# SHARP BOUNDARY TRACE INEQUALITIES

GILES AUCHMUTY

ABSTRACT. This paper describes sharp inequalities for the trace of Sobolev functions on the boundary of a bounded region $\Omega \subset \mathbb{R}^N$. The inequalities bound (semi-)norms of the boundary trace by certain norms of the function and its gradient on the region and two specific constants $k_\rho$ and $k_\Omega$ associated with the domain and a weight function. These inequalities are sharp in that there are functions for which equality holds. Explicit inequalities in some special cases when the region is a ball, or the region between two balls, are evaluated.

## 1. INTRODUCTION

This paper describes some elementary, but sharp, trace inequalities. It will be proved that if $\rho$ is a positive weight function on the boundary $\partial\Omega$ of a bounded region $\Omega$, then there are positive constants $k_\rho, k_\Omega$ such that

$$\int_{\partial\Omega} \rho \, |\gamma(u)| \, d\sigma \ \leq \ k_\rho \ \int_\Omega \ |u| \ dx \ + \ k_\Omega \ \int_\Omega \ |\nabla u| \ dx \tag{1.1}$$

for all $u \in W^{1,1}(\Omega)$. Here $\gamma$ is the boundary trace operator and mild conditions on the boundary $\partial\Omega$ and the function $\rho$ are required. $k_\rho, k_\Omega$ are characterized explicitly and equality holds here when $u \equiv c$ on $\overline{\Omega}$.

Moreover, when $r > 1$, inequalities of the form

$$\int_{\partial\Omega} \ \rho \, |\gamma(u)|^p \, d\sigma \ \leq \ k_\rho \int_\Omega \ |u|^p \ dx \ + \ p \, k_\Omega \, \|u\|_q^{p-1} \, \|\nabla u\|_r \tag{1.2}$$

hold for $u \in W^{1,r}(\Omega)$, certain ranges of $p$ (depending only on $r$) and for a specific value of $q$ that depends on $r$ and $p$. See theorems 5.1 and 6.3 below. The constants $k_\rho$ and $k_\Omega$ are the same in these two inequalities and may be regarded as geometrical quantities associated with the region $\Omega$ and the weight function $\rho$. In particular, $k_\Omega$ is the $L^\infty$ norm of the gradient of a function $w_\rho$ characterized by the region and $\rho$. This function will be called the *trace weight function* and is the solution of a variational problem for the torsion equation.

When $\rho \equiv 1$ on $\partial\Omega$ these inequalities can be used to estimate the norms of the trace operator $\gamma : W^{1,r}(\Omega) \to L^p(\partial\Omega, \, d\sigma)$. Taking $\rho$ to be the characteristic function of a proper subset of the boundary leads to restricted trace estimates.

The inequalities described here are much simpler than the trace results described in texts such as those of DiBenedetto [5], chapter IX.18 or Leoni [9], chapter 15. Grisvard [8], theorem 1.5.1.10 describes a family of trace inequalities that follow from the case where $p = q = r > 1$ in (1.2) above.

A number of papers have treated issues regarding the "best" constant for various Sobolev trace inequalities - and whether there are Sobolev functions that attain these optimal values. That is, they have proved results about the norm of the trace operator either as a map of $W^{1,1}(\Omega)$ to $L^1(\partial\Omega, d\sigma)$ or from $W^{1,p}(\Omega)$ to various $L^q(\partial\Omega, d\sigma)$. See Andreu, Mazon, Rossi [1], Border, Rossi, Ferreira [3], Motron [11] for example. These inequalities do not have the scaling properties, or geometrical interpretations, of (1.1) or (1.2) and assume $\rho \equiv 1$. Inequalities such as some of those treated in Auchmuty [2] or (2) of Maggi and Vilani [10] are quite different to the ones to be investigated here - even though they involve the same functionals.

## 2. Definitions and Notation

A region is a non-empty open connected set in $\mathbb{R}^N$ with $N \geq 2$. Let $L^p(\Omega)$, $p \in [1, \infty]$ be the usual real Lebesgue spaces of functions on $\Omega$. All functions here take values in $\bar{\mathbb{R}} := [-\infty, \infty]$ and the norm on $L^p(\Omega)$ is denoted $\|.\|_p$. A function is said to be positive on a set E if $f(x) \geq 0$ for all $x$ in E. It is strictly positive if $f(x) > 0$ for all $x$ in E. When $\Omega$ has finite Lebesgue measure, then $\bar{u} := |\Omega|^{-1} \int_\Omega u \, dx$ will be the mean value of $u$ on $\Omega$.

