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PRESSURE SCHUR COMPLEMENT PRECONDITIONERS FOR THE DISCRETE OSEEN PROBLEM*

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Abstract. We consider several preconditioners for the pressure Schur complement of the discrete steady Oseen problem. Two of the preconditioners are well known from the literature and the other is new. Supplemented with an appropriate approximate solve for an auxiliary velocity subproblem, these approaches give rise to a family of the block preconditioners for the matrix of the discrete Oseen system. In the paper we critically review possible advantages and difficulties of using various Schur complement preconditioners. We recall existing eigenvalue bounds for the preconditioned Schur complement and prove such for the newly proposed preconditioner. These bounds hold both for LBB stable and stabilized finite elements. Results of numerical experiments for several model two-dimensional and three-dimensional problems are presented. In the experiments we use LBB stable finite element methods on uniform triangular and tetrahedral meshes. One particular conclusion is that in spite of essential improvement in comparison with "simple" scaled mass-matrix preconditioners in certain cases, none of the considered approaches provides satisfactory convergence rates in the case of small viscosity coefficients and a sufficiently complex (e.g., circulating) advection vector field.

Key words. Navier–Stokes equations, Oseen equations, finite element, iterative methods, preconditioning, pressure Schur complement

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1. Introduction.

We consider the numerical solution of the steady Oseen problem: Given a divergence free advection velocity $\mathbf{w} : \Omega \to \mathbb{R}^d$, force field $\mathbf{f} : \Omega \to \mathbb{R}^d$, and boundary data $\mathbf{g} : \partial\Omega \to \mathbb{R}^d$, find a velocity $\mathbf{u} : \Omega \to \mathbb{R}^d$ and a pressure $p : \Omega \to \mathbb{R}$ such that

$$-\nu\Delta\mathbf{u} + (\mathbf{w}\cdot\nabla)\,\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \;\; \Omega, \tag{1.1}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in} \;\; \Omega, \tag{1.2}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on} \;\; \partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded, connected domain with a piecewise smooth Lipschitz boundary $\partial\Omega$. The coefficient $\nu > 0$ is a given kinematic viscosity. For the sake of uniqueness of p one may impose some additional condition, such as $\int_\Omega p\,dx = 0$.

A necessity of solving Oseen equations numerically is commonly related to using Picard-type iterative methods to find a solution to steady Navier–Stokes problems. In this case $\mathbf{w}$ is an approximation of the velocity from previous iterative steps, so it is updated on every nonlinear iteration. Among another applications we mention Uzawa-type algorithms for the augmented variational inequality approach to the modeling of

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Bingham fluids; see, e.g., [24]. Again one may need to solve discrete Oseen systems many times with different $\mathbf{w}$ and $\mathbf{f}$. Thus there is a demand for efficient iterative solvers for the discrete Oseen problem. We note that besides (1.3) other boundary conditions may be imposed in various models. Furthermore we remark that when using different boundary conditions special attention may be required.

In this paper we consider a finite element method to discretize (1.1)–(1.3). However, the linear algebraic solvers discussed here can be applied in a finite difference or finite volume context in the same manner. We assume that the finite element velocity (not necessarily conforming) and pressure spaces $\mathbb{V}_h$ and $\mathbb{Q}_h$ approximate $\mathbf{H}_0^1(\Omega)$ and $L_2^0(\Omega) := \{q \in L_2(\Omega) : (q,1) = 0\}$, respectively. Consider the following finite element problem: Find $\mathbf{u}_h \in \mathbb{V}_h$ and $p_h \in \mathbb{Q}_h$ satisfying

$$a_h(\mathbf{u}_h, \mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) + (q_h, \operatorname{div} \mathbf{u}_h) + c_h(p_h, q_h) = (\mathbf{f}_h, \mathbf{v}_h) + (g_h, q_h) \quad \forall\, \mathbf{v}_h \in \mathbb{V}_h,\ q_h \in \mathbb{Q}_h. \tag{1.4}$$

The bilinear form $a_h(\cdot,\cdot)$ may include some stabilizing terms for the advection dominated case. The nonnegative bilinear form $c_h(\cdot,\cdot)$ may be included in the finite element formulation if $\mathbb{V}_h$ and $\mathbb{Q}_h$ form an LBB unstable pair [21]; otherwise, $c_h(\cdot,\cdot) \equiv 0$. Denote by $(\cdot,\cdot)_{\mathbb{V}}$ the energy scalar product on $\mathbb{V}_h$ satisfying $(\psi,\phi)_{\mathbb{V}} = (\nabla\psi, \nabla\phi)$ for $\psi, \phi \in \mathbb{V}_h \cap \mathbf{H}_0^1$. For the bilinear forms a_h and c_h we assume ellipticity, continuity, and stability conditions

$$\alpha_1 \|\mathbf{v}_h\|_{\mathbb{V}}^2 \le a_h(\mathbf{v}_h, \mathbf{v}_h), \qquad a_h(\mathbf{v}_h, \mathbf{u}_h) \le \alpha_2 \|\mathbf{v}_h\|_{\mathbb{V}} \|\mathbf{u}_h\|_{\mathbb{V}} \quad \forall\, \mathbf{v}_h, \mathbf{u}_h \in \mathbb{V}_h, \tag{1.5}$$

$$\gamma_1^2 \, \|q_h\|^2 \le \sup_{\mathbf{v}_h \in \mathbb{V}_h} \frac{(q_h, \operatorname{div} \mathbf{v}_h)^2}{\|\mathbf{v}_h\|_{\mathbb{V}}^2} + c_h(q_h, q_h) \quad \forall\, q_h \in \mathbb{Q}_h, \tag{1.6}$$

$$c_h(q_h, p_h) \le \gamma_2 \|q_h\| \|p_h\|, \quad (q_h, \operatorname{div} \mathbf{v}_h) \le \gamma_3 \|q_h\| \|\mathbf{v}_h\|_{\mathbb{V}} \quad \forall\, q_h, p_h \in \mathbb{Q}_h,\ \mathbf{v}_h \in \mathbb{V}_h, \tag{1.7}$$

with positive mesh-independent constants α_1, α_2, γ_1, γ_2, and γ_3. We note that conditions (1.6) and (1.7) are common for the pressure stabilized finite element methods; see, e.g., the recent studies in [8]. For the LBB stable pairs (1.6) and (1.7) trivially hold.

