Probability

Part 2 - Basic Rules of Probability

1. References

(1) R. Durrett, *The Essentials of Probability*, Duxbury.
(2) L.L. Helms, *Probability Theory with Contemporary Applications*, Freeman.
(3) J.J. Higgins and S. Keller-McNulty, *Concepts in Probability and Stochastic Modeling*, Duxbury.
(4) R.V. Hogg and E.A. Tanis, *Probability and Statistical Inference*, Prentice-Hall.

2. Solved Problems

Problem 1. Part of the specification of a probability model of a random experiment is to describe its sample space and the collection of events. What are the axioms satisfied by the collection of events?

Solution:

The sample space S is the set of all possible outcomes of the experiment. An event is a subset of the sample space. If S is countable, then it is almost always the case that every subset of S is taken to be an event. If S is uncountable, for example an interval of real numbers, then not every subset is called an event. The collection $\mathcal{A}$ of all events must satisfy the following axioms.

(1) $S \in \mathcal{A}$.

(2) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

(3) If $A_1, A_2, \cdots$ is a finite or infinite sequence of events, then $\cup_i A_i \in \mathcal{A}$.

Problem 2: Show that if $\mathcal{A}$ is the collection of events in a probability model and if $A_1, A_2, \cdots$ is a finite or infinite sequence of events, then $\cap_i A_i \in \mathcal{A}$.

Solution:

DeMorgan's law says that $(\cap_i A_i)^c = \cup_i A_i^c$. Since each $A_i \in \mathcal{A}$, each $A_i^c \in \mathcal{A}$ by (2) above. Therefore, by (3), $\cup_i A_i^c \in \mathcal{A}$. Hence, $(\cap_i A_i)^c \in \mathcal{A}$ and by (2) again, $\cap_i A_i \in \mathcal{A}$.

We can summarize the properties of the collection of events by saying that it contains the entire sample space and it is closed under finite or countably infinite unions, finite or countably infinite intersections, and complementation.

Problem 3: Suppose that the sample space of a random experiment is the set $\mathbb{R}$ of all real numbers. Suppose also that the collection $\mathcal{A}$ of all events contains all the closed half lines of the form $(-\infty, b]$, where $b \in \mathbb{R}$. Show that all sets of the following types are also events:

(1) all open half lines of the form (a, ∞), where $a \in \mathbb{R}$,

(2) all half open intervals of the form $(a, b]$, where $a, b \in \mathbb{R}$,

(3) all singleton sets $\{a\}$, where $a \in \mathbb{R}$,

(4) all bounded open intervals (a, b) and all bounded closed intervals $[a, b]$.

(5) all open sets and all closed sets.

Solution:

The open half line (a, ∞) is the complement of the closed half line $(-\infty, a]$. Therefore, all the half lines in (1) are events. The bounded half open interval $(a, b]$ is the intersection of $(-\infty, b]$ and (a, ∞). Thus, it is an event. A singleton set $\{a\} = \cap_n (a - 1/n, a]$ is a countable intersection of events, so it is an event. Bounded open intervals (a, b) and bounded closed intervals $[a, b]$ may be obtained by adjoining or deleting singletons to or from $(a, b]$, so they are events. Finally, every open set is a countable union of intervals, so all open sets are events. Closed sets are the complements of open sets, so they are events.

Events formed by applying a sequence of set operations to intervals are called Borel sets. They constitute the most natural collection of events for describing a great many random experiments.

Problem 4: In addition to a sample space S and a collection $\mathcal{A}$ of events, a probability model for a random experiment must have a probability measure that assigns a number between 0 and 1 to each event. Thus, the probability measure is a function $P : \mathcal{A} \longrightarrow [0, 1]$. What are the essential properties of a probability measure?

Solution:

A probability P must have the following properties:

(1) $P(S) = 1$,

(2) If $A_1, A_2, \cdots$ is a sequence of pairwise disjoint events, then $P(\cup_i A_i) = \sum_i P(A_i)$.

Problem 5: Suppose S is countable and $\mathcal{A}$ is the collection of all subsets of S. Show that for each $A \in \mathcal{A}$,

$$P(A) = \sum_{a \in A} P(\{a\}).$$

Solution:

$A = \cup_{a \in A}\{a\}$ is a countable union of pairwise disjoint events. The result follows from property (2) of a probability measure.

Problem 6: Prove the following properties of probability measures:

(1) If $A \subseteq B$, then $P(A) \leq P(B)$.

(2) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

(3) $P(A^c) = 1 - P(A)$.

Solution:

(1) $B = A \cup (B \sim A)$ is a disjoint union. Therefore, $P(B) = P(A) + P(B \sim A) \geq P(A)$. (2) As is customary, write AB for $A \cap B$. $A \cup B = A \cup (B \sim AB)$. The right hand side is a disjoint union. Therefore, $P(A \cup B) = P(A) + P(B \sim AB) = P(A) + P(B) - P(AB)$. (3) $S = A \cup A^c$. Therefore, $P(S) = 1 = P(A) + P(A^c)$.

