a counting problem, we may have to use one or a combination of these principles. The counting principles we have studied are:

- **Inclusion-exclusion principle**: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
- **Complement Rule** $n(A') = n(U) - n(A)$.
- **Multiplication principle**: If I can break a task into r steps, with m_1 ways of performing step 1, m_2 ways of performing step 2 (no matter what I do in step 1), ..., m_r ways of performing step r (no matter what I do in the previous steps), then the number of ways I can complete the task is

$$m_1 \cdot m_2 \cdots m_r.$$

(This also applies if step i of task amounts to selecting from set A_i with m_i elements.)
- **Addition principle**: If I must choose exactly one activity to complete a task from among the (disjoint) activities $A_1, A_2, \ldots, A_r$ and I can perform activity 1 in m_1 ways, activity 2 in m_2 ways, ..., activity r in m_r ways, then I can complete the task in

$$m_1 + m_2 + \cdots + m_r$$

ways. (This also applies if task amounts to selecting one item from r disjoint sets $A_1, A_2, \ldots, A_r$ with $m_1, m_2, \ldots, m_r$ items respectively.)
- **Permutations**: The number of arrangements of n objects taken r at a time is

$$\mathbf{P}(n,r) = n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}.$$

- **Permutations of objects with some alike:**
  - The number of different permutations (arrangements), where order matters, of a set of n objects (taken n at a time) where r of the objects are identical is

$$\frac{n!}{r!}.$$

  - Consider a set of n objects which is equal to the disjoint union of k subsets, $A_1, A_2, \ldots, A_k$, of objects in which the objects in each subset A_i are identical and the objects in different subsets A_i and A_j, $i \neq j$ are not identical. Let r_i denotes the number of objects in set A_i, then the number of different permutations of the n objects (taken n at a time) is

$$\frac{n!}{r_1!r_2!\ldots r_n!}.$$

  This can also be considered as an application of the technique of **"overcounting"** where we count a larger set and then divide.
- **Combinations:** The number of ways of choosing a subset of (or a sample of) r objects from a set with n objects, where order does not matter, is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}.$$

  Note this was also an application of the technique of "overcounting".

Math 4389 (Author unknown!)

Probability Theory: Counting

The Product Rule:

Suppose a procedure can be broken down into a sequence of m tasks. If there are n_i ways to perform task i, then there are

$$n_1 n_2 \ldots n_m$$

ways to perform the procedure.

Ex: The chairs in an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

26 10

Ex: Each user on a computer system has a password which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

6 char. $\underset{\text{digit}}{\underline{10}} \cdot \underline{36} \cdot \underline{36} \cdot \underline{36} \cdot \underline{36} \cdot \underline{36}$ $\qquad 6 \cdot 10 \cdot 36^5$

7 char $\qquad + \underline{\qquad\qquad}$

8 char. $\qquad + \underline{\qquad\qquad}$

at the end, subtract off duplicates

Combinations:

Def: The number of ways to select a subset $\{s_1, s_2, \ldots, s_r\}$ of r elements of the set $S = \{1, 2, 3, \ldots, n\}$ is called the *combination of n items taken r at a time* and is given by

$$C\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Permutations:

Def: The number of ways to choose a sequence $s_1, s_2, s_3, \ldots s_r$ of distinct elements of the set $S = \{1, 2, 3, \ldots, n\}$ is called the *r-permutation of n items* and is given by

$$n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

Ex: How many permutations of the letters ABCDEFGH contain the string ABC?

$$\underbrace{\underset{1 \text{ item}}{ABC}\ D\ E\ F\ G\ H}_{6 \text{ items}} \qquad 6!$$

Ex: How many different poker hands of five cards can be dealt from a standard deck of 52 cards?

$C(52, 5)$

Ex: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

$C(30, 6)$

Ex: What if the crew of 6 above is split into a captain, a co-captain, and 4 crew members (who each share the same job)?

$C(30,6) \cdot 6 \cdot 5$

Indistinguishable Objects:

Thm: The number of different permutations of n objects, where there are n_i objects of Type i for $i = 1, 2, 3, \ldots, k$ is given by

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

Ex: How many different strings can be made by reordering the letters of the word *SUCCESS*?

$$\frac{7!}{3! \cdot 2!} = \frac{7 \cdot \cancel{6} \cdot 5 \cdot 4 \cdot 3 \cdot \cancel{2} \cdot 1}{\cancel{6} \cdot \cancel{2}} = 7 \cdot 60 = 420$$

How about MISISSIPPI?

MISSISSIPPI

$$\frac{11!}{4!\,4!\,2!}$$

$$P(A) = \frac{\text{\# of desired outcomes}}{\text{\# of all outcomes}}$$

PROBABILITY THEORY

An *experiment* is a procedure that yields one of a given set of possible outcomes.