Let $\{\phi_i\}_{1\le i\le n}$ and $\{\psi_j\}_{1\le j\le m}$ be bases of $\mathbb{V}_h$ and $\mathbb{Q}_h$, respectively. Define the following matrices:

$$A_{i,j} = a_h(\phi_j, \phi_i), \quad B_{i,j} = -(\operatorname{div} \phi_j, \psi_i), \quad C_{i,j} = c_h(\psi_j, \psi_i).$$

The linear algebraic system corresponding to (1.4) (*the discrete Oseen system*) takes the form

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{1.8}$$

We are interested in solving (1.8) by a preconditioned iterative method. Following the conventional approach [21, 29, 12] we consider the block triangular preconditioner for the system (1.8):

$$\mathcal{P} = \begin{pmatrix} \hat{A} & B^T \\ O & -\hat{S} \end{pmatrix}. \tag{1.9}$$

The matrix $\hat{A}$ is a preconditioner for the matrix A, such that $\hat{A}^{-1}$ may be considered as an inexact solver for linear systems involving A. The matrix $\hat{S}$ is a preconditioner for the pressure Schur complement of (1.8) $S = BA^{-1}B^T + C$. In an iterative algorithm one needs the actions of $\hat{A}^{-1}$ and $\hat{S}^{-1}$ on subvectors, rather than the matrices $\hat{A}$, $\hat{S}$ explicitly. Once good preconditioners for A and S are given, an appropriate Krylov subspace iterative method for (1.8) with the block preconditioner (1.9) is an efficient solver. In the literature one can find geometric or algebraic multigrid (see, e.g., [21] and references therein) or domain decomposition [23, 35] iterative algorithms which provide effective preconditioners $\hat{A}$ for a wide range of ν and various meshes. However, despite considerable recent effort and progress, building a preconditioner for S, which is robust for a wide range of parameters (especially viscosity), discretizations, and meshes, is still a challenge.

In this paper we recall two recent approaches to construct a preconditioner for S. One is due to Kay, Loghin, and Wathen [28], and another is from Elman and coauthors [16, 18]. Furthermore, in attempting to overcome some difficulties associated with these approaches we consider a new preconditioner for S. We give motivation for different choices of $\hat{S}^{-1}$, prove eigenvalue bounds, and present results of several numerical experiments.

As already mentioned, a good preconditioner $\hat{S}$ is necessary for building the block triangular preconditioner (1.9). Furthermore, there exist other numerical methods for the incompressible Navier–Stokes equations, where finding a proper approximation to S is vital. These methods are based on the Uzawa method and its variants [22, 40, 10], Arrow–Hurwicz [22, 1], SOR [14], and special factorizations [3, 5] for the linearized problems. Moreover, the performance of the widely used splitting algorithms like SIMPLE, projection, or pressure correction methods for the time integration of unsteady problem is also closely related with the issue of the Schur complement preconditioning [34, 38, 37].

There are also iterative methods for solving (1.8) which do not require consideration of the Schur complement S or its preconditioner, at least explicitly. Among these are coupled multigrid methods of Vanka type [27, 38], methods based on Hermitian splitting [4], augmented Lagrangian based preconditioning [6], and implicit-factorization preconditioning [15, 13]; see also the review article [5]. It is not the intention of this paper to discuss or compare these methods. We note only that implementing some of them in a purely algebraic manner may involve serious difficulties [41]. Thus the pressure Schur complement based block solvers remain attractive for treating “real-life” engineering problems and have potential for developing black-box algorithms.

The remainder of the paper is organized as follows. In section 2 we consider two well-known preconditioners for S and present a new approach. In section 3 we prove eigenvalue bounds. For the new preconditioner the h-independent bounds both for the LBB stable and the pressure stabilized discretizations are shown. In section 4 results of numerical experiments with different preconditioners are given for two-dimensional (2D) and three-dimensional (3D) problems discretized on simplicial meshes.

2. Schur complement preconditioners. In this paper all variants of preconditioner $\hat{S}$ are defined through their inverses. Before proceeding to the preconditioners we define the pressure mass, velocity mass, and Laplacian matrices:

$$(M_p)_{i,j} = (\psi_j, \psi_i), \quad (M_u)_{i,j} = (\phi_j, \phi_i), \quad L_{i,j} = (\phi_j, \phi_i)_{\mathbb{V}}.$$

2.1. Pressure convection-diffusion. We first consider the *pressure convection-diffusion* (PCD) preconditioner, proposed by Kay, Loghin, and Wathen [28] and studied by these and other authors (see [21]):

$$\hat{S}^{-1} := \hat{M}_p^{-1} A_p L_p^{-1}. \tag{2.1}$$

Here $\hat{M}_p^{-1}$ denotes an approximate solve with the pressure mass matrix. Matrices A_p and L_p are approximations to convection-diffusion and Laplacian operators in $\mathbb{Q}_h$, respectively. Note that both A_p and L_p need some boundary conditions to be prescribed.

In discretizations of (1.1)–(1.3) with continuous pressure approximations, one can use the conforming discretization of the pressure Poisson problem with Neumann boundary conditions. The corresponding finite element formulation for the case of the Neumann conditions is standard: Find $p_h \in \mathbb{Q}_h$ such that

$$(\nabla p_h, \nabla q_h) = (f, q_h) \qquad \forall\, q_h \in \mathbb{Q}_h \subset \mathrm{H}^1(\Omega).$$

Neumann boundary conditions are conventionally set for the convection-diffusion problem on $\mathbb{Q}_h$.

