Linear Algebra and Matrix Theory

Part 4 - Eigenvalues and Eigenvectors

1. References

(1) S. Friedberg, A. Insel and L. Spence, *Linear Algebra*, Prentice-Hall.

(2) M. Golubitsky and M. Dellnitz, *Linear Algebra and Differential Equations Using Matlab*, Brooks-Cole.

(3) K. Hoffman and R. Kunze, *Linear Algebra*, Prentice-Hall.

(4) P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, Academic Press.

(5) G. Strang, *Linear Algebra and Its Applications*, Academic Press.

2. Solved Problems

1. Find the eigenvalues and the corresponding eigenspaces of the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Solution: A scalar λ is an eigenvalue if and only if the kernel of $\lambda I - A$ is a nontrivial subspace of $\mathbb{F}^{3\times 1}$. That subspace is the corresponding eigenspace. Recall that row equivalent matrices have the same null space. Use symbolic row-reduction on the matrix $\lambda I - A$ to obtain a row equivalent matrix whose kernel is easy to find. Several steps may be combined.

$$\lambda I - A = \begin{pmatrix} \lambda-1 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-1 \\ \lambda-1 & 1 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-1 \\ 0 & -\lambda^2+3\lambda-1 & -(\lambda-1) \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-1 \\ 0 & 0 & \lambda(\lambda-1)(\lambda-3) \end{pmatrix}$$

The last matrix has a nontrivial null space if and only if its determinant is zero. Obviously, that occurs for $\lambda = 0, 1, 3$. These are the eigenvalues. Let us first find the eigenspace for $\lambda = 0$. From above, substituting $\lambda = 0$, we need the solutions of

$$\begin{pmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

All solutions are of the form $x_1 = x_2 = x_3$, so the first eigenspace is

$$S(0) = sp\{(1,1,1)^t\}.$$

For $\lambda = 1$ the eigenspace $S(1)$ is the set of all solutions of

$$\begin{pmatrix} 1 & -1 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

Now the solutions are of the form $x_2 = 0$, $x_3 = -x_1$. Thus,

$$S(1) = sp\{(1, 0, -1)^t\}$$

The eigenspace for $\lambda = 3$ is

$$S(3) = sp\{(1, -2, 1)^t\}$$

The details are left to you.

2. Tell why it is possible to diagonalize the matrix A of problem 1. Then find a similarity transformation that diagonalizes it.

Solution: The answer to the question depends on two theorems. One theorem asserts that eigenvectors corresponding to distinct eigenvalues are linearly independent. Since we have three distinct eigenvalues: 0, 1, and 3, any three vectors (other than the zero vector) belonging to their respective eigenspaces are linearly independent. In particular, $v_1 = (1, 1, 1)^t$, $v_2 = (1, 0, -1)^t$, and $v_3 = (1, -2, 1)^t$ are linearly independent. Therefore, they form a basis for $\mathbb{R}^{3\times 1}$.

The second theorem alluded to above asserts that a matrix (or linear transformation on a finite dimensional vector space) is diagonalizable if and only if there is a basis for the space consisting of eigenvectors of the matrix or transformation. Clearly, $\{v_1, v_2, v_3\}$ above is a basis for $\mathbb{R}^{3\times 1}$, so A is diagonalizable. This means that A is similar to a diagonal matrix, or that there is a nonsingular matrix Q such that $Q^{-1}AQ = \Lambda$, a diagonal matrix. The diagonal entries of Λ must be the eigenvalues and the correspondingly numbered columns of Q must be associated eigenvectors. Therefore, we may take

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

The inverse of Q is

$$Q^{-1} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & -1/2 \\ 1/6 & -1/3 & 1/6 \end{pmatrix}$$

You can verify that $Q\Lambda Q^{-1} = A$.

