

Math 4389

(Author unknown!)

Probability Theory: Counting

The Product Rule:

Suppose a procedure can be broken down into a sequence of m tasks. If there are n_i ways to perform task i , then there are

$$n_1 n_2 \dots n_m$$

ways to perform the procedure.

Ex: The chairs in an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Ex: Each user on a computer system has a password which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

26

10

6 char.

$$\frac{10}{\text{digit}} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26}$$

$$6 \cdot 10 \cdot 26^5$$

7 char

+

8 char.

+

$$\underline{\hspace{2cm}}$$

Combinations:

Def: The number of ways to select a subset $\{s_1, s_2, \dots, s_r\}$ of r elements of the set $S = \{1, 2, 3, \dots, n\}$ is called the *combination of n items taken r at a time* and is given by

$$C \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Permutations:

Def: The number of ways to choose a sequence $s_1, s_2, s_3, \dots, s_r$ of distinct elements of the set $S = \{1, 2, 3, \dots, n\}$ is called the *r -permutation of n items* and is given by

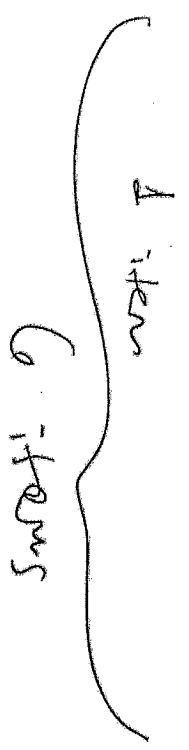
$$n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}$$

Ex: How many permutations of the letters **ABCDEFGHI** contain the

string **ABC**?

A B C D E F G H

6!



Ex: How many different poker hands of five cards can be dealt from a standard deck of 52 cards?

$$C(52, 5)$$

Ex: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

$$C(30, 6)$$

Ex: What if the crew of 6 above is split into a captain, a co-captain, and 4 crew members (who each share the same job)?

$$C(30, 6) \cdot 6 \cdot 5$$

Indistinguishable Objects:

Thm: The number of different permutations of n objects, where there are n_i objects of Type i for $i = 1, 2, 3, \dots, k$ is given by

$$\frac{n!}{n_1! n_2! \dots n_k!}$$

Ex: How many different strings can be made by reordering the letters of the word **SUCCESS**?

How about MISSISSIPPI?

$$\frac{7!}{3! \cdot 2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 2} = 7 \cdot 60 = 420$$

M I S S I S S I P P I

$$7! =$$

$$4! \cdot 4! \cdot 2! =$$

PROBABILITY THEORY

An *experiment* is a procedure that yields one of a given set of possible outcomes.

The *sample space* of the experiment is the set of possible outcomes.

An *event* is a subset of the sample space.

$$P(A) = \frac{\# \text{ of desired outcomes}}{\# \text{ of all outcomes}}$$

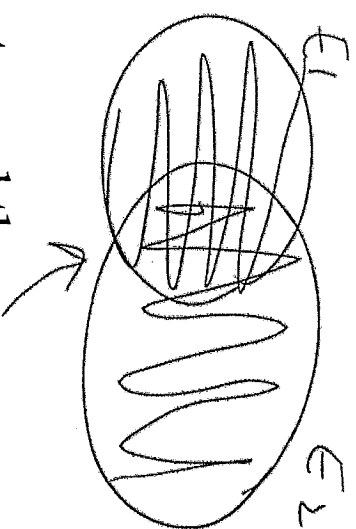
Def: If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is $E \subseteq S$, then the *probability* of E is

$$\Pr(E) = \frac{|E|}{|S|}$$

Complements and Unions:

If S is the sample space for an experiment and E, E_1, E_2 are events, the following are true:

1. $\Pr(S) = 1$
2. $\Pr(E^c) = 1 - \Pr(E)$
3. $\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2)$



If $\Pr(E_1 \cap E_2) = 0$, then they are said to be *disjoint* events, and then

$$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2)$$

(10g) → 2 girls
12b

22 people → 5 students

Ex: A team of 5 students will be selected from a class of 10 girls and 12 boys.

What is the probability that there are 2 girls in this team?