For $1 \leq p < \infty$, $W^{1,p}(\Omega)$ is the Sobolev space of Lebesgue measurable functions on $\Omega$ that are $L^p$ and whose weak derivatives $D_j u$ are again in $L^p$ for each $j \in I_N := \{1, 2, \ldots, N\}$. $W^{1,p}(\Omega)$ is a real Banach space with respect to the norm defined by

$$\| u \|_{1,p}^p := \int_\Omega [\, |u(x)|^p + |\nabla u(x)|^p \,] \, dx. \tag{2.1}$$

Here $\nabla u := (D_1 u, D_2 u, \ldots, D_N u)$ is the gradient of the function $u$ and $|v|$ will denote the 2-norm of a vector v.

The definitions and terminology of Evans and Gariepy [6], will generally be followed except that $\sigma$, $d\sigma$, respectively, represent Hausdorff $(N-1)-$dimensional measure and integration with respect to this measure. Also Hausdorff $(N-1)-$dimensional measure will be called surface area measure in this paper. In particular we require that a unique unit outward normal function $\nu : \partial\Omega \to S_1$ be defined $\sigma \, a.e.$ on $\partial\Omega$. Here $S_1$ is the unit sphere in $\mathbb{R}^N$.

$H^1(\Omega)$ will denote the standard Sobolev space of Lebesgue measurable functions on $\Omega$ that are $L^2$ and whose weak derivatives $D_j u$ are again in $L^2$ for each $j \in I_N := \{1, 2, \ldots, N\}$. $H^1(\Omega)$ is a real Hilbert space under the standard $H^1-$ inner product

$$[u, v]_{1,2} := \int_\Omega [\, u(x) \, v(x) + \nabla u(x) \cdot \nabla v(x) \,] \, dx. \tag{2.2}$$

and the associated norm is denoted $\|u\|_{1,2}$.

When $K$ is a compact subset of $\mathbb{R}^N$, then $C(K)$ will denote the Banach space of continuous real valued functions on $K$ with the maximum norm. When $\Omega$ is an open set then $C_b^k(\Omega)$ is the space of uniformly bounded $C^k-$functions on $\Omega$.

A bounded region $\Omega$ is said to satisfy the *Sobolev imbedding theorem* provided the imbedding of $W^{1,p}(\Omega)$ into $L^q(\Omega)$ is continuous when $1 \le p < N$ and $1 \le q \le p_S$ with $p_S = pN/(N-p)$. $p_S$ is called the *Sobolev conjugate of $p$*. When $p = N$ this imbedding should be continuous for all $1 \le q < \infty$. These theorems hold under a variety of conditions on the boundary $\partial\Omega$. See [6] or DiBenedetto [5] for statements and proofs of such results.

This paper will describe some inequalities for certain boundary (semi-)norms of Sobolev functions in terms of norms of the functions on $\Omega$. The following condition will be required throughout.

**Condition B1:** *$\Omega$ is a bounded region in $\mathbb{R}^N$ and its boundary $\partial\Omega$ is the union of a finite number of disjoint closed Lipschitz surfaces; each surface having finite surface area.*

Let $C_c^1(\mathbb{R}^N)$ be the space of functions that are $C^1$ on $\mathbb{R}^N$ and have compact support. The Gauss-Green (or divergence) theorem holds on a region $\Omega$ provided for any $v \in C_c^1(\mathbb{R}^N)$,

$$\int_\Omega D_j v \; dx \;\; = \;\; \int_{\partial\Omega} v \, \nu_j \; d\sigma \quad \text{for } j \in I_N. \tag{2.3}$$

See [6], chapter 5 section 5, for conditions on the region $\Omega$ and its boundary for which (2.3) holds.

Given a function $v \in C_c^1(\mathbb{R}^N)$, its restriction to $\overline{\Omega}$ will again be denoted $v$ and the restriction operator is $R_\Omega$. The space of all such restrictions will be denoted $C_R^1(\overline{\Omega})$ and is a subspace of $W^{1,\infty}(\Omega)$. Define $G^{1,p}(\Omega)$ to be the closure of $C_R^1(\overline{\Omega})$ in the $(1,p)$ norm.

Grisvard [8] theorem 1.4.2.1 quotes a result that $G^{1,p}(\Omega) = W^{1,p}(\Omega)$ when the region $\Omega$ has a "continuous boundary". DiBenedetto [5] propositions 18.1 and 19.1 shows that $G^{1,p}(\Omega) = W^{1,p}(\Omega)$ when the region $\Omega$ satisfies a segment property and provides a counterexample with a disconnected boundary. Many of the standard extension theorems for $C^1$ functions on $\Omega$ imply this condition. See the discussion of extension theorems in Brezis [4] for example. Whether or not $W^{1,p}(\Omega) = G^{1,p}(\Omega)$ is a regularity condition on the boundary $\partial\Omega$.