An alternative way to define L_p is to set $L_p = (B\hat{M}_u^{-1}B^T)$, where $\hat{M}_u$ is a diagonal approximation to the velocity mass matrix. Although a diagonal matrix might be a poor approximation to M_u in the case of anisotropic grids, this operator can be seen as a mixed discretization of the pressure Poisson problem with Neumann boundary conditions. Using $(B\hat{M}_u^{-1}B^T)$ is convenient in the case of discontinuous pressures. It is not straightforward to define A_p for the case of discontinuous pressures; see section 4 for a definition of A_p in the case of isoP_2-P_0 elements.

Analysis of the PCD preconditioner and numerical results found in the literature (see also section 4) recover several specific features of the method. In particular, the advantages of using this preconditioner are the following:

1. The PCD preconditioner provides mesh-independent convergence rates for moderate values of ν.
2. Dependence of the convergence rates on ν^{-1} is significantly improved in comparison to a scaled pressure mass matrix.
3. Numerical results [42, 30] suggest that the preconditioner is not very sensitive to grid anisotropy, at least for some discretizations. Thus in [42] the PCD preconditioning was used to solve the Oseen problem on every step of Picard iterations for the 3D driven cavity problem discretized with Q_2-Q_1 finite elements on a regular grid. The number of iterations was essentially independent of the aspect ratio of elements. In [30] the preconditioner was successfully used to compute a flow around an obstacle in a 3D cube, using isoP_2-P_1 elements on unstructured grid and allowing elements with high aspect ratios.
4. Some theoretical analysis of the preconditioner is available in [20], where the authors prove eigenvalue estimates for the preconditioned system. We recall these results in section 3.
5. The preconditioner can be used both for LBB stable and pressure stabilized discretizations.

There are also some open questions associated with using this method:

1. Degradation of the convergence rates as $\nu \to 0$ is easily seen even for the simplest constant flow: $\mathbf{w} = (1, 0)$ or $\mathbf{w} = (1, 0, 0)$.

2. The issue of proper boundary conditions in L_p and A_p is not very well understood. We address this question below in more detail.
3. The A_p matrix does not naturally arise in the original problem. Some effort may be needed to build it, especially for discontinuous pressure approximations.
4. The preconditioner is specifically oriented to the Oseen problem. Its extension to similar problems, like Newton linearization, quasi-Newtonian fluids, the Navier–Stokes system coupled with other equations, or general problems having the same 2×2 structure as (1.8), is not clear.

The choice of boundary conditions for the definition of A_p and L_p depends on boundary conditions in the Oseen problem. In [21] it is recommended that Neumann boundary conditions should be prescribed on those parts of $\partial\Omega$ where in the original formulation of the Oseen problem one has Dirichlet boundary conditions for $\mathbf{u}$, and Dirichlet boundary conditions in A_p and L_p should be used on those parts of $\partial\Omega$ where in the original formulation one has outflow boundary conditions for the stress tensor. A motivation for these recommendations comes from the symmetric case of the unsteady Stokes problem, where this choice proved to work well for the Cahouet–Chabard preconditioner [11, 32]. Our experiments and analysis suggest that for the case of dominating convection ($\nu \to 0$) this choice is not optimal. Below we give arguments that on the inflow boundary for small ν one may prefer using a Dirichlet homogeneous fictitious boundary for constructing A_p. Experimentally we observed that the change of boundary conditions on the outflow part of the boundary is not crucial. On the characteristic parts of $\partial\Omega$ and outflow parts with conditions on the stress tensor (the so-called do-nothing boundary conditions), Neumann boundary conditions in A_p and L_p are more appropriate.

We consider the continuous counterpart of the pressure Schur complement operator: $S := -\mathrm{div}\,(-\nu\Delta+(\mathbf{w}\cdot\nabla))_0^{-1}\nabla$, where $(-\nu\Delta+(\mathbf{w}\cdot\nabla))_0^{-1}$ is the solution operator for the vector convection-diffusion problem with Dirichlet boundary conditions. The operator S defines a mapping of L_0^2 onto L_0^2. For a given function $p \in L_0^2$ denote $\mathbf{v} = (-\nu\Delta + (\mathbf{w}\cdot\nabla))_0^{-1}\nabla p$, $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$; then one can rewrite the relation $q = S\,p$ in the form

$$-\nu\Delta\mathbf{v} + (\mathbf{w}\cdot\nabla)\,\mathbf{v} - \nabla p = \mathbf{0}, \qquad -\mathrm{div}\,\mathbf{v} = q \quad \text{in } \Omega, \tag{2.2}$$

$$\mathbf{v}|_{\partial\Omega} = \mathbf{0}. \tag{2.3}$$

Let $\nu = 0$ and prescribe boundary condition (2.3) only on the inflow part Γ_{in} of $\partial\Omega$, i.e., $\Gamma_{in} = \{\mathbf{x} \in \partial\Omega \,:\, \mathbf{w}(\mathbf{x})\cdot\mathbf{n} < 0\}$. For the sake of simplicity we assume the following: the plain patch Γ_{in} is orthogonal to the x-axis, and $\mathbf{w}$ is sufficiently smooth, orthogonal to Γ_{in} at each point, and stays parallel to the x-axis in some neighborhood $\mathcal{O} \subset \Omega$ of Γ_{in}. For simplicity, we consider the 2D case (the 3D case is considered similarly). Integrating the first equation in (2.2) along characteristics in $\mathcal{O}$, using condition $\mathbf{v}|_{\Gamma_{in}} = 0$ and assumptions $\nu = 0$ and $\mathbf{w} = (1,0)$ in $\mathcal{O}$ one gets the equality

$$\mathbf{v}(x,y) = \int_0^x \nabla p\,(s,y)\,d\,s \quad \text{in } \mathcal{O}. \tag{2.4}$$

