Linear Algebra and Matrix Theory

Part 5 - Inner Products, Normal Matrices, Projections, etc.

1. References

(1) S. Friedberg, A. Insel and L. Spence, *Linear Algebra*, Prentice-Hall.

(2) M. Golubitsky and M. Dellnitz, *Linear Algebra and Differential Equations Using Matlab*, Brooks-Cole.

(3) K. Hoffman and R. Kunze, *Linear Algebra*, Prentice-Hall.

(4) P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, Academic Press.

(5) G. Strang, *Linear Algebra and Its Applications*, Academic Press.

2. Solved Problems

1. Show that the eigenvalues of a complex hermitian matrix are real.

Solution: Recall that A^* denotes the conjugate transpose of a matrix. Also recall that $(AB)^* = B^*A^*$ whenever AB is defined. Let $x \in \mathbb{C}^{n\times 1}$, $x \neq 0$, be an eigenvector corresponding to an eigenvalue λ of $A \in \mathbb{C}^{n\times n}$. Then

$$Ax = \lambda x$$

$$x^*Ax = \lambda x^*x$$

$$\overline{x^*Ax} = \overline{\lambda}x^*x \text{ (because } x^*x > 0.)$$

$$x^*A^*x = \overline{\lambda}x^*x$$

$$x^*Ax = \overline{\lambda}x^*x \text{ (because } A^* = A.)$$

Hence, $\overline{\lambda} = \lambda$ and λ is real.

2. Let $v_1 = (1,1,0)^t$, $v_2 = (0,1,1)^t$ and $v_3 = (1,0,1)^t$. With respect to the standard inner product of $\mathbb{R}^{3\times 1}$, find orthonormal vectors u_1, u_2, and u_3 such that $u_i \in sp\{v_1, \cdots, v_i\}$ for $i = 1,2,3$.

Solution: Given a linearly independent set of vectors $\{v_1, \cdots, v_m\}$ in an inner product space, the Gram-Schmidt procedure produces a sequence $\{u_1, \cdots, u_m\}$ of mutually orthogonal unit vectors such that $u_i \in sp\{v_1, \cdots, v_i\}$ for $i = i, \cdots, m$. The vectors u_i are defined inductively by

$$u_1 = v_1/||v_1||$$

and for $k > 1$ by

$$\tilde{v}_k = v_k - \sum_{i=1}^{k-1} (v_k|u_i)u_i$$

$$u_k = \tilde{v}_k/||\tilde{v}_k||$$

The expression $(v|u)$ denotes the inner product of the vectors v and u. $||v||$ is the norm of the vector v, defined as $v = \sqrt{(v|v)}$. In the present case, $(v|u) = u^*v$ and $||v_1|| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$. Hence,

$$u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Omitting the detailed calculations, the remaining steps are:

$$(v_2|u_1) = \frac{1}{\sqrt{2}}$$

$$\tilde{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$u_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$(v_3|u_1) = \frac{1}{\sqrt{2}}$$

$$(v_3|u_2) = \frac{1}{\sqrt{6}}$$

$$\tilde{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$u_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

3. Show that the transition matrix from one orthonormal basis of a finite-dimensional inner product space to another is unitary.

Solution: When we say that a matrix $Q \in \mathbb{C}^{n\times n}$ is unitary we mean that its columns are orthonormal with respect to the standard inner product on

$\mathbb{C}^{n\times 1}$. Another way of expressing this is $Q^*Q = QQ^* = I$. Thus, the inverse of Q is Q^*. Real unitary matrices are commonly called orthogonal matrices. We will use the "dot product" notation to denote the standard inner product

$$x \cdot y = y^* x = \sum_{i=1}^{n} x_i \bar{y}_i.$$

Let $\{u_1, \cdots, u_n\}$ and $\{v_1, \cdots, v_n\}$ be orthonomal bases. This means that $(v_i|v_j) = 0$ if $i \neq j$ and $= 1$ if $i = j$. The u's satisfy the same equations. If $Q = (q_{i,j})$ is the transition matrix from the u-basis to the v-basis,

$$v_j = \sum_{i=1}^{n} q_{i,j} u_i.$$

Thus,

$$\begin{aligned}
(v_k|v_j) &= (\sum_{l=1}^{n} q_{l,k} u_l | \sum_{i=1}^{n} q_{i,j} u_i) \\
&= \sum_{l=1}^{n} \sum_{i=1}^{n} q_{l,k} \bar{q}_{i,j} (u_l|u_i) \\
&= \sum_{l=1}^{n} q_{l,k} \bar{q}_{l,j} \\
&= q_{\cdot,k} \cdot q_{\cdot,j}
\end{aligned}$$

Since the v's are orthonormal, so are the columns of Q.