When $u$ is a continuous function on $\overline{\Omega}$, then the boundary trace operator is the linear map $\gamma : C(\overline{\Omega}) \to C(\partial\Omega)$ that evaluates the function pointwise on the boundary. Our interest here is in the properties of this operator when the domain is a Sobolev space $W^{1,r}(\Omega)$. When $r > N$, this trace operator is well-defined from Morrey's imbedding theorem. For $1 \le r \le N$, $\gamma$ may be extended to be a continuous linear transformation of $W^{1,p}(\Omega)$ into $L^1(\partial\Omega, d\sigma)$ provided the boundary satisfies some regularity conditions. See [5], [6] or [8] for descriptions of such results..

The region $\Omega$ is said to satisfy the $W^{1,r}$-*trace theorem* provided the trace mapping has an extension $\gamma$ that is a continuous linear map of $W^{1,r}(\Omega)$ into $L^q(\partial\Omega, d\sigma)$ for all $1 \le r \le N$

and

$$1 \ \le \ q \ \le \ p_T(r) \ := \ \frac{r(N-1)}{N-r} \quad \text{for } r < N, \text{ and} \quad 1 \ \le \ q \ < \ \infty \quad \text{when } r \ = \ N. \quad (2.4)$$

The quantity $p_T(r)$ is called the *critical trace index* associated with $r$. Proofs of such results may be found in a number of texts including DiBenedetto [5], section IX.18 and Grisvard [8], section 1.5.

Our basic assumption is that the region $\Omega$ has a boundary $\partial\Omega$ such that

**Condition B2:** The region $\Omega$ is such that (B1), the Sobolev imbedding and $W^{1,p}$-trace theorems hold and $W^{1,p}(\Omega) = G^{1,p}(\Omega)$.

For regions obeying this condition (B2), the Gauss-Green theorem (2.3) holds for all $v \in W^{1,1}(\Omega)$ with $\gamma v$ replacing $v$ on the right hand side and the derivatives taken in a weak sense; see [6], chapter 5. In the following the trace operator will generally be understood when integrals over $\partial\Omega$ are taken.

## 3. The Trace Weight Function

When (B1) holds, the boundary $\partial\Omega$ is a finite union of compact connected subsets in $\mathbb{R}^N$ whose Hausdorff $(N-1)$ dimensional measure is finite and positive. Let $\rho : \partial\Omega \to [0,\infty]$ be a Borel function that is in $L^s(\partial\Omega, d\sigma)$ for some $s \ge 1$ and $M := \int_{\partial\Omega} \ \rho \, d\sigma$.

The sharp inequalities to be derived here involve a function that will be called the *trace weight function* $w_\rho$ of $\rho$. $w_\rho$ is defined as the unique solution of a simple variational principle and may be regarded as a solution of the torsion equation subject to Neumann boundary data. When $\rho \equiv 1$ on $\partial\Omega$ this function is a geometrical quantity associated with the region $\Omega$. Torsion functions , obeying zero Dirichlet boundary conditions, were recently used by Giorgi and Smits [7]. to obtain upper and lower bounds on the principal eigenvalues of Dirichlet-Laplacians on $\Omega$. Their function also has a probabilistic interpretation.

Let $H^1_m(\Omega)$ be the subspace of $H^1(\Omega)$ of all functions that satisfy $\bar{u} = 0$. It is a closed subspace of $H^1(\Omega)$ and condition (B2) implies that Poincare's inequality holds. Namely there is a $c_\Omega > 0$ such that

$$\int_\Omega \ |\nabla u|^2 \, dx \ \ge \ c_\Omega \ \int_\Omega (u - \bar{u})^2 \, dx, \quad \text{for all} \quad u \in H^1(\Omega). \quad (3.1)$$

Thus $H^1_m(\Omega)$ is a real Hilbert space  with respect to  the inner product defined by

$$\langle u, w \rangle_\nabla \ := \ \int_\Omega \nabla u \cdot \nabla w \, dx.$$

Given $\rho$ as above, define $\mathcal{E} : H^1_m(\Omega) \to \mathbb{R}$ by

$$\mathcal{E}(w) \ := \ \int_\Omega \ |\nabla w|^2 \, dx \ - \ 2 \int_{\partial\Omega} \ \rho \, w \, d\sigma. \quad (3.2)$$

Consider the variational problem $(\mathcal{P}_\rho)$ of minimizing $\mathcal{E}$ on $H^1_m(\Omega)$ and evaluating

$$\alpha(\rho) := \inf_{w \in H^1_m(\Omega)} \mathcal{E}(w).$$

**Theorem 3.1.** *Assume (B2) holds, $N \geq 3$ and $\rho \in L^s(\partial\Omega, d\sigma)$ with $s \geq s_c := 2(1-1/N)$. Then there is a unique minimizer $w_\rho$ of $\mathcal{E}$ on $H^1_m(\Omega)$. When $N = 2$, we require $s > 1$. $w_\rho$ satisfies*

$$\int_\Omega \nabla w \cdot \nabla v \, dx = \int_{\partial\Omega} \rho \, v \, d\sigma \quad \text{for all} \quad v \in H^1_m(\Omega), \text{ and} \tag{3.3}$$

$$\alpha(\rho) = - \|\nabla w_\rho\|^2_2 = - \int_{\partial\Omega} \rho \, w_\rho \, d\sigma. \tag{3.4}$$

