{R}^{3\times 1}$ by

$$T(x_1, x_2, x_3)^t = (x_1, x_1 + x_2, x_1 + x_2 + x_3)^t.$$

Show that T is a linear transformation. Find $[T]$, the matrix of T relative to the standard basis $\mathcal{B}$ of $\mathbb{R}^{3\times 1}$. Find the matrix $[T]'$ of T relative to the basis

$$\mathcal{B}' = \{(1,0,0)^t, (1,1,0)^t, (1,1,1)^t\}$$

2. In problem 1, find the transition matrix Q from $\mathcal{B}$ to $\mathcal{B}'$. Verify that $[T]' = Q^{-1}[T]Q$.

3. Let $\mathcal{V}_n$ be the vector space of real polynomials of degree $\leq n$ and let $\mathcal{B}_n$ be the usual basis of $v_i(t) = t^{i-1}$. Define $T : \mathcal{V}_2 \longrightarrow \mathcal{V}_3$ by

$$Tv(x) = \int_0^x v(t)dt$$

Let $D : \mathcal{V}_3 \longrightarrow \mathcal{V}_2$ be the differentiation operator. (Note that this is slightly different from the definition of D in Solved Problem no. 1). Find the matrices $[T]$ and $[D]$ with respect to the given bases. Find TD and DT and their matrices. Verify that $[DT] = [D][T]$ and $[TD] = [T][D]$.

4. Verify the theorem on rank and nullity for the linear transformation T in problem 3.