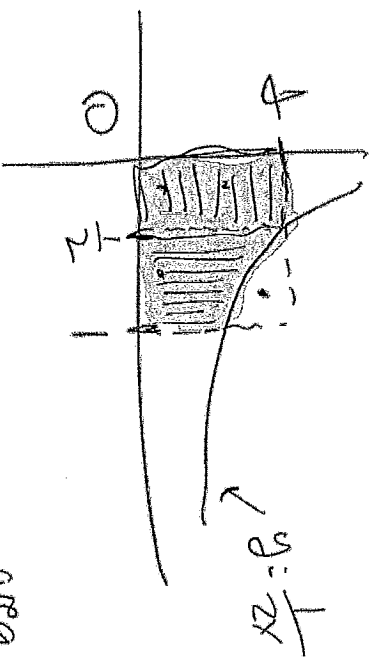
$$P(2 \text{ girls} \text{ \& } 3 \text{ boys}) = \frac{C(10,2) \cdot C(12,3)}{C(22,5)}$$

What is the probability that there is at least one girl in the team?

$$P(1 \text{ girl} \text{ \& } 4 \text{ b} \text{ or } 2 \text{ girls} \text{ \& } 3 \text{ b} \text{ or } 3 \text{ girls} \text{ \& } 2 \text{ b} \text{ or } 4 \text{ girls} \text{ \& } 1 \text{ b} \text{ or } 5 \text{ girls} \text{ \& } 0 \text{ b})$$
$$= \frac{C(10,1) \cdot C(12,4) + C(10,2) \cdot C(12,3) + \dots + C(22,5)}{C(22,5)}$$

$$P(\text{at least 1g}) = 1 - P(0 \text{ girls}) = 1 - \frac{C(10,0) \cdot C(12,5)}{C(22,5)}$$

Example: Two points x and y are selected at random in the interval $[0, 1]$.
 What is the probability that the product xy is less than $1/2$? ~~(*)~~



$$x \cdot y < \frac{1}{2} \quad (x, y)$$

$$y < \frac{1}{2 \cdot x} = \frac{1}{2} \left(\frac{1}{x} \right)$$

y

$$P(x \cdot y < \frac{1}{2}) = \frac{\text{area of desired region}}{\text{area of all}}$$

$$= \frac{\frac{1}{2} + \int_{1/2}^1 \frac{1}{2x} dx}{1}$$

$$= \frac{1}{2} + \left[\frac{1}{2} \ln(x) \right]_{1/2}^1 = \frac{1}{2} + (0 - \frac{1}{2} \ln(\frac{1}{2}))$$

$$= \frac{1}{2} - \frac{1}{2} \ln(\frac{1}{2})$$

$$= \frac{1}{2} + \frac{1}{2} \ln 2$$

$$= \frac{1}{2} (1 + \ln 2)$$

Independence:

Def: Two events, E_1 and E_2 , are independent if and only if

$$\Pr(E_1 \cap E_2) = \Pr(E_1) \Pr(E_2)$$

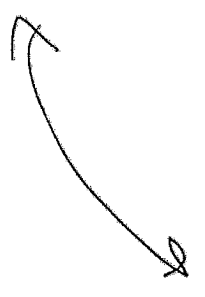
} 1-5 red
1-5 purple

Ex: 5 red balls numbered 1-5 and five purple balls numbered 1-5 are placed in an urn. Let E be the event that an even number is drawn and let F be the event that a red ball is drawn. Are these events independent?

$$P(E) = \frac{4}{10} = \frac{2}{5} \qquad P(F) = \frac{5}{10} = \frac{1}{2}$$

$$P(E \cap F) = P(\text{even \& red}) = \frac{2}{10} = \frac{1}{5}$$

$$P(E) \cdot P(F) = \frac{2}{5} \cdot \frac{1}{2} = \frac{1}{5}$$



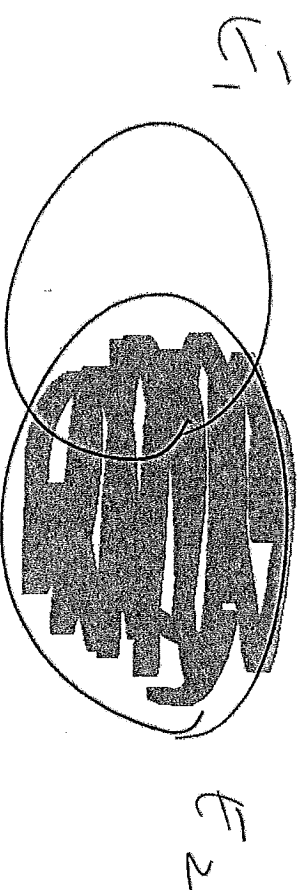
Conditional Probability:

Def: If E_1 and E_2 are events, the conditional probability of E_1 given E_2 is given by

$$\Pr(E_1 | E_2) = \frac{\Pr(E_1 \cap E_2)}{\Pr(E_2)}, \text{ provided } \Pr(E_2) \neq 0$$

ex: $\Pr(E_1) = 0.2$, $\Pr(E_2) = 0.5$, $\Pr(E_1 \cap E_2) = 0.1$

$$\Pr(E_1 | E_2) = \frac{\Pr(E_1 \cap E_2)}{\Pr(E_2)}$$



Ex: Twenty marbles, numbered 1-20, are placed in an urn. What is the probability that an 8 or 10 is drawn given that it is known that an even number is drawn?

$$P(8 \text{ or } 10 \mid \text{even}) = \frac{2}{10} = \frac{1}{5}$$

What is the probability that a 3 or 5 is drawn given that a prime number is drawn?

$$P(3 \text{ or } 5 \mid \text{prime}) = \frac{2}{8} = \frac{1}{4}$$

$$P(3 \text{ or } 4 \mid \text{prime}) = \frac{1}{8}$$

The Law of Total Probability

Law of Total Probability

Suppose that $F_1, F_2, F_3, \dots, F_n$ are events such that

$$F_i \cap F_j = \emptyset, \text{ whenever } i \neq j \text{ and } F_1 \cup F_2 \cup F_3 \cup \dots \cup F_n = S.$$

Then for any event E ,

$$\Pr(E) = \Pr(E | F_1) \Pr(F_1) + \Pr(E | F_2) \Pr(F_2) + \dots + \Pr(E | F_n) \Pr(F_n)$$

Ex:

An auto insurer classifies its policyholders as either average or substandard risks. Assume 70% of the policy holders are average risks. During the year, 1% of the average risks have an accident, and 5% of the substandard risks have an accident. What fraction of policyholders are involved in an accident during the year?

$$\begin{aligned} P(\text{acc.}) &= P(\text{acc} | \text{avg}) \cdot P(\text{avg}) + P(\text{acc} | \text{sub.}) \cdot P(\text{sub.}) \\ &= 0.01 \cdot 0.70 + 0.05 \cdot 0.30 \end{aligned}$$

A *binomial random variable*, X , represents the number of successes in n independent trials, where the probability of success on an individual trial is p . (For example: The number of heads obtained during 10 coin flips.)

Conditions under which X is binomial:

1. There are n “trials,” and n is fixed before the trials begin.
2. Each trial results in either a “success” (S) or a “failure” (F).
3. The trials are independent (the outcome of one trial does not influence the outcome of any other trial.
4. The probability of success, $\Pr(S) = p$, is constant from trial to trial.

$$C(n, r) \cdot p^r \cdot q^{n-r}$$

$$C(12, 5) \cdot \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^7$$

A random variable, X , is said to have a binomial distribution with parameters n and p if its probability mass function is given by

$$P_x(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \dots$$

where n is a positive integer, and p is a number in the interval $[0, 1]$.

For a binomial random variable, X , with parameters n and p

$$E[X] = np, \text{ and } Var(X) = np(1-p)$$

St. deviation:

$$\sqrt{npq}$$

$$\text{variance: } npq = 6 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{2}$$

expected value (mean) $\mu = n \cdot p = 6 \cdot \frac{1}{6} = \boxed{1}$

Example: A fair die is rolled 6 times. What is the probability of getting

a) Exactly 2 fives?

$$n=6$$

success: getting 5

b) No more than 2 fives?

$$P = \frac{1}{6} \quad q = 1 - p = \frac{5}{6}$$

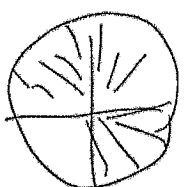
$$P(2 \text{ successes in } 6 \text{ trials}) = C(6, 2) \cdot p^2 \cdot q^4$$

$$= \frac{6!}{4! \cdot 2!} \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^4$$

b)

$$P(2 \text{ suc. or less. or } 0 \text{ suc.})$$

$$= C(6, 2) \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^4 + C(6, 1) \cdot \left(\frac{1}{6}\right) \cdot \left(\frac{5}{6}\right)^5 + C(6, 0) \cdot \left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^6$$