Now we compute $\mathrm{div}\,\mathbf{v}$ in $\mathcal{O}$. If p is sufficiently smooth, the equality (2.4) yields

$$\mathrm{div}\,\mathbf{v} = \frac{\partial}{\partial x}\int_0^x \frac{\partial}{\partial s}p\,(s,y)\,d\,s + \frac{\partial}{\partial y}\int_0^x \frac{\partial}{\partial y}p\,(s,y)\,d\,s = \frac{\partial p}{\partial x} + \int_0^x \frac{\partial^2 p}{\partial y^2}\,d\,s \quad \text{in } \mathcal{O}.$$

The second equation in (2.2) implies $q = -\partial p/\partial x - \int_0^x \partial^2 p/\partial y^2 d\,s$ in $\mathcal{O}$. Therefore, we have in the sense of traces

$$q = -\frac{\partial p}{\partial x} \quad \text{on} \quad \Gamma_{in}. \tag{2.5}$$

On the other hand, the PCD approach suggests approximating $p = S^{-1}q$ by

$$p = -(\mathbf{w}\cdot\nabla)\Delta^{-1}q, \tag{2.6}$$

where Δ^{-1} is a solution operator to the Poisson problem with *some* boundary conditions. We want to define these conditions on Γ_{in} in a way consistent with (2.5). To this end we rewrite (2.6) in $\mathcal{O}$:

$$p = -\frac{\partial r}{\partial x}, \qquad \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = q. \tag{2.7}$$

Relations (2.5) and (2.7) lead to $\partial^2 r/\partial y^2 = 0$ on Γ_{in}. The corresponding homogeneous boundary condition in the definition of Δ^{-1} is $r = 0$ on Γ_{in}. Therefore, for the practically important case when the flow $\mathbf{w}$ is orthogonal to Γ_{in} and ν is small, the reasonable boundary conditions for the PCD preconditioner on the *inflow* are *homogeneous Dirichlet boundary conditions*.

Although the above analysis studies the limit case of $h \to 0$ and $\nu \to 0$ (continuous inviscid problem), we believe it to be useful for understanding the issue of proper boundary conditions in the discrete preconditioner (2.1). In particular, it suggests that the choice of the optimal boundary conditions in $\hat{S}^{-1}$ depends not only on the type of conditions in (1.3), but also on ν and $\mathbf{w}$ (it may be different for inflow, outflow, and characteristic parts of $\partial\Omega$). From the practical standpoint there are still open questions. Thus in the continuous counterpart of the preconditioner ($\hat{S}^{-1} = (\nu\Delta - \mathbf{w}\cdot\nabla)\Delta^{-1}$) one has to prescribe boundary conditions *only* for the Poisson problem solution operator Δ^{-1}, while in the discrete case some boundary conditions are involved in the definition of *both* matrices A_p and L_p. The guess that conditions in A_p and L_p should be the same for optimal performance is not fully confirmed by our numerical experiments. In particular, the best convergence rates were observed with Neumann boundary conditions in L_p and different boundary conditions in A_p depending on ν and $\mathbf{w}$. We do not have a clear explanation of this phenomenon. Furthermore, from the implementation standpoint, Dirichlet boundary conditions for A_p may not be imposed on the nodes at Γ_{in} since these nodes have to contribute to the set of pressure degrees of freedom. For this reason one introduces outside Ω a fictitious one-cell layer attached to Γ_{in}. Dirichlet boundary conditions are assigned at layer nodes not belonging to Γ_{in}. Technical details can be found in section 4.

Results of numerical experiments suggest that the choice of boundary conditions on $\Gamma_{out} = \{\mathbf{x} \in \partial\Omega \,:\, \mathbf{w}(\mathbf{x})\cdot\mathbf{n} > 0\}$ in the preconditioner does not affect its performance in any substantial way. If in the Oseen problem (1.1)–(1.3) instead of Dirichlet conditions one sets the normal component of the stress tensor equal to zero on the outflow,

$$-\nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)\cdot\mathbf{n} + 2p\,\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{out}, \tag{2.8}$$

then for $\nu \to 0$ one gets $p = 0$ on Γ_{out}. Now relation (2.6) immediately gives $(\mathbf{w}\cdot\nabla)r = 0$ on Γ_{out}. For the case when $\mathbf{w}$ is orthogonal to Γ_{out} this results in the *homogeneous Neumann boundary conditions* for the matrices in the PCD preconditioner.

Using similar arguments one can show that Neumann boundary conditions "inside" $\hat{S}$ are appropriate for the characteristic part of $\partial\Omega$.

2.2. BFBt. Next we consider the BFBt preconditioner, proposed by Elman [16] and further developed by Elman and coauthors in [18]. The best available modification from [18] is

$$\hat{S}^{-1} := (B\hat{M}_u^{-1}B^T)^{-1}B\hat{M}_u^{-1}A\hat{M}_u^{-1}B^T(B\hat{M}_u^{-1}B^T)^{-1}, \tag{2.9}$$

where $\hat{M}_u$ is a diagonal approximation of M_u. In the case of continuous pressure elements the BFBt preconditioner may be reformulated as

$$\hat{S}^{-1} := L_p^{-1}B\hat{M}_u^{-1}A\hat{M}_u^{-1}B^T L_p^{-1}.$$

Below some observations on the BFBt preconditioner are listed. On the positive side one has the following:

1. In contrast to the PCD method, the preconditioner (2.9) is built from the matrices readily available. Indeed, matrices A and B are already in the system (1.8), and $\hat{M}_u$ is fairly easy to construct from M_u by the lumping procedure.
2. The preconditioner can be defined for general problems like (1.8), although in the general case one may need to find some other scaling matrices instead of the velocity mass matrix.
3. We found the BFBt preconditioner to be robust with respect to ν for the simplest parallel constant wind, $\mathbf{w} = (1, 0)$, and continuous pressure elements.
4. The issue of proper pressure boundary conditions does not arise explicitly.

