Math 4389–Topology and metric space language

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Math 4389

Abstract

Our interest in this part of the survey course/test preparation is really in using a common language that applies to $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^n$, metric spaces, and topology (which are successively more general). Thus we should know, for example, things like 'a union of open sets is open and the intersection of two open sets is open'. Or interior, neighborhood, boundary, limit/accumulation points, closure, compact sets , etc. We will visit these below, but the point is that you know these words, and something about such things from 3333 and elsewhere, the results should all be quite familiar sounding. There are often a few questions on topology or the associated notion of metrics on the Field test of Math GRE.

- A topological space is a set X with a collection of subsets of it which we have decided to call open sets. This collection of open sets is called a topology on X. The union of any number of open sets is open, and the intersection of two open sets is open. The empty set and X are open. We can generalize much of Analysis to such spaces, as we shall see briefly here.

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IF YOU GET LOST BELOW WHEN YOU HEAR THE WORD TOPOLOGY OR METRIC SPACE, JUST TAKE THAT SPACE TO BE THE REAL NUMBERS, OR $\mathbb{R}^n$. (THAT PROBABLY GOES FOR THE RELATED QUESTIONS ON THE FIELD TEST OR MATH GRE TOO.)

- A neighborhood of a point x is an open set U containing x. The interior int S of a set S are the points x which have a (open) neighborhood which is a subset of S. The interior of a set S is the biggest open set contained in S. And S is open iff int $S = S$. The boundary of a set S are the points x such that every (open) neighborhood of x contains points in S and in S^c (that is, points not in S).

- An open set is a set that contains none of its boundary points.
- An accumulation (or limit or cluster) point of a set A is an element x such that every neighborhood of x contains at least one point in $A \setminus \{x\}$ (or equivalently contains infinitely many points in A. If $x \notin A$ then x is an accumulation point iff it is a boundary point.
- An isolated point of a set A is a point x in A for which there exists a neighborhood of x which does not contain any other points of A.
- A closed set is the complement U^c of an open set U, and also can be defined as a set that contains all its boundary (or limit/accumulation) points. A set is clopen iff it is both open and closed. Both the empty set and its complement (the universe) are clopen. Arbitrary intersections of closed sets are closed. Finite unions of closed sets are closed.

- The closure $\overline{A}$ of A, is the intersection of all closed subsets of X, containing A. We take $\overline{\phi} = \phi, \overline{X} = X$.
- Note that $\overline{A}$ is the smallest closed subset of X containing A. Also A is closed iff $A = \overline{A}$.
- Test: $x \in \overline{A}$ iff $U \cap A \neq \phi$ for all open sets U with $x \in U$. In a metric space you can take U here to be an open ball center x (defined on next slide).
- The boundary of A equals $\bar{A} \setminus \operatorname{int}(A)$; and $\bar{A}$ is the union of A and its boundary (or the union of the interior of A and its boundary).

- A metric space is a set X, with a function $d : X \times X \to [0, \infty)$, called a metric, satisfying the following properties:

(i) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(ii) (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$,

(iii) $d(x, y) = 0$ if and only if $x = y$.

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- A normed (vector) space is a vector space X over the real or complex scalars, with a function $\|\cdot\| : X \to [0, \infty)$, called a *norm*, satisfying the following properties:
  (i) $\|x\| = 0$ iff $x = 0$,
  (ii) $\|\lambda x\| = |\lambda|\|x\|$ for all scalars λ and $x \in X$,
  (iii) (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,

In a metric space (X, d), an open ball is $B(x, r) = \{y \in X : d(x, y) < r\}$, for some $r > 0$.

Proposition Every normed space is a metric space with metric $d(x, y) = ||x - y||$. *Proof.* (i) $d(x, y) = 0$ iff $||x - y|| = 0$ iff $x - y = 0$ iff $x = y$.
(ii) $d(x, y) = ||x - y|| = ||(-1)(y - x)|| = ||y - x|| = d(y, x)$.
(iii) $d(x, z) = ||x - z|| = ||(x - y) + (y - z)|| \le ||x - y|| + ||y - z|| = d(x, y) + d(y, z)$. $\square$

- In a metric space (X, d), an open ball is $B(x, r) = \{y \in X : d(x, y) < r\}$, for some $r > 0$.
- Open sets in a metric space X are the unions of open balls. This is the topology for X.

- A set U in a metric space is open iff $\forall x \in U\ \exists r > 0$ s.t. $B(x, r) \subset U$.
- An interior point of a set S is a point x for which $B(x, r) \subset S$ for some $r > 0$.

Thus: Every metric space (X, d) has an associated topology, called the metric topology. Putting this together with Proposition 1, we see that every normed space, $(X, ||\cdot||)$, has an associated topology, called the norm topology.