Example: For a biased coin, the odds of getting heads is 3 to 1. If the coin is flipped 10 times. What is the probability of getting at least one head?

$$P(\text{heads}) = \frac{3}{3+1} = \frac{3}{4} \quad q = 1 - \frac{3}{4} = \frac{1}{4}$$

$$P(1H \text{ or } 2H \text{ or } 3H \text{ or } \dots \text{ or } 10H)$$

$$= 1 - P(\underline{\underline{0H}}) = 1 - C(10,0) \cdot \left(\frac{3}{4}\right)^0 \cdot \left(\frac{1}{4}\right)^{10}$$

$$= 1 - 1 \cdot 1 \cdot \frac{1}{4^{10}} = 1 - \frac{1}{4^{10}}$$

Continuous Variables and Probability density function:

Let X be a continuous random variable. Then, a probability density function of X is a function $f(x)$ such that for any two number with $a <= b$,

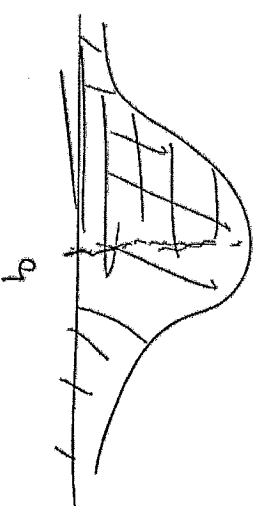
$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

Properties:

1. $f(x) \geq 0$ for all x .

2. $\int_{-\infty}^{\infty} f(x) dx = 1$

Note: $P(X > a) = 1 - P(X \leq a)$



Definition: The cumulative distribution function $F(x)$ for a continuous random variable X is defined by:

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt$$

$$P(X \leq 2) = \int_{-\infty}^2 f(t) dt$$

Note: $F'(x) = f(x)$

Using $F(x)$:

$$P(X > a) = 1 - F(a)$$

$$P(a \leq X \leq b) = F(b) - F(a)$$

Expected Value:

$$\mu = E(X) = \int_{-\infty}^{\infty} tf(t)dt$$

The variance:

$$\sigma^2 = \text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx$$

Standard deviation: square root of variance.

Example: A company hires a marketing consultant who determines that the length of time (in minutes) that a consumer spends on the company's website is a random variable, X , whose probability density function is:

$$f_X(t) = \begin{cases} \frac{1}{6}e^{-t/6}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$P(X > 10) = 1 - \underbrace{P(X \leq 10)}_{= 1 - \int_0^{10} f(t) dt}$$

What's the probability that a consumer will spend more than 10 minutes on the company's website?

$$\int t \cdot \frac{1}{6} e^{-t/6} dt$$

$$\text{mean: } \frac{1}{\lambda} = 6$$

$$= 1 - \int_0^{10} \frac{1}{6} e^{-t/6} dt$$

$$= 1 - \left[\frac{1}{6} \cdot \frac{e^{-t/6}}{(-1/6)} \right]_0^{10}$$

$$= 1 - \left[-e^{-t/6} \right]_0^{10}$$

$$= 1 - [e^{-5/3} + e^0]$$

$$= 1 + e^{-5/3} - 1 = \boxed{e^{-5/3}}$$

Exercise: Let X be a random variable with probability density function given below. Find the mean of X .

$$f_X(t) = \begin{cases} \frac{1}{4}t^3, & 0 < t < 2 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} t \cdot f(t) dt = \int_0^2 t \cdot \left(\frac{1}{4}t^3\right) dt$$

$$= \int_0^2 \frac{1}{4}t^4 dt = \left[\frac{1}{4} \cdot \frac{t^5}{5} \right]_0^2 = \frac{32}{20} - 0 = \boxed{\frac{8}{5}}$$

The Exponential Distribution

A random variable X is said to have an *exponential distribution* with parameter λ if its probability density function is given by

$$f_x(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

where λ is a positive real number.

If X is an *exponential random variable* with parameter λ , then

$$E[X] = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2}$$

Example: Let X be a random variable with probability density function given below.

$$f_X(t) = \begin{cases} \frac{1}{4}e^{-t/4}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Then, the mean of X is 4 and variance is 16.