*Proof.* When the trace theorem holds on $\Omega$ then $w \in H^1_m(\Omega)$ implies that $w \in L^p(\partial\Omega)$ for $p \leq p_T(2) := 2(N-1)/(N-2)$ (and $N \geq 3$). Thus $\mathcal{E}_2(w) := \int_{\partial\Omega} \rho \, w \, d\sigma$ will be a continuous linear functional on $H^1_m(\Omega)$ provided $\rho \in L^s(\partial\Omega, d\sigma)$ with $s \geq s_c := 2(1-1/N)$. When $N = 2$, $\mathcal{E}_2$ is a continuous linear functional when $s > 1$ using the corresponding trace theorem.

Thus our conditions imply that $\mathcal{E}$ is continuous, strictly convex and coercive on $H^1_m(\Omega)$. Hence there is a unique minimizer of $\mathcal{E}$ on $H^1_m(\Omega)$. It is easily verified that $\mathcal{E}$ is Gateaux differentiable on $H^1_m(\Omega)$ with

$$\langle D\mathcal{E}(w), v \rangle = 2 \int_\Omega \nabla w \cdot \nabla v \, dx - 2 \int_{\partial\Omega} \rho \, v \, d\sigma.$$

Thus the minimizer of $\mathcal{E}$ on $H^1_m(\Omega)$ satisfies (3.3).

Put $v = w_\rho$ in (3.3) and substitute in the expression (3.2). Then (3.4) follows. $\square$

**Corollary 3.2.** *Under the conditions of theorem 3.1, the function $w_\rho$ is a weak solution of*

$$\Delta w = k_\rho := M/|\Omega| \quad \text{a.e. on } \Omega \text{ and} \quad D_\nu w = \rho \quad \text{on } \partial\Omega. \tag{3.5}$$

*Proof.* Equation (3.3) implies that there is a $c \in \mathbb{R}$ such that

$$\int_\Omega \nabla w \cdot \nabla v \, dx - \int_{\partial\Omega} \rho \, v \, d\sigma = c \int_\Omega v \, dx \quad \text{for all} \quad v \in H^1(\Omega)$$

from a standard orthogonality argument. Take $v \equiv 1$ on $\Omega$ in this equation to see that $c = -M/|\Omega|$. In particular this holds for all $v \in H^1_0(\Omega)$ and thus $w_\rho$ is a weak solution of the equation

$$-\Delta w = c \quad \text{on } \Omega.$$

Using the Gauss-Green theorem and this result we find that

$$\int_{\partial\Omega} v \, [D_\nu w - \rho] \, d\sigma = 0 \quad \text{for all} \quad v \in H^1(\Omega).$$

Since this holds for all $v \in H^1(\Omega)$, the boundary condition follows. $\square$

This shows that the function $w_\rho$ may be regarded as a weak solution of the torsion equation on $\Omega$ subject to a Neumann boundary condition. Note that $v(x) := w_\rho(x) - c\,|x|^2$ is a harmonic function on $\Omega$ when $c = M/(2N|\Omega|)$ and $v$ obeys

$$D_\nu v = \rho_1 := \rho - c\,(x \cdot \nu) \qquad \text{on } \partial\Omega. \tag{3.6}$$

Thus $v$, and also $w_\rho$, will be $C^\infty$ on $\Omega$ as $v$ is harmonic on $\Omega$. For our following analysis the essential condition on the function $\rho$ will be

**Condition B3:**   $\rho \in L^s(\partial\Omega, d\sigma)$ with $s \geq s_c$ if $N \geq 3$ (or s $> 1$ when N $= 2$), and $w_\rho \in W^{1,\infty}(\Omega)$.

The requirement that $w_\rho \in W^{1,\infty}(\Omega)$ holds under various regularity conditions on $\rho$ and $\partial\Omega$. In particular when $\rho \equiv 1$ on $\partial\Omega$, it holds when $\partial\Omega$ satisfies (B2).

## 4. Weighted Trace Inequalities.

In this section some simple sharp trace inequalities will be derived using the divergence theorem and the trace weight function. The inequalities are for the quantity $I_p(u) := \int_{\partial\Omega} \rho\,|\gamma(u)|^p\,d\sigma$ with $p \geq 1$. This integral is the $p$−th power of a (semi-)norm on a class of functions defined on $\partial\Omega$. Our analysis is based on the following result for bounded $C^1$ functions on $\Omega$.