- We say that a sequence (x_n) in a metric space (X, d) converges to x if $d(x_n, x) \to 0$. Then a set A in a metric space is closed iff it contains the limits of all convergent sequences with terms in A. Also $\bar{A}$ is the set of limits of all convergent sequences with terms in A.

- If Y is a subset of a topological space X, then the *subspace topology* or *relative topology* on Y is when the open sets of Y are the intersections $U \cap Y$ of open sets U in X. Sets of this form are also called 'relatively open' sets in Y.

- An open cover of a subset A of a topological space X, is a collection $\mathcal{C}$ of open sets whose union contains A. A subcover is a subset of this collection $\mathcal{C}$ whose union also contains A. A set is compact if every open cover of it has a finite subcover.

- A topological space X is connected if it does not have a clopen set besides $\emptyset$ and X. A subset of a topological space X is connected if it is connected in the subspace topology.

- A subset A of $\mathbb{R}^n$ (or a metric space) is connected if for all nonempty disjoint open sets U, V whose union contains A, either $A \subset U$ or $A \subset V$.

- The connected sets in $\mathbb{R}$ are just the intervals.

Theorem (Heine-Borel) A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

Theorem (Bolzano-Weierstrass) Every nonempty bounded set in $\mathbb{R}^n$ has an accumulation point. Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

- Let $f : X \to Y$ be a function between topological spaces (we sometimes call a function a 'map'). We say that f is continuous if $f^{-1}(V)$ is open in X for all open V in Y. (We may take V to be an open ball here in the metric space case.) We say that f is continuous at a point $x \in X$ if for every open neighborhood V of $f(x)$ in Y, there exists an open neighborhood U of x in X such that $f(U) \subset V$. If A is a subset of X then we say that f is continuous on A if f is continuous at every point in A.

Theorem Let $f : X \to Y$ be a function between topological spaces. TFAE:

(i) f is continuous.

(ii) $f^{-1}(C)$ is closed in X $\forall$ closed $C \subset Y$.

(iii) f is continuous at every $x \in X$.

We have the usual results, such as compositions of continuous functions are continuous, etc.

Theorem Let $f : X \to Y$ be a function between metric spaces, and $x \in X$. TFAE:

(i) f is continuous at x.

(ii) whenever $x_n \to x$ in X then $f(x_n) \to f(x)$ in Y.

(iii) Given $\epsilon > 0$ there exists a $\delta > 0$ such that $d(f(y), f(x)) < \epsilon$ whenever $d(y, x) < \delta$.

A few more definitions: A function $f : X \to Y$ (between topological or metric spaces) is called **open** if $f(U)$ is open in Y for all open U in X. We say that f is a **homeomorphism** if f is one-to-one, onto, continuous, and open (note that a one-to-one, onto function is open iff f^{-1} is continuous). Two topological spaces X and Y are said to be **homeomorphic** if there exists a homeomorphism between them.

Theorem If $f : X \to Y$ is continuous, and X is compact, then $f(X)$ is compact.

Corollary (Extreme value/Min-Max theorem) If $f : X \to \mathbb{R}$ is continuous on a topological space X, and given a compact subset A of X, then f achieves a minimum and a maximum on A. That is, there exist $x_1, x_2 \in X$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in A$. In particular, f is bounded (the range of f is closed and bounded, i.e. compact).

[Proof: Exercise using last theorem, and results from 3333 like the Heine-Borel theorem.]

- For sets in $\mathbb{R}^n$, path connectedness or convexity is often more important than connectedness, and if you are taking the math subject GRE you could look up these on wikipedia, but we will treat connectedness here.

Theorem If $f : X \to Y$ is continuous, and X is connected, then $f(X)$ is connected.

Corollary (Intermediate Value Theorem (IVT)) If $f : X \to \mathbb{R}$ is continuous and X is connected, then $f(X)$ is an interval. Hence if c is a number between $f(x_1)$ and $f(x_2)$, for $x_1, x_2 \in X$, then $\exists z \in X$ such that $f(z) = c$.

If, in addition, X is compact, then $f(X) = [a, b]$, some $a \leq b$ in $\mathbb{R}$.

Proof. Use the last theorem and the fact that connected sets in the $\mathbb{R}$ are intervals. So $f(X)$ is an interval. By the Extreme value/Min-Max theorem above (or the Theorem stated before that), it is a compact interval. But the only compact intervals are $[c, d]$, some c, d in R. Since Range f equals $[c, d]$, if $f(x_1) < t < f(x_2)$ for $x_1, x_2 \in X$ then $t \in [c, d] = f(X)$, so there exists $z \in X$ with $f(z) = t$. $\square$

As time permits, you could read more of the online notes on Analysis, and the two “Multivariable” pdf’s on the course notes link under the analysis tab.