On the other hand, one has the following:

1. The dependence on ν^{-1} is still observed for more complicated flows, like circulating flows.
2. h-independence of the convergence rate is observed for the special case of small ν, parallel constant wind with isoP_2-P_1 elements. Otherwise the BFBt method shows some h-dependence.
3. Two pressure Poisson problems should be solved instead of one, as for the PCD preconditioner.

We have no clear explanation of why, for certain discretizations, the mesh-dependent convergence rates occur for the BFBt preconditioner (see some arguments in section 2.3). In [16] and [17] Elman observed some h-dependence in the convergence rate of the GMRES method using the BFBt preconditioner for the finite difference (MAC) and finite element Q_2-Q_1 discretizations. Some h-dependence was also observed by Vainikko and Graham in [39] with Q_2-P_{-1} elements and by Hemmingsson and Wathen in [26] with a finite difference method. In all these papers the variant of the preconditioner with identity matrices instead of $\hat{M}_u$ in (2.9) was used. In the recent paper [18] it was noted that introducing $\hat{M}_u$ in (2.9) improves the situation significantly in the case of Q_2-Q_1 finite element discretizations, leading to virtually no h-dependence. The explanation of this phenomenon was partially heuristic. Furthermore, if one considers a uniform grid and applies finite differences or finite elements with piecewise linear velocity, then $\hat{M}_u$ is a scaled identity matrix (at least for the case of Dirichlet boundary conditions in (1.3)). Thus in this case the matrix $\hat{M}_u$ in (2.9) has no effect on the preconditioner. We note that a possible deterioration of convergence rates is consistent with available eigenvalue estimates for this preconditioner (see (3.2)) which also show the h-dependence.

An attempt to extend the preconditioner for the pressure-stabilized elements was recently made by Elman and coauthors [19]. It leads to a somewhat more complicated definition of $\hat{S}^{-1}$.

We have not found results in the literature about the performance of the BFBt preconditioner in the case of stretched grids.

2.3. More preconditioners. Our motivation is to build a preconditioner based on available matrices (blocks) in the spirit of BFBt with similar "robustness" properties with respect to ν, but without possible convergence failures for small h. To this end, let us consider the preconditioner (2.9) once again. If one ignores the velocity stabilization terms in A, the continuous counterpart of the BFBt preconditioner can be written as

$$\Delta_N^{-1}\mathrm{div}\,(-\nu\Delta + \mathbf{w}\cdot\nabla)\,\nabla\,\Delta_N^{-1}, \tag{2.10}$$

where Δ_N^{-1} is the solution operator for the Poisson problem with Neumann boundary conditions. Consider the operator (2.10) acting on some $p \in L_2(\Omega)$. Trouble might be caused by the lack of a proper tangential boundary condition for $\mathbf{v} = \nabla\,\Delta_N^{-1} p$. Indeed, for the normal component of $\mathbf{v}$ we have $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$, but the tangential component does not necessarily vanish on $\partial\Omega$. At the same time the computing of $(\nu\Delta + \mathbf{w}\cdot\nabla)\mathbf{v}$ requires $\mathbf{v} = 0$ on $\partial\Omega$, since the discrete counterpart of $(\nu\Delta + \mathbf{w}\cdot\nabla)$—matrix A—was built assuming homogeneous Dirichlet boundary conditions for velocity. Similarly, for the vector function $\mathbf{u} = (\nu\Delta + \mathbf{w}\cdot\nabla)\mathbf{v}$ the normal condition $\mathbf{u}\cdot\mathbf{n} = 0$ is not ensured, although this condition was used to define the matrix B—a discrete counterpart of div. Another relevant question is whether the operator in (2.10) is well defined in a bounded domain as an operator from L_0^2 to L_0^2. It can be shown that this requires the H^2-regularity assumption for the Poisson problem and proper homogeneous boundary conditions for the intermediate function $\mathbf{u}$ as discussed above.

It is necessary to point out that such regularity and boundary condition issues formally do not arise on the discrete level. However, the failure of the discrete operator to approximate a well-posed continuous counterpart as $h \to 0$ may be a reason for the h-dependence of the BFBt preconditioner for some discretizations.

The suggested remedy is to commute ∇ and div with Δ_N^{-1} in (2.10). We obtain

$$\mathrm{div}\,\Delta_0^{-1}(-\nu\Delta + \mathbf{w}\cdot\nabla)\,\Delta_0^{-1}\nabla, \tag{2.11}$$

where Δ_0^{-1} is a solution operator for the velocity vector Poisson problem with Dirichlet boundary conditions. Note that instead of the pressure Poisson problem like in the PCD and BFBt preconditioners the new approach involves the velocity Poisson problem. This allows one to avoid issues of pressure boundary conditions and regularity. The commutation property which we used holds only with special boundary conditions, e.g., periodic; this property is even more questionable for discrete operators. However, variants of such commutation arguments are often used in the literature to deduce PCD and BFBt preconditioners. The discrete operator corresponding to (2.11) is

$$\hat{S}^{-1} := \hat{M}_p^{-1} B L^{-1} A L^{-1} B^T \hat{M}_p^{-1}. \tag{2.12}$$

Here L^{-1} is an approximate solve for the discrete velocity vector Poisson problem.

Our observations about this preconditioner are the following. On the positive side one has the following:

1. The preconditioner (2.12) is built from the matrices already available: matrix L is the diffusion part of matrix A.
2. The action of L^{-1} may be performed using the same technology as that of $\hat{A}^{-1}$ (MG, AMG, etc.).