**Lemma 4.1.** *Suppose $\Omega$ is a bounded region obeying (B2) and $\rho$ is such that (B3) holds. When $p \in [1, \infty)$,*

$$\int_{\partial\Omega} \rho\,|\gamma(u)|^p\,d\sigma \;\leq\; k_\rho \int_\Omega |u|^p\,dx \;+\; p \int_\Omega |u|^{p-1}\,|\nabla u|\,|\nabla w_\rho|\,dx \tag{4.1}$$

*for all $u \in C(\overline{\Omega}) \cap C_b^1(\Omega)$.*

*Proof.*    Let $w_\rho$ be the trace weight function for $\Omega$ as described in the preceding section. The divergence theorem yields

$$\int_\Omega \text{div } (|u|^p\,\nabla w_\rho)\,dx \;=\; \int_{\partial\Omega} \rho\,|u|^p\,d\sigma$$

upon using the Neumann condition from (3.5). When $p > 1$, this integral also equals

$$\int_\Omega |u|^p\,\Delta w_\rho\,dx \;+\; p \int_\Omega |u|^{p-2}\,u\,\nabla u \cdot \nabla w_\rho\,dx$$

using a product rule for weak derivatives. Then (4.1) follows from the Poisson equation in (3.5) and elementary inequalities.

When $p = 1$, the integral over $\Omega$ becomes

$$\int_\Omega |u|\,\Delta w_\rho\,dx \;+\; \int_\Omega \nabla|u| \cdot \nabla w_\rho\,dx$$

with these derivatives taken in a weak sense. Then (4.1) follows from Corollary 3.2 and the fact that $|\nabla |u|| = |\nabla u|$ *a.e.*.     □

For convenience we shall write $X$ for the space $C(\overline{\Omega}) \cap C_b^1(\Omega)$. The preceding result extends to $u \in W^{1,1}(\Omega)$ using common Sobolev function approximation arguments. In the following we will use the quantity

$$k_\Omega \; := \; \| \, |\nabla w_\rho| \, \|_\infty. \tag{4.2}$$

It is worth noting that when $|\nabla w_\rho|^2$ is continuous on $\overline{\Omega}$, then it is subharmonic on $\Omega$ so it attains its maximum on the boundary $\partial\Omega$. Thus, when $w_\rho$ is a classical solution and $\rho$ is continuous on the boundary, then $\sup_{z \in \partial\Omega} \rho(z) \leq k_\Omega$ from Corollary 3.2.

**Theorem 4.2.** *Suppose $\Omega$ is a bounded region obeying (B2) and $\rho \in L^s(\partial\Omega, d\sigma)$ with $s \geq s_c$. Then, for $u \in H^1(\Omega)$,*

$$\int_{\partial\Omega} \rho \, |\gamma(u)| \, d\sigma \; \leq \; k_\rho \int_\Omega |u| \, dx \; + \; \| \nabla w_\rho \|_2 \, \| \nabla u \|_2. \tag{4.3}$$

*When (B3) also holds then, for all $u \in W^{1,1}(\Omega)$,*

$$\int_{\partial\Omega} \rho \, |\gamma(u)| \, d\sigma \; \leq \; k_\rho \int_\Omega |u| \, dx \; + \; k_\Omega \int_\Omega |\nabla u| \, dx. \tag{4.4}$$

*Proof.*     When $p = 1$ the inequalities hold for $u \in X$ from the lemma, properties of the weak derivative and the Cauchy - Schwarz inequality. When $u \in H^1(\Omega)$ there is a sequence $\{u_m\} \subset X$ that converges to $u$ in $H^1(\Omega)$. The trace operator $\gamma$ is continuous for the region $\Omega$ as (B2) holds, so $\gamma(u_m)$ converges to $\gamma(u)$ in $L^2(\partial\Omega, d\sigma)$ and there is a subsequence that converges pointwise to $\gamma(u)$ $\sigma$ *a.e.* on $\partial\Omega$. Since the inequality (4.3) holds for each $m$, it holds in the limit upon using Fatou's lemma on this subsequence.

Similarly given a $u \in W^{1,1}(\Omega)$, there is a sequence $\{u_m\} \subset X$ that converges to $u$ in $W^{1,1}(\Omega)$. The trace operator $\gamma$ is continuous as a map of $W^{1,1}(\Omega)$ to $L^1(\partial\Omega, d\sigma)$ from condition (B2) and (4.4) holds for each $u_m$. Fatou's lemma applied to a pointwise convergent subsequence then yields (4.4) for all $u \in W^{1,1}(\Omega)$.     □

Note that these inequalities are sharp and become equalities when $u \equiv c$ on $\Omega$. When the quantity $k_\Omega < \infty$, then $I_1(u)$ is finite whenever $u \in W^{1,1}(\Omega)$.