The *sample space* of the experiment is the set of possible outcomes.

An *event* is a subset of the sample space.

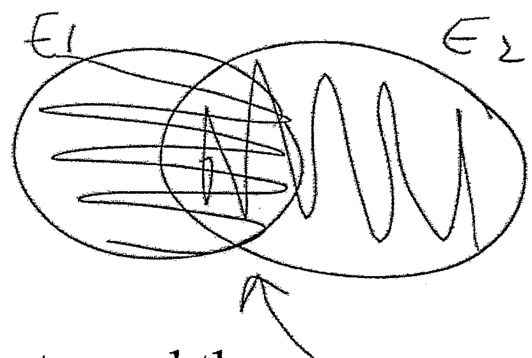
Def: If S is a finite nonempty sample space of *equally likely* outcomes, and E is an event, that is $E \subseteq S$, then the *probability* of E is

$$\Pr(E) = \frac{|E|}{|S|}$$

Complements and Unions:

If S is the sample space for an experiment and E, E_1, E_2 are events, the following are true:

1. $\Pr(S) = 1$
2. $\Pr(E^c) = 1 - \Pr(E)$
3. $\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2)$



If $\Pr(E_1 \cap E_2) = 0$, then they are said to be *disjoint* events, and then

$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2)$

$$10g \to 2 \text{ girls}$$
$$12b$$
$$22 \text{ people} \to 5 \text{ students}$$

Ex: A team of 5 students will be selected from a class of 10 girls and 12 boys.

What is the probability that there are 2 girls in this team?

$$P(2 \text{ girls } \& \text{ 3 boys}) = \frac{C(10,2) \cdot C(12,3)}{C(22,5)}$$

What is the probability that there is at least one girl in the team?

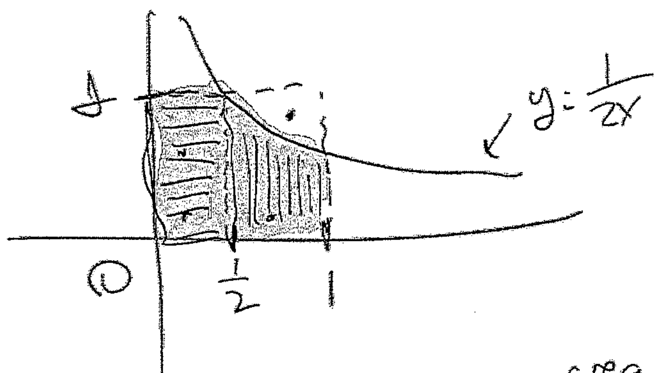
$$P(1 \text{ girl } \& \text{ 4b} \text{ or } 2 \text{ girls } \& \text{ 3b} \text{ or } 3 \text{ girls } \& \text{ 2b} \text{ or } 4g \ \& \text{ 1b} \text{ or } 5g \ \& \text{ 0b})$$

$$= \frac{C(10,1) \cdot C(12,4) + C(10,2) \cdot C(12,3) + \cdots}{C(22,5)}$$

$$P(\text{at least } 1g) = 1 - P(0g \ \& \ 5b) = 1 - \frac{C(10,0) \cdot C(12,5)}{C(22,5)}$$

Example: Two points x and y are selected at random in the interval [0,1]. What is the probability that the product xy is less than 1/2?

$(\cancel{xy}$



$$x \cdot y < \frac{1}{2}$$

(x, y)

$$y < \frac{1}{2 \cdot x} = \frac{1}{2}\left(\frac{1}{x}\right)$$

y

$$P\left(x \cdot y < \frac{1}{2}\right) = \frac{\text{area of desired region}}{\text{area of all}}$$

$$= \frac{\frac{1}{2} + \int_{1/2}^{1} \frac{1}{2x}\,dx}{1}$$

$$= \frac{1}{2} + \left[\frac{1}{2}\ln(x)\right]_{1/2}^{1} = \frac{1}{2} + \left(0 - \frac{1}{2}\ln\left(\frac{1}{2}\right)\right)$$

$$= \frac{1}{2} - \frac{1}{2}\ln\left(\frac{1}{2}\right)$$

$$= \frac{1}{2} + \frac{1}{2}\ln 2$$

$$= \frac{1}{2}(1 + \ln 2)$$

Independence:

Def: Two events, E_1 and E_2, are *independent* if and only if $\Pr(E_1 \cap E_2) = \Pr(E_1)\Pr(E_2)$.

