MATH 4389–THE BASIC DE FACTS/CHEATSHEET

1. First order ODEs

- If linear, $y' + p(x)y = g(x)$, then the solution is

$$y = \frac{1}{\mu(x)}\,[\int^x \mu(s)\,g(s)\,ds + C], \qquad \text{where } \mu(x) = e^{\int^x p(t)\,dt}\,.$$

A linear equation is *homogeneous* if the right side $g(x)$ is 0.

- If nonlinear, is it: Separable: $M(x)\,dx = N(y)\,dy$. Integrate.

- Exact: $M(x,y)\,dx + N(x,y)\,dy = 0$ where $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Here solution is $g(x,y) = C$ where $\frac{\partial g}{\partial x} = M$ and $\frac{\partial g}{\partial y} = N$. Solve these by integrating.

- Integrating factor: if there exists $\mu(x,y)$ with $(\mu M)\,dx + (\mu N)\,dy = 0$ exact. Two easy special cases: 1) If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x. Solve for $\mu(x)$ in

$$\frac{d\mu}{dx} = \left(\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\right)\mu(x).$$

2) If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$ is a function of y. Solve for $\mu(y)$ in

$$\frac{d\mu}{dy} = \left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\right)\mu(y).$$

In either case, $(\mu M)\,dx + (\mu N)\,dy = 0$ is now exact.

- Homogeneous in the sense that $\frac{dy}{dx} = F(\frac{y}{x})$ (call this ‘homogeneous sense 2’). Let $v = \frac{y}{x}$ and solve $\frac{dx}{x} = \frac{dv}{F(v)-v}$.

- Bernoulli equation: $y' + p(x)y = g(x)y^n$ with $n \neq 0, 1$. Substitute $v = y^{1-n}$ to get a linear ODE.

2. Second order ODEs

2.1. Nonlinear second order ODEs.

- If of form $y'' = f(x, y')$. Substitute $v = y', v' = y''$, and get a first order ODE $v' = f(x, v)$. Solve this for v, then integrate v to get y.
- If of form $y'' = f(y, y')$. Substitute $v = y', v' = y''$, and get a first order ODE $v\frac{dv}{dy} = f(y, v)$. Solve this for $v(y)$, then solve $\frac{dy}{dx} = v(y)$ to get y.

2.2. Linear second order ODEs.

- Linear second order: $a(x)\, y'' + b(x)\, y' + p(x)y = g(x)$.
- A linear equation is *homogeneous* if the right side $g(x)$ is 0. Otherwise it is called *nonhomogeneous*.
- Linear homogeneous second order with two linearly independent solutions $y_1(x), y_2(x)$, then the general solution is $y = c_1\, y_1(x) + c_2\, y_2(x)$. You can test if solutions $y_1(x), y_2(x)$ are linearly independent by Wronskian $W(y_1(x), y_2(x)) = y_1 y_2' - y_1' y_2 \neq 0$.
- Linear second order homogeneous constant coefficients: $ay''+by'+cy = 0$, with $a \neq 0$. Solve the *characteristic equation* $ar^2 + br + c = 0$ to get solutions r_1, r_2.
- If r_1, r_2 real and distinct, then $y = c_1\, e^{r_1 x} + c_2\, e^{r_2 x}$ is solution.
- If r_1, r_2 real and equal, then $y = c_1\, e^{r_1 x} + c_2\, x e^{r_1 x}$ is solution.
- If $r_1, r_2 = \alpha \pm i\beta$ then $y = c_1\, e^{\alpha x} \cos(\beta x) + c_2\, e^{\alpha x} \sin(\beta x)$ is solution.
- If linear homogeneous but not constant coefficients, and know one solution $y_1(x)$, can use *reduction of order* to find a second linearly independent solution $y_2(x)$:

$$y_2(x) = y_1(x) \int^x \frac{e^{-\int^s p(t)\, dt}}{y_1(s)^2}\; ds.$$

2.3. Linear second order nonhomogeneous ODEs.

- Linear second order nonhomogeneous ODE $a(x)\,y'' + b(x)\,y' + p(x)y = g(x)$: If you can find one *particular solution* $y_p(x)$, and if the *associated homogeneous equation* $a(x)\,y'' + b(x)\,y' + p(x)y = 0$ has two linearly independent solutions $y_1(x), y_2(x)$, then the general solution is $y = c_1\,y_1(x) + c_2\,y_2(x) + y_p(x)$.
- Linear second order nonhomogeneous constant coefficients: $ay'' + by' + cy = g(x)$: Finding a particular solution by undetermined coefficients *table:*
  - If $g(x)$ is an nth degree polynomial: try $y_p(x)$ of form
  $$y_p(x) = x^s(A_0 + A_1x + \cdots + A_nx^n).$$
  - If $g(x)$ is an nth degree polynomial times $e^{\alpha x}$: try
  $$y_p(x) = x^s(A_0 + A_1x + \cdots + A_nx^n)e^{\alpha x}.$$
  - If $g(x)$ is an nth degree polynomial times $e^{\alpha x}$ times $\cos(\beta x)$ or $\sin(\beta x)$: try $y_p(x)$ of form
  $$x^s e^{\alpha x}[(A_0 + A_1x + \cdots + A_nx^n)\cos(\beta x) \;+\; (B_0 + B_1x + \cdots + B_nx^n)\sin(\beta x)].$$

    Here $s = 0, 1, 2, \cdots$ is the smallest whole number so that no 'term' of y_p above is a solution of the associated homogeneous equation. (By a 'term' we mean e.g. $x^sA_nx^n$ above.)
  - Other *undetermined coefficients method–method of annihilators.*
- Linear second order nonhomogeneous ODE $y'' + b(x)\,y' + p(x)y = g(x)$ by variation of parameters: Find two linearly independent solutions $y_1(x), y_2(x)$ of the associated homogeneous equation. Then solve
$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g(x) \end{cases}$$
for u_1' and u_2', then integrate to get u_1, u_2. Finally $y_p(x) = u_1y_1 + u_2y_2$.
- Alternative way to get u_1, u_2 in the previous item: $u_1 = -\int \frac{y_2\,g}{W}\,dx,\ u_2 = \int \frac{y_1\,g}{W}\,dx$ where W is the Wronskian $W(y_1(x), y_2(x))$.