The results show that the boundary integral $I_1(u)$ is well-defined and finite whenever $u \in H^1(\Omega)$ and $\rho \in L^s(\partial\Omega, d\sigma)$ with $s \geq s_c$. In particular (4.3) holds without any further regularity assumptions about $\nabla w_\rho$. The following sections will describe various sharp $p-$norm analogues of the inequality (4.4).

## 5. Boundary Trace Inequalities when $r > N$.

Generalizations of the preceding inequality to values $p > 1$ depend essentially on proving the continuity of, and finding upper bounds on, the functional $G_p$ defined by

$$G_p(u) := \int_\Omega |u|^{p-1} |\nabla u| \, dx. \tag{5.1}$$

Note that $G_p$ is positive and homogeneous of degree $p$.

Let $X_{pr}$ be the class of all $L^p$ functions on $\Omega$ with gradient $|\nabla u| \in L^r(\Omega)$. For $p, r \geq 1$, this is a real Banach space with respect to the usual intersection norm;

$$\|u\|_{pr} := \|u\|_p + \| |\nabla u| \|_r.$$

$G_p$ is said to be *locally bounded* on the space $X_{pr}$ if it is bounded on bounded subsets of $X_{pr}$.

**Theorem 5.1.** *Suppose (B2), (B3) hold and $u \in X_{pr}$. If $p > 1$ then*

$$\int_{\partial\Omega} \rho \, |\gamma(u)|^p \, d\sigma \; \leq \; k_\rho \int_\Omega |u|^p \, dx \; + \; p \, k_\Omega \, G_p(u) \tag{5.2}$$

*whenever $G_p(u)$ is continuous and locally bounded on $X_{pr}$. If $p \geq p_0(r) := 2 - 1/r$ and $N < r < \infty$, then*

$$\int_{\partial\Omega} \rho \, |\gamma(u)|^p \, d\sigma \; \leq \; k_\rho \int_\Omega |u|^p \, dx \; + \; p \, k_\Omega \, \|u\|_q^{p-1} \, \|\nabla u\|_r \tag{5.3}$$

*for $u \in X_{pr}$ and $q = \frac{r(p-1)}{r-1} \geq 1$. When $p > 1$ and $r = \infty$, then (5.3) holds with $q = p - 1$.*

*Proof.* The first inequality holds by an approximation argument. Choose a sequence $\{u_m\} \subset X$ that converges to $u$ in $X_{pr}$ so that the functions converge in $L^p$ and their gradients converge to $\nabla u$ in $L^r(\Omega; \mathbb{R}^N)$. Now (5.2) holds for each $u_m$. From the proof of lemma 4.1, we have that

$$\int_{\partial\Omega} \rho \, |u_m - u_k|^p \, d\sigma \; \leq \; k_\rho \|u_m - u_k\|_p^p \; + \; p \, k_\Omega \, G_p(u_m - u_k). \tag{5.4}$$

Thus if $G_p$ is continuous at 0, this left hand side is a Cauchy sequence and thus $I_p(u_m)$ converges to a unique limit and (5.2) holds by continuity.

Use Holder's inequality in (5.1), to see that

$$0 \; \leq \; G_p(u) \; \leq \; \|u\|_q^{p-1} \, \|\nabla u\|_r \qquad \text{with } q \; = \; \frac{r(p-1)}{r-1}. \tag{5.5}$$

So $G_p$ is locally bounded on $X_{pr}$ when $q \geq 1$ or $p \geq 2 - 1/r$.

Suppose that the sequence $\{u_m\}$ converges to u in $X_{pr}$, then

$$G_p(u) - G_p(u_m) \; = \; \int_\Omega |u|^{p-1} [|\nabla u| - |\nabla u_m|] \, dx + \int_\Omega |\nabla u_m| [v - v_m] dx$$

where $v = |u|^{p-1}$, $v_m = |u_m|^{p-1}$. Apply Holder's inequality to each of these integrals to obtain

$$|G_p(u) - G_p(u_m)| \leq \|u\|_q^{p-1} \|| \nabla u| - | \nabla u_m|\|_r + \||\nabla u_m|\|_r \|v - v_m\|_{r'}. \tag{5.6}$$

When $r > N$, then Morrey's inequality and the fact that $\Omega$ is a 1-extension region implies that each $u \in X_{pr}$ may be taken to be a continuous bounded function on $\overline{\Omega}$. Then the sequence $\{u_m\}$ converges to $u$ in the supremum norm on $\overline{\Omega}$, so $v_m$ also converges to $v$ in this norm and thus $G_p(u_m)$ converges to $G_p(u)$ from this inequality as $|\Omega|$ is finite. Hence (5.3) follows from the first part of this theorem,.