1-5 red
1-5 purple

Ex: 5 red balls numbered 1-5 and five purple balls numbered 1-5 are placed in an urn. Let E be the event that an even number is drawn and let F be the event that a red ball is drawn. Are these events independent?

$$P(E) = \frac{4}{10} = \frac{2}{5} \qquad P(F) = \frac{5}{10} = \frac{1}{2}$$

$$P(E \cap F) = P(\text{even \& red}) = \frac{2}{10} = \frac{1}{5}$$

$$P(E) \cdot P(F) = \frac{\cancel{2}}{5} \cdot \frac{1}{\cancel{2}} = \frac{1}{5}$$

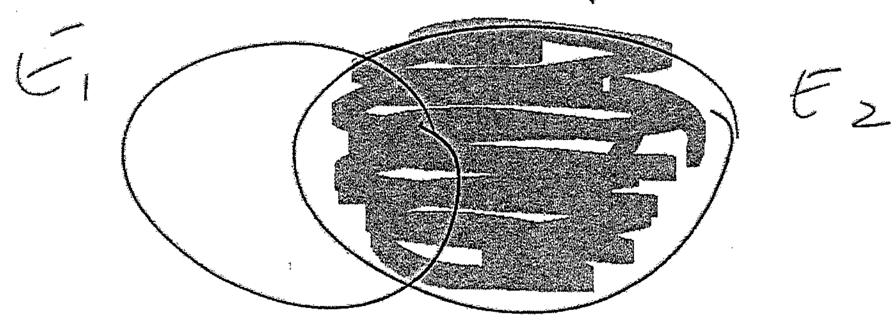
Conditional Probability:

Def: If E_1 and E_2 are events, the *conditional probability of* E_1 *given* E_2 is given by

$$\Pr(E_1 \mid E_2) = \frac{\Pr(E_1 \cap E_2)}{\Pr(E_2)}, \text{ provided } \Pr(E_2) \neq 0$$

ex: $P(E_1) = 0.2$, $P(E_2) = 0.5$, $P(E_1 \cap E_2) = 0.1$

$$P(E_1 \mid E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$



Ex: Twenty marbles, numbered 1-20, are placed in an urn. What is the probability that an 8 or 10 is drawn given that it is known that an even number is drawn?

$$P(8 \text{ or } 10 \mid \text{even}) = \frac{2}{10} = \frac{1}{5}$$

What is the probability that a 3 or 5 is drawn given that a prime number is drawn?

$$P(3 \text{ or } 5 \mid \text{prime}) = \frac{2}{8} = \frac{1}{4}$$

$$P(3 \text{ or } 4 \mid \text{prime}) = \frac{1}{8}$$

The Law of Total Probability

Law of Total Probability

Suppose that $F_1, F_2, F_3, \ldots, F_n$ are events such that

$F_i \cap F_j = \varnothing$, whenever $i \neq j$ and $F_1 \cup F_2 \cup F_3 \cup \ldots \cup F_n = S$.

Then for any event E,

$$\Pr(E) = \Pr(E \mid F_1)\Pr(F_1) + \Pr(E \mid F_2)\Pr(F_2) + \ldots + \Pr(E \mid F_n)\Pr(F_n)$$

Ex:

An auto insurer classifies its policyholders as either average or substandard risks. Assume 70% of the policy holders are average risks. During the year, 1% of the average risks have an accident, and 5% of the substandard risks have an accident. What fraction of policyholders are involved in an accident during the year?

$$P(\text{acc.}) = P(\text{acc} \mid \text{avg}) \cdot P(\text{avg}) + P(\text{acc} \mid \text{sub.}) \cdot P(\text{sub})$$

$$= 0.01 \cdot 0.70 + 0.05 \cdot 0.30$$

A ***binomial random variable***, X, represents the number of successes in n independent trials, where the probability of success on an individual trial is p. (For example: The number of heads obtained during 10 coin flips.)

Conditions under which X is binomial:

1. There are n "trials," and n is fixed before the trials begin.
2. Each trial results in either a "success" (S) or a "failure" (F).
3. The trials are independent (the outcome of one trial does not influence the outcome of any other trial.
4. The probability of success, $\Pr(S) = p$, is constant from trial to trial.

$C(n,r) \cdot p^r \cdot q^{n-r}$ $\qquad$ $C(12,5) \cdot \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^7$

A random variable, X, is said to have a binomial distribution with parameters n and p if its probability mass function is given by

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1,2,...$$

where n is a positive integer, and p is a number in the interval [0,1].