3. Since the preconditioner does not use a discrete pressure Poisson solver, the issue of appropriate boundary conditions does not arise.
4. Preconditioner (2.12) can be easily extended for more general linearized Navier–Stokes-type problems.
5. In our experiments preconditioner (2.12) shows h-independent convergence results in a wider set of cases: both FE choices, various convection fields, up to $\nu = 10^{-3}$. This is supported by the h-independent eigenvalue estimates, which we prove in the next section.

On the other hand, one has the following:

1. Dependence on ν^{-1} is observed for more complicated flows, like circulating flows.
2. For the diffusion dominated case the condition number of the preconditioned matrix $\hat{S}^{-1}S$ is squared compared to the optimal mass matrix preconditioner. Indeed, if $\mathbf{w} = 0, \nu = 1$, and L^{-1} is the exact inverse, we have $A = L$ and

$$\hat{S}^{-1}S = \hat{M}_p^{-1}BL^{-1}AL^{-1}B^T\hat{M}_p^{-1}S = (\hat{M}_p^{-1}S)^2.$$

   This results in nearly doubling the iteration numbers for the diffusion dominated case.
3. Compared to the PCD and BFBt preconditioners, the matrix L in (2.12) has a larger dimension than L_p or $(B\hat{M}_u^{-1}B^T)$.

Remark 2.1. Similar to BFBt, the new preconditioner (2.12) cannot be immediately used for the LBB unstable finite elements. However, for this case it admits a simple modification:

$$S^{-1} := \hat{M}_p^{-1}(BL^{-1}AL^{-1}B^T + C)\hat{M}_p^{-1}.$$

It is easy to show that for the symmetric case this modification ensures h-independent convergence. For the nonsymmetric case h-independent eigenvalue bounds will be proved in the next section.

Let us also mention a method from [9] for constructing a preconditioner $\hat{S}$. The approach is quite different from the techniques considered above. It is based on a hierarchical matrix technique for building approximate inverses for matrices. However, the numerical experiments in [9] show a large setup time in this approach, which makes it rather expensive in practice. Finally, we remark that effective pressure Schur complement preconditioners can be built for the linearized Navier–Stokes equations with nonlinear terms written in the rotation form [33, 31].

3. Eigenvalues estimates. It is well known that characterizing the rate of convergence of nonsymmetric preconditioned iterations can be a difficult task. In particular, eigenvalue information alone may not be sufficient to give meaningful estimates of the convergence rate of a method like preconditioned GMRES [25]. The situation is even more complicated for a method like BiCGStab, for which virtually no convergence theory exists. Nevertheless, experience shows that for many linear systems arising in practice, a well-clustered spectrum (away from zero) usually results in rapid convergence of the preconditioned iteration. Therefore, in this section we recall some known estimates for the eigenvalues of the preconditioned Schur complement with the PCD (2.1) and BFBt (2.9) preconditioners for LBB stable elements. Bounds for the PCD preconditioning will be extended to the pressure stabilized case. Also we prove analogous estimates for the new preconditioning (2.12).

Below we use the following notation: $\|\cdot\|$, $\langle\cdot,\cdot\rangle$ denotes the Euclidean norm and scalar product. We also define the norm $\|q\|_* := \langle M_p^{-1}q, q\rangle^{\frac{1}{2}}$. Note that for a matrix

$D \in \mathbb{R}^{m\times m}$ and corresponding matrix norms it holds that $\|D\|_* = \|M_p^{-\frac{1}{2}} D M_p^{\frac{1}{2}}\|$. Furthermore, the PCD preconditioner from (2.1) we denote by S_1, the BFBt preconditioner from (2.9) we denote by S_2, and S_3 will be the new preconditioner from (2.12).

Assume a quasi-uniform discretization (partition into triangles or quadrilaterals) of Ω. Let h denote a maximum diameter of elements. Assume that finite element spaces $\mathbb{V}_h$ and $\mathbb{Q}_h$ satisfy standard approximation properties and inverse inequalities. For the LBB stable case in [20] the following bounds for the eigenvalues of the preconditioned Schur complement were proved:

$$c_1 \le |\lambda(S{S_1}^{-1})| \le C_1, \tag{3.1}$$

$$c_2 \le |\lambda(S{S_2}^{-1})| \le C_2 h^{-2} \tag{3.2}$$

with positive constants c_1, c_2, C_1, C_2 independent of the meshsize h. For the pressure stabilized case ($C \neq 0$) we do not find in the literature any eigenvalue estimates with the PCD preconditioner S_1. Hence we give the proof of such bounds below in Theorem 3.2. As we already mentioned, using the BFBt preconditioner S_2 in the pressure stabilized case is not straightforward and requires some modifications [19]; this modified preconditioner is not considered in the paper. Also no eigenvalue estimates are known for the modified preconditioning.

Remark 3.1. The constants in (3.1), (3.2) may depend on other parameters; in particular, they depend on viscosity ν. It is hard to find this dependence in an optimal way. At the same time, one-dimensional analysis from [16] suggests that the upper bound in (3.2) may be tight with respect to h at least for some discretizations.

The following theorem extends results in (3.1) for the LBB unstable case and provides h-independent bounds for preconditioner (2.12). For the sake of brevity we prove the theorem assuming the exact inverses M_p^{-1} and L^{-1}. More practical choices involving the use of spectrally equivalent preconditioners do not change the result.

THEOREM 3.2. *For a quasi-uniform discretization of* Ω *assume that finite element spaces* $\mathbb{V}_h$ *and* $\mathbb{Q}_h$*, forming a* not *necessarily LBB stable pair, satisfy standard approximation and inverse inequalities. Assume conditions* (1.5)–(1.7) *for the bilinear forms of the discrete Oseen problem. The following estimates hold:*

$$c_1 \le |\lambda(S{S_1}^{-1})| \le C_1, \tag{3.3}$$

$$c_3 \le |\lambda(S{S_3}^{-1})| \le C_3 \tag{3.4}$$

with positive constants c_1, C_1, c_3, C_3 *independent of the meshsize* h*.*

To prove the theorem we will need several auxiliary estimates, which we put together in the following lemma.