The last sentence follows from (5.2) upon using Holder's inequality in (5.1). $\qquad\square$

Note that equality holds in these inequalities for constant functions. When $\rho(z) \equiv 1$ on $\partial\Omega$, these inequalities provide bounds on trace norms of functions in $X_{pr}$ and $W^{1,r}(\Omega)$. This inequality is much simpler than the trace estimates described in the literature such as chapter 9, section 18 of DiBenedetto [5]. In particular it shows that the trace norms depend essentially just on the two geometrical quantities $k_\rho$ and $k_\Omega$ - and in a simple manner on $p, r$.

When the Sobolev imbedding theorem is invoked this inequality becomes

$$\int_{\partial\Omega} \rho \, |u|^p \, d\sigma \leq k_\rho \int_\Omega |u|^p \, dx + C_{rq} \, k_\Omega \, \|u\|_{1,r}^p \quad \text{for all} \quad u \in W^{1,r}(\Omega) \tag{5.7}$$

where $C_{rq}$ depends on the norm of the imbedding of $W^{1,r}(\Omega)$ to $L^q(\Omega)$.

When $r > N$ and $p = 2 - 1/r$, then $q = 1$ in (5.3) and $X_{pr} = W^{1,r}(\Omega)$. When $1 < p < 2 - 1/r$, then there is a bound on the boundary 1-seminorm of a function $u \in W^{1,r}(\Omega)$ from the inequality (4.4) and on the (2-1/r)-seminnorm from theorem 5.1. Inequalities for the intermediate p-seminorms may then be found using standard interpolation inequalities.

## 6. Boundary Trace Inequalities when $1 < r \leq N$.

Here the preceding inequalities will be extended to situations with $1 < r \leq N$. First consider the case where $p = r$, so $X_{pr} = W^{1,p}(\Omega)$.

**Theorem 6.1.** *Suppose (B2), (B3) hold and $p > 1$. Then,*

$$\int_{\partial\Omega} \rho \, |\gamma(u)|^p \, d\sigma \leq k_\rho \int_\Omega |u|^p \, dx + p \, k_\Omega \, \|u\|_p^{p-1} \, \|\nabla u\|_p \quad \text{for } u \in W^{1,p}(\Omega). \tag{6.1}$$

*Proof.* In this case the functional $G_p$ defined by (5.1) satisfies

$$0 \leq G_p(u) \leq \| u \|_p^{p-1} \| \nabla u \|_p$$

upon use of Holder's inequality.

Let $\{u_m\} \subset X$ be a sequence that converges to $u$ in $W^{1,p}(\Omega)$, then the inequality (5.6) holds with $r = p$ and the associated sequence $\{v_m\}$ defined as in the proof of theorem 5.2 is a Cauchy sequence in $L^{p'}$. Thus the sequence $I_p(u_m)$ is a Cauchy sequence as in (5.4)

above. Hence $I_p(u)$ is well-defined for each $u \in W^{1,p}(\Omega)$ and the fact that (6.1) holds for each $u_m$ implies that it also holds for such $u$. $\qquad\square$

(6.1) is sharp as it becomes an equality for constant functions on $\Omega$. When Young's inequality (with an $\epsilon$) is used on the last term here, an inequality of the form given in theorem 1.5.1.10 of Grisvard [8] follows. Grisvard's result involves a general constant $K$ rather than the specific quantities of (6.1). This result is different to the results described in [3] as the expression on the right hand side of (6.1) is not the standard $(1, p)-$norm.

When $r \leq N$ then the boundary trace of a function in $W^{1,r}(\Omega)$ need not be a bounded function on $\partial\Omega$. Suppose $p_0(r) := 2 - 1/r$ and $p_T(r)$ is the critical trace index associated with $r$ from (2.4), then there are bounds on $G_p$ for a range of values of $p$.

**Lemma 6.2.**     *Suppose (B2)-(B3) hold and $r \in (1, N]$ with $N \geq 3$, then $G_p$ is locally bounded on $W^{1,r}(\Omega)$ when $p \in [p_0(r), p_T(r)]$. When $N = 2$, $r \in (1, 2]$, then $G_p$ is locally bounded on $W^{1,r}(\Omega)$ for all $p \in [p_0(r), \infty)$. For these values,*

$$0 \ \leq \ G_p(u) \ \leq \ \|u\|_q^{p-1} \, \|\nabla u\|_r \qquad with \ q \ = \ \frac{r(p-1)}{r-1} \in [1, r_S]. \qquad (6.2)$$

*Proof.*     This follows from Holder's inequality applied to (5.1) and yields this value of $q$. As $p$ increases from $p_0(r)$ to $p_T(r)$, the associated values of $q$ increase from 1 to the Sobolev conjugate $r_S$. $\qquad\square$

This enables the following boundary trace inequality.