3. Find the characteristic polynomial and the minimum polynomial of the matrix

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution: The characteristic polynomial of A is the polynomial

$$p_A(\lambda) = det(\lambda I - A).$$

It is a monic polynomial whose degree is the same as the dimension of A. In this case,

$$p_A(\lambda) = det \begin{pmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda-1 & -1 \\ 0 & 0 & \lambda-1 \end{pmatrix}.$$

Since this is triangular it is easy to see that

$$p_A(\lambda) = (\lambda+1)(\lambda-1)^2.$$

The Hamilton-Cayley theorem asserts that the characteristic polynomial of a matrix annihilates the matrix, i.e., that $p_A(A) = 0$. In the present case,

$$p_A(A) = (A+I)(A-I)^2$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2$$

You can verify that this is the zero matrix.

The minimum polynomial of a matrix is defined to be the monic polynomial of smallest degree that annihilates the matrix. So, if m_A is the minimum polynomial, $m_A(A) = 0$ and no polynomial of smaller degree satisfies this matrix equation. The minimum polynomial must divide the characteristic polynomial, and each irreducible factor of the characteristic polynomial must be a factor of the minimum polynomial, possibly to a smaller power.

With these considerations, it is clear that the only possibilities for the minimum polynomial of the matrix A above are $m_A(\lambda) = (\lambda + 1)(\lambda - 1)$ or $m_A(A) = (\lambda+1)(\lambda-1)^2 = p_A(\lambda)$. The first is not the minimum polynomial because

$$(A + I)(A - I) = A^2 - I \neq 0$$

Therefore, $m_A = p_A$.

4. Show that A in problem 3 is not diagonalizable.

Solution: A matrix is diagonalizable if and only if its minimum polynomial factors into distinct first degree factors. Since this is not the case for the matrix A above, it is not diagonalizable.

A more direct approach is to show that there is no basis of eigenvectors of A. The eigenvalues are 1 and -1. The eigenspaces $S(-1)$ and $S(1)$ are both one-dimensional. In fact, $S(1) = sp\{(0, 1, 0)^t\}$, even though 1 is a repeated zero of the characteristic polynomial. Therefore, there cannot be more than two linearly independent eigenvectors.

5. Let $\mathcal{V} = \mathcal{C}^\infty(\mathbb{R})$ be the real vector space of all infinitely differentiable real-valued functions on $\mathbb{R}$. Let D be the differentiation operator. Show that every real number is an eigenvalue of D and find the associate eigenspace.

Solution: $\lambda \in \mathbb{R}$ is an eigenvalue if and only if there is a nonzero vector $v \in \mathcal{V}$ such that $Dv = \lambda v$. There is such a vector (function), namely, $v(t) = e^{\lambda t}$. The eigenspace is the set of all real multiples of this vector, i.e., the set of all functions of the form $v(t) = ce^{\lambda t}$, where c is an arbitrary real number.

6. Let $A \in \mathbb{F}^{n \times n}$ and let q be a polynomial with coefficients in $\mathbb{F}$. Show that if $\lambda \in \mathbb{F}$ is an eigenvalue of A, then $q(\lambda)$ is an eigenvalue of $q(A)$.

Solution: There is a nonzero vector v such that $Av = \lambda v$. $A^2 v = A(Av) = A(\lambda v) = \lambda A v = \lambda^2 v$. Arguing in the same manner, $A^3 v = \lambda^3 v$, and by induction $A^k v = \lambda^k v$ for all $k \in \mathbb{N}$. The rest of the proof is left to you.

3. Unsolved Problems

Unless otherwise stated, the scalar field is $\mathbb{C}$, even if the entries of the matrix are real.

1. Find the eigenvalues and associated eigenspaces of the matrix

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

2. Find a diagonal matrix Λ and a nonsingular matrix Q such that $Q^{-1}AQ = \Lambda$, where

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

3. Find the minimum polynomial of the matrix in problem 2.

4. A matrix A is *nilpotent* if there is a positive integer k such that $A^k = 0$. Show that the only eigenvalue of a nilpotent matrix is 0. Find an example of a 2×2 nilpotent matrix that is not the zero matrix.

5. Show that the eigenvalues of a triangular matrix are its diagonal entries.

6. Show that A^t and A have the same eigenvalues. Show that the eigenvalues of A^* are conjugates of the eigenvalues of A.