LEMMA 3.3. *The ellipticity, continuity, and stability assumptions* (1.5)–(1.7) *yield the following estimates involving matrices $A, L, B, C,$ and M_p:*

$$\alpha_1\alpha_2^{-2}\|z\|^2 \le \langle A^{-1}L^{\frac12}z, L^{\frac12}z\rangle \qquad \forall\, z\in\mathbb{R}^n, \tag{3.5}$$

$$\|L^{\frac12}A^{-1}L^{\frac12}\| \le \alpha_1^{-1}, \tag{3.6}$$

$$\|L^{-\frac12}AL^{-\frac12}\| \le \alpha_2, \tag{3.7}$$

$$\|M_p^{-\frac12}BL^{-\frac12}\| = \|L^{-\frac12}B^TM_p^{-\frac12}\| \le \gamma_3, \tag{3.8}$$

$$\|M_p^{-\frac12}CM_p^{-\frac12}\| \le \gamma_2, \tag{3.9}$$

$$\langle (BL^{-1}B^T + C)q, q\rangle \ge \gamma_1^2\langle M_pq, q\rangle \qquad \forall\, q\in\mathbb{R}^m. \tag{3.10}$$

Proof. Note that the continuity and ellipticity estimates from (1.5) can be rewritten in the matrix-vector notation

$$\langle Au, v\rangle \le \alpha_2\|L^{\frac12}u\|\|L^{\frac12}v\| \qquad \forall\, u, v\in\mathbb{R}^n, \tag{3.11}$$

$$\alpha_1\|L^{\frac12}u\|^2 \le \langle Au, u\rangle \qquad \forall\, u\in\mathbb{R}^n. \tag{3.12}$$

For arbitrary $z\in\mathbb{R}^n$ consider $u = A^{-1}z$ and $v = L^{-1}z$. Due to (3.11) and (3.12) one gets

$$\begin{aligned}\langle L^{-1}z, z\rangle = \langle v, z\rangle = \langle v, Au\rangle \le \alpha_2\|L^{\frac12}u\|\|L^{\frac12}v\| = \alpha_2\|L^{\frac12}u\|\langle L^{-1}z, z\rangle^{\frac12}\\ \le \alpha_1^{-\frac12}\alpha_2\langle Au, u\rangle^{\frac12}\langle L^{-1}z, z\rangle^{\frac12} = \alpha_1^{-\frac12}\alpha_2\langle A^{-1}z, z\rangle^{\frac12}\langle L^{-1}z, z\rangle^{\frac12}.\end{aligned}$$

This yields $\langle L^{-1}z, z\rangle \le \alpha_1^{-1}\alpha_2^2\langle A^{-1}z, z\rangle$ for any $z\in\mathbb{R}^n$ which is equivalent to (3.5). Furthermore, consider the following relations:

$$\begin{aligned}\langle A^{-1}z, z\rangle = \langle u, z\rangle = \langle u, Lv\rangle \le \|L^{\frac12}u\|\|L^{\frac12}v\| \le \alpha_1^{-\frac12}\langle Au, u\rangle^{\frac12}\|L^{\frac12}v\|\\ = \alpha_1^{-\frac12}\langle A^{-1}z, z\rangle^{\frac12}\langle L^{-1}z, z\rangle^{\frac12}.\end{aligned}$$

Thus we obtain

$$\alpha_1\langle A^{-1}z, z\rangle \le \langle L^{-1}z, z\rangle \qquad \forall\, z\in\mathbb{R}^n.$$

We use this inequality and (3.12) to check

$$\langle A^{-1}z, y\rangle \le \alpha_1^{-1}\langle L^{-1}z, z\rangle^{\frac12}\langle L^{-1}y, y\rangle^{\frac12} \qquad \forall\, y, z\in\mathbb{R}^n. \tag{3.13}$$

Indeed, denoting $u = A^{-1}z$ and $v = L^{-1}y$, we have

$$\begin{aligned}\langle A^{-1}z, y\rangle = \langle u, y\rangle = \langle u, Lv\rangle \le \|L^{\frac12}u\|\|L^{\frac12}v\| \le \alpha_1^{-\frac12}\langle Au, u\rangle^{\frac12}\langle L^{-1}y, y\rangle^{\frac12}\\ = \alpha_1^{-\frac12}\langle A^{-1}z, z\rangle^{\frac12}\langle L^{-1}y, y\rangle^{\frac12} \le \alpha_1^{-1}\langle L^{-1}z, z\rangle^{\frac12}\langle L^{-1}y, y\rangle^{\frac12}.\end{aligned}$$

Now (3.6) follows from (3.13) through

$$\|L^{\frac12}A^{-1}L^{\frac12}\| = \sup_{x\ne0}\sup_{y\ne0}\frac{\langle L^{\frac12}A^{-1}L^{\frac12}x, y\rangle}{\|x\|\|y\|} = \sup_{x\ne0}\sup_{y\ne0}\frac{\langle A^{-1}x, y\rangle}{\|L^{-\frac12}x\|\|L^{-\frac12}y\|} \le \alpha_1^{-1}.$$

In the same way (3.7) follows from (3.11):

$$\|L^{-\frac{1}{2}}AL^{-\frac{1}{2}}\| = \sup_{u\neq 0}\sup_{v\neq 0}\frac{\langle L^{-\frac{1}{2}}AL^{-\frac{1}{2}}u,v\rangle}{\|u\|\|v\|} = \sup_{u\neq 0}\sup_{v\neq 0}\frac{\langle Au,v\rangle}{\|L^{\frac{1}{2}}u\|\|L^{\frac{1}{2}}v\|} \leq \alpha_2.$$

Finally, thanks to (1.7) we get

$$\|L^{-\frac{1}{2}}B^TM_p^{-\frac{1}{2}}\|$$

$$= \sup_{q\neq 0}\frac{\|L^{-\frac{1}{2}}B^Tq\|}{\|M_p^{\frac{1}{2}}q\|} = \sup_{q\neq 0}\sup_{v\neq 0}\frac{\langle L^{-\frac{1}{2}}B^Tq,v\rangle^{\frac{1}{2}}}{\|M_p^{\frac{1}{2}}q\|\|v\|} = \sup_{q\neq 0}\sup_{v\neq 0}\frac{\langle q,Bv\rangle^{\frac{1}{2}}}{\|M_p^{\frac{1}{2}}q\|\|L^{\frac{1}{2}}v\|} \leq \gamma_3$$

and $\|M_p^{-\frac{1}{2}}BL^{-\frac{1}{2}}\| = \|(L^{-\frac{1}{2}}B^TM_p^{-\frac{1}{2}})^T\| \leq \gamma_3$.