**Theorem 6.3.**     *Suppose (B2), (B3) hold, $r \in (1, N], N \geq 3$ and $p \in [p_0(r), p_T(r)]$. Then, for $u \in W^{1,r}(\Omega)$,*

$$\int_{\partial\Omega} \ \rho \, |\, \gamma(u) \,|^p \, d\sigma \ \leq \ k_\rho \int_\Omega \ |u|^p \, dx \ + \ p \, k_\Omega \, \|u\|_q^{p-1} \, \|\nabla u\|_r. \qquad (6.3)$$

*Here $q \ = \ r'(p-1) \in [1, r_S]$ with $r'$ the conjugate index of $r$. When $N$=2, $r \in (1, 2]$, then this holds for all $p \in [p_0(r), \infty)$.*

*Proof.*     To prove this, from theorem 5.1, we need only show that $G_p$ is continuous on $W^{1,r}(\Omega)$ since it is locally bounded for this range of $p$ from lemma 6.2. However, just as in (5.6),

$$|G_p(u) \ - \ G_p(w)| \ \leq \ \|u\|_q^{p-1} \, \||\nabla u| - | \nabla w|\|_r \ + \ \||\nabla w|\|_r \, \||u|^{p-1} - |w|^{p-1}\|_{r'}. \qquad (6.4)$$

This implies that $G_p$ is continuous when $r'(p-1) \geq 1$. $\qquad\square$

When $N \geq 3$ and $r_S$ is the Sobolev conjugate of $r$, then there is a constant $C_r$ such that

$$\|u\|_{r_S} \ \leq \ C_r \|u\|_{1,r} \quad \text{for all} \quad u \in W^{1,r}(\Omega).$$

When $q \in [p, r_S]$, the value of $\|u\|_q$ can be bounded in terms of the values of $\|u\|_p$ and $\|u\|_{r_S}$ by the usual interpolation inequalities

$$\|u\|_q \ \leq \ \|u\|_p^{1-\theta} \, \|u\|_{r_S}^\theta \qquad \text{with} \quad \theta \ = \ \frac{q-p}{r_S-p}.$$

Substituting such bounds into (6.3) one obtains bounds on $I_p(u)$ in terms of the $p-$norm of $u$, the $r-$norm of $\nabla u$ and the (1,r)- norm of $u$.

When $r \in (1, N]$ and $p \in (1, r)$, then upper bounds on $I_p(u)$ may be found using interpolation between bounds on $I_1(u)$ from (4.4) and those of $I_r(u)$ from (6.1) as $X_{pr} = W^{1,r}(\Omega)$.

## 7. EXPLICIT TRACE INEQUALITIES WHEN $\Omega$ IS A BALL.

The preceding inequalities provide some simple bounds when the region $\Omega = B_R$ is a ball of radius $R$ in $\mathbb{R}^N$ and the function $\rho(z) \equiv 1$ on $\partial\Omega$.

In this case the constants $k_\rho = N/R$ and $k_\Omega = 1$ since the trace weight function is

$$w_\rho(x) = c + \frac{k_\rho r^2}{2N}$$

where c is a constant. Thus the $W^{1,1}(\Omega)$ trace inequality becomes, from (4.4),

$$\int_{S_R} |u|\, d\sigma \le \frac{N}{R} \int_{B_R} |u|\, dx + \int_{B_R} |\nabla u|\, dx. \tag{7.1}$$

Here $S_R$ is the boundary of $B_R$.

For $N \ge 3$, the various cases described above yield that the boundary traces satisfy

$$\int_{S_R} |u|^p\, d\sigma \le \frac{N}{R}\, \|u\|_p^p + p\, \|u\|_q^{p-1}\, \|\nabla u\|_r \tag{7.2}$$

for $u \in W^{1,r}(\Omega), q = r'(p-1)$ and either (i) $r \in (1, N]$, and $p \in [p_0(r), p_T(r)]$, or (ii) $r \in (N, \infty)$ and $p \ge 2 - 1/r$, or (iii) $r = \infty, p > 1$. Here each of the norms on the right hand side is with respect to the ball $B_R$ and equality holds here for constant functions.

When $\Omega := B_R \setminus \bar{B}_1$ with $R > 1$ is the region between balls of radius R and radius 1, then these results yield simple inequalities for both $\int_{S_1} |u|^p\, d\sigma$ and $\int_{S_R} |u|^p\, d\sigma$ as well as for the integral over the whole boundary. To obtain these inequalities take $\rho$ to be the characteristic function of either the interior boundary $S_1$, the exterior boundary $S_R$ or the whole boundary respectively. The trace weight functions can be found explicitly in each of these cases and the corresponding values of $k_\rho$ will be different, namely:

$$\frac{N}{R^N - 1}, \qquad \frac{NR^{N-1}}{R^N - 1} \qquad \frac{N(R^{N-1} + 1)}{R^N - 1}$$

respectively but $k_\Omega = 1$ for each of these choices of the boundary integral.

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Department of Mathematics, University of Houston, Houston, Tx 77204-3008 USA

*E-mail address*: auchmuty@uh.edu