For a binomial random variable, X, with parameters n and p

$\underbrace{E[X] = np}$, and $\underbrace{Var(X)} = np(1-p)$

st. deviation: $\sqrt{\text{variance}}$ $\quad \sqrt{npq}$

variance: $npq = 6 \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{6}$

expected value (mean) $\mu = n \cdot p = 6 \cdot \frac{1}{6} = \boxed{1}$

Example: A fair die is rolled $\boxed{\text{6 times}}$. What is the probability of getting

$n=6$

a) Exactly 2 fives?

b) No more than 2 fives?

success: getting 5

$p = \frac{1}{6}$ $\quad q = 1-p = \frac{5}{6}$

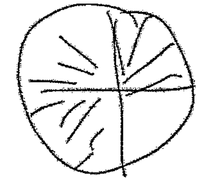
$$P(\text{2 successes in 6 trials}) = C(6,2) \cdot p^2 \cdot q^4$$

$$= \frac{6!}{4!\,2!} \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^4$$

b)

$P($2 suc. or 1 suc. or 0 suc$)$

$$= C(6,2) \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^4 + C(6,1) \cdot \left(\frac{1}{6}\right) \cdot \left(\frac{5}{6}\right)^5 + C(6,0) \cdot \left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^6$$



Example: For a biased coin, the odds of getting heads is 3 to 1. If the coin is flipped 10 times. What is the probability of getting at least one head?

$$\text{prob(heads)} = \frac{3}{3+1} = \frac{3}{4} \qquad q = 1 - \frac{3}{4} = \frac{1}{4}$$

$$P(1H \text{ or } 2H \text{ or } 3H \text{ or } \dots \text{ or } 10H)$$

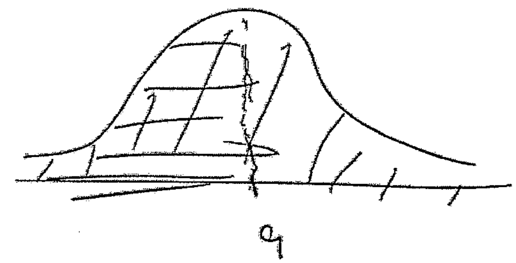
$$= 1 - P(\underline{\underline{0H}}) = 1 - C(10,0)\cdot\left(\tfrac{3}{4}\right)^0\cdot\left(\tfrac{1}{4}\right)^{10}$$

$$= 1 - 1\cdot 1\cdot \frac{1}{4^{10}} = 1 - \frac{1}{4^{10}}$$

Continuous Variables and Probability density function:

Let X be a continuous random variable. Then, a probability density function of X is a function f(x) such that for any two number with a<=b,

$$P(a \le X \le b) = \int_a^b f(t)\,dt$$



Properties:

1. $f(x) \ge 0$ for all x.
2. $\boxed{\int_{-\infty}^{\infty} f(x)dx = 1}$

Note: $P(X > a) = 1 - P(X \le a)$

Definition: The cumulative distribution function F(x) for a continuous random variable X is defined by:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$$

$$P(X \le 2) = \int_{-\infty}^{2} f(t)dt$$

Note: $F'(x) = f(x)$

Using F(x):

$$P(X > a) = 1 - F(a)$$

$$P(a \le X \le b) = F(b) - F(a)$$

Expected Value:

$$\mu = E(X) = \int_{-\infty}^{\infty} tf(t)dt$$

The variance:

$$\sigma^2 = Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

Standard deviation: square root of variance.

Example: A company hires a marketing consultant who determines that the length of time (in minutes) that a consumer spends on the company's website is a random variable, X, whose probability density function is:

$$f_X(t) = \begin{cases} \frac{1}{6} e^{-t/6}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$P(X > 10) = 1 - \underbrace{P(X \leq 10)}$$

$$= 1 - \int_{-\infty}^{10} f(t)\, dt$$

What's the probability that a consumer will spend more than 10 minutes on the company's website?

$$= 1 - \int_0^{10} \frac{1}{6} e^{-t/6}\, dt$$

$$\boxed{\int t \cdot \frac{1}{6} e^{-t/6}\, dt}$$

$$= 1 - \left[\frac{1}{6} \cdot \frac{e^{-t/6}}{-1/6} \right]_0^{10}$$

mean: $\frac{1}{\lambda} = 6$

$$= 1 - \left[-e^{-t/6} \right]_0^{10}$$

$$= 1 - \left[-e^{-5/3} + e^0 \right]$$

$= 1 + e^{-5/3} - 1 = \boxed{e^{-5/3}}$

Exercise: Let X be a random variable with probability density function given below. Find the mean of X.

$$f_X(t) = \begin{cases} \frac{1}{4}t^3, & 0<t<2 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} t \cdot f(t)\, dt = \int_0^2 t \cdot \left(\frac{1}{4}t^3\right) dt$$

$$= \int_0^2 \frac{1}{4}t^4\, dt = \left[\frac{1}{4} \cdot \frac{t^5}{5}\right]_0^2 = \frac{32}{20} - 0$$

$$= \boxed{\frac{8}{5}}$$

The Exponential Distribution

A random variable X is said to have an ***exponential distribution*** with parameter λ if its probability density function is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

where λ is a positive real number.

If X is an ***exponential random variable*** with parameter λ, then

$$E[X] = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2}$$

Example: Let X be a random variable with probability density function given below.

$$f_X(t) = \begin{cases} \frac{1}{4}e^{-t/4}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Then, the mean of X is 4 and variance is 16.