4. Find an orthonormal basis for $\mathbb{R}^{3\times 1}$ consisting of eigenvectors of

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Solution: Given a hermitian matrix $A \in \mathbb{C}^{n\times n}$, there is an orthonormal basis for $\mathbb{C}^{n\times 1}$ consisting of unit eigenvectors of A. If A is real, it has real eigenvectors (recall that the eigenvalues of a hermitian matrix are always real). Furthermore, eigenvectors belonging to different eigenvalues of a hermitian matrix are always orthogonal, so if the given matrix has distinct eigenvalues, we need only select unit eigenvectors for them.

Using the techniques of a previous section,

$$det(\lambda I - A) = det \begin{pmatrix} \lambda - 2 & 0 & -1 \\ 0 & \lambda - 2 & 1 \\ -1 & 1 & \lambda - 1 \end{pmatrix}$$

$$= \lambda(\lambda - 2)(\lambda - 3).$$

For $\lambda_1 = 0$ an eigenvector is $v_1 = (-1, 1, 2)^t$, so a unit eigenvector is

$$u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

Similarly, unit eigenvectors for $\lambda_2 = 2$ and $\lambda_3 = 3$ are

$$u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

and

$$u_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

So, $\{u_1, u_2, u_3\}$ is an orthonormal basis for $\mathbb{R}^{3\times 1}$.

5. For the matrix A in problem 4, find a real orthogonal matrix U and a diagonal matrix Λ such that $U^*AU = \Lambda$.

Solution: This illustrates the general theorem that a hermitian matrix is *unitarily similar* to a diagonal matrix. The equation just above may be rewritten $AU = U\Lambda$, or $Au_i = \lambda_i u_i$, where λ_i is the i^{th} diagonal entry of Λ and u_i is the i^{th} column of U. Therefore, we take the columns of U to be orthonormal eigenvectors of A and the diagonal entries of Λ to be the corresponding eigenvalues. Since A here is real symmetric, all these objects will be real.

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and

$$U = \begin{pmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

6. On $\mathbb{C}^{n\times 1} \times \mathbb{C}^{n\times 1}$ let $f(x, y) = y^*Hx$, where $H \in \mathbb{C}^{n\times n}$. Under what conditions will f be an inner product?

Solution: Three conditions must be satisfied:

(1) $f(\alpha x + \beta z, y) = \alpha f(x, y) + \beta f(z, y)$ for all $\alpha, \beta \in \mathbb{C}$ and all $x, y, z \in \mathbb{C}^{n\times 1}$.

(2) $f(y, x) = \overline{f(x, y)}$ for all $x, y \in \mathbb{C}^{n\times 1}$.

(3) $f(x, x) > 0$ for all $x \neq 0$ in $\mathbb{C}^{n\times 1}$.

The first of these holds for any H. For the second,

$$\overline{f(x, y)} = \overline{y^* H x} = x^* H^* y,$$

while

$$f(y, x) = x^* H y.$$

These are equal for all x, y if and only if $H^* = H$, that is, if and only if H is hermitian. The third condition holds if and only if $x^* H x > 0$ for all $x \in \mathbb{C}^{n\times 1}$. If this is so, we say that H is *positive definite.* A necessary and sufficient condition for a hermitian matrix to be positive definite is that all its eigenvalues be positive.

3. Unsolved Problems

1. Find an orthonormal basis for $\mathbb{R}^{2\times 1}$ of eigenvectors of the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

2. Find the transition matrix from the standard basis to this basis and show that it is unitary.

3. Show that the eigenvalues of a unitary matrix all have absolute value (or complex modulus) 1.

4. Find all $n \times n$ matrices which are both hermitian positive definite and unitary.

5. Apply the Gram-Schmidt procedure to convert $\{v_1, v_2, v_3\}$ to an orthonormal basis for $\mathbb{R}^{3\times 1}$, where $v_1 = (1, 0, 0)^t$, $v_2 = (1, 1, 0)^t$, and $v_3 = (1, 1, 1)^t$.