Problem 7: Toss a coin 3 times and assume that all outcomes are equally likely. Let A_i denote the event of a head on the i^{th} toss. Show that the events A_1, A_2, A_3 are independent.

Solution:

For convenience, let us indicate a head by the numeral 1 and a tail by the numeral 0. The sample space of the experiment is the set of all sequences of 0's and 1's of length 3. Hence, S has 2^3 elements and if outcomes are equally likely, $P(\{a\}) = 1/8$ for each outcome a, e.g., $a = (1, 0, 1)$. A verbal description of an event, such as we gave for A_i, must be capable of translation into a precise definition of a subset of S, although often it is not necessary to do so. Here, for example,

$$A_1 = \{(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$$

$$A_2 = \{(0,1,0),(0,1,1),(1,1,0),(1,1,1)\}$$

$$A_3 = \{(0,0,1),(0,1,1),(1,0,1),(1,1,1)\}$$

$$A_1A_2 = \{(1,1,0),(1,1,1)\}$$

$$A_1A_3 = \{(1,0,1),(1,1,1)\}$$

$$A_2A_3 = \{(0,1,1),(1,1,1)\}$$

$$A_1A_2A_3 = \{(1,1,1)\}$$

Now it is easy to see that $P(A_iA_j) = 1/4 = P(A_i)P(A_j)$ for $i \neq j$ and $P(A_1A_2A_3) = 1/8 = P(A_1)P(A_2)P(A_3)$. This shows that A_1, A_2, A_3 are independent events. In general, events in a collection $\{A_i\}$ are independent if for each finite subcollection $\{A_i | i \in F\}$,

$$P(\bigcap_{i\in F} A_i) = \prod_{i\in F} P(A_i).$$

Problem 8: Two cards are drawn in succession, without replacement, from a standard 52-card deck. All outcomes are equally likely. Let A_i denote the event that the i^{th} card is red, i.e., a heart or a diamond. Show that A_1 and A_2 are dependent events.

Solution:

We must show that $P(A_1A_2) \neq P(A_1)P(A_2)$. The number of 2 term nonrepeating sequences that begin with a red card is 26×51. The number of outcomes of the experiment is 52×51. Therefore, $P(A_1) = (26 \times 51)/(52 \times 51) = 1/2$. $P(A_1A_2) = (26 \times 25)/(52 \times 51) = 25/102$. To find $P(A_2)$, write $A_2 = (A_2A_1) \cup (A_2A_1^c)$. Then $P(A_2) = P(A_2A_1) + P(A_2A_1^c) = (26 \times 25)/(52 \times 51) + (26 \times 26)/(52 \times 51) = 1/2$. Thus, $P(A_1A_2) \neq P(A_1)P(A_2)$ and A_1 and A_2 are dependent.

Problem 9: For the situation described in Problem 8, find the conditional probabilities $P(A_2|A_1)$ and $P(A_1|A_2)$.

Solution:

The conditional probability of event A_2, given event A_1, is defined as

$$P(A_2|A_1) = \frac{P(A_2A_1)}{P(A_1)}$$

Thus, $P(A_2|A_1) = (25/102)/(1/2) = 25/51$. Likewise,$P(A_1|A_2) = 25/51$. This is another way of seeing that A_1 and A_2 are dependent. Two events C and D with nonzero probabilities are independent if and only if $P(C|D) = P(C)$.

Problem 10: Four cards are drawn in succession without replacement from a standard deck. At each draw, all the remaining cards in the deck are equally likely. What is the probability of the color sequence red-red-black-red?

Solution:

Let R_i denote the event of a red card on the i^{th} draw and let $B_i = R_i^c$ denote the event of a black card. We want to find $P(R_1R_2B_3R_4)$. It can be shown that if $A_1, A_2, \cdots, A_k$ are events in a probability model,

$$P(A_1 \cdots A_k) = P(A_k|A_1 \cdots A_{k-1})P(A_{k-1}|A_1 \cdots A_{k-2}) \cdots P(A_2|A_1)P(A_1).$$

Thus,

$$\begin{aligned} P(R_1R_2B_3R_4) &= P(R_4|R_1R_2B_3)P(B_3|R_1R_2)P(R_2|R_1)P(R_1) \\ &= \frac{24}{49}\frac{26}{50}\frac{25}{51}\frac{26}{52} = .0624. \end{aligned}$$

3. Unsolved Problems

Problem 1: Show that $P(\phi) = 0$.

Problem 2: Find and prove an expression for $P(A_1 \cup A_2 \cup A_3)$.

Problem 3: Show that a finite or countably infinite set of real numbers is a Borel set.

Problem 4: Draw two cards in succession without replacement from a standard deck. Let $A =$ "Red on first draw" and $B =$ "King on the second draw". Are A and B independent or dependent events?

Problem 5: Show that if A is any event and B is an event such that $P(B) = 1$, then A and B are independent.

Problem 6: By induction, show that

$$P(A_1 \cdots A_k) = P(A_k|A_1 \cdots A_{k-1}) \cdots P(A_2|A_1)P(A_1).$$

Problem 7: Modify Solved Problem no. 10 so that the cards are drawn with replacement.