Inequalities (3.9) and (3.10) easily follow from the conditions (1.7) and (1.6), respectively. ∎

Now we are in position to prove Theorem 3.2.

Proof. The proof uses the technique of norm equivalence developed in [20]. In particular, we will show that

$$c_5\|M_pq\|_* \leq \|Sq\|_* \leq C_5\|M_pq\|_* \qquad \forall\, q \in \mathbb{Q}_h \tag{3.14}$$

and prove the estimates for S_k:

$$c_4\|M_pq\|_* \leq \|S_kq\|_* \leq C_4\|M_pq\|_*, \quad k=1,3, \qquad \forall\, q \in \mathbb{Q}_h. \tag{3.15}$$

From (3.15) and (3.14) one obtains the norm equivalence

$$c\|S_kq\|_* \leq \|Sq\|_* \leq C\|S_kq\|_*, \quad k=1,3, \qquad \forall\, q \in \mathbb{Q}_h \tag{3.16}$$

with mesh-independent positive constants c and C. To complete the proof one may consider the obvious inequalities

$$\|S_kS^{-1}\|_*^{-1} \leq |\lambda(SS_k{}^{-1})| \leq \|SS_k{}^{-1}\|_*. \tag{3.17}$$

As a consequence of (3.16) and (3.17) the estimate (3.4) follows from the eigenvalues of $S_k^{-1}S$ with some constants c_k, C_k independent of the meshsize h.

Therefore, we can focus on checking (3.14) and (3.15). First we prove (3.14). The upper bound in (3.14) follows from

$$\|SM_p^{-1}\|_* = \|M_p^{-\frac{1}{2}}(BA^{-1}B^T + C)M_p^{-\frac{1}{2}}\|$$

$$\leq \|M_p^{-\frac{1}{2}}BL^{-\frac{1}{2}}\|\|L^{\frac{1}{2}}A^{-1}L^{\frac{1}{2}}\|\|L^{-\frac{1}{2}}B^TM_p^{-\frac{1}{2}}\| + \|M_p^{-\frac{1}{2}}CM_p^{-\frac{1}{2}}\| \leq \gamma_3\alpha_1^{-1} + \gamma_2;$$

here we used estimates (3.6), (3.8), and (3.9). Next we find a lower bound for $\frac{\|SM_p^{-1}q\|_*}{\|q\|_*}$:

$$\inf_{q\neq 0}\frac{\|SM_p^{-1}q\|_*}{\|q\|_*} = \inf_{q\neq 0}\frac{\|M_p^{-\frac{1}{2}}SM_p^{-\frac{1}{2}}q\|}{\|q\|} \geq \inf_{q\neq 0}\frac{\langle M_p^{-\frac{1}{2}}SM_p^{-\frac{1}{2}}q,q\rangle}{\|q\|^2} \tag{3.18}$$

$$= \inf_{q\neq 0}\frac{\langle A^{-1}B^TM_p^{-\frac{1}{2}}q, B^TM_p^{-\frac{1}{2}}q\rangle + \langle CM_p^{-\frac{1}{2}}q, M_p^{-\frac{1}{2}}q\rangle}{\|q\|^2}$$

with use of the Cauchy–Schwarz inequality. Denoting $u = B^T M_p^{-\frac{1}{2}} q$ and applying (3.5) and (3.10), we continue with (3.18):

$$\inf_{q\neq 0} \frac{\|SM_p^{-1}q\|_*}{\|q\|_*} \geq \inf_{q\neq 0} \frac{\alpha_1\alpha_2^{-2}\langle L^{-1}B^T M_p^{-\frac{1}{2}}q, B^T M_p^{-\frac{1}{2}}q\rangle + \langle CM_p^{-\frac{1}{2}}q, M_p^{-\frac{1}{2}}q\rangle}{\|q\|^2}$$

$$= \inf_{q\neq 0} \frac{\langle(\alpha_1\alpha_2^{-2}BL^{-1}B^T + C)q, q\rangle}{\|M_p^{\frac{1}{2}}q\|^2} \geq \min\{\alpha_1\alpha_2^{-2}, 1\}\gamma_1^2.$$

Thus the lower bound in (3.14) is proved with the h-independent constant $c_5 = \min\{\alpha_1\alpha_2^{-2}, 1\}\gamma_1^2$.

For $k = 1$ inequalities (3.15) were proved in [20]. The proof in [20] does not use the LBB stability assumption. Thus we can use this result and conclude that eigenvalue bounds (3.3) hold with h-independent constants c_1, C_1.

To show (3.15) for $k = 3$ we first prove an upper bound on $\|M_p S_3^{-1}\|_*$. Thanks to (3.7), (3.8), and (3.9), one gets

$$\|M_p S_3^{-1}\|_* = \|M_p^{-\frac{1}{2}}(BL^{-1}AL^{-1}B^T + C)M_p^{-\frac{1}{2}}\|$$

$$\leq \|M_p^{-\frac{1}{2}}BL^{-\frac{1}{2}}\|\|L^{-\frac{1}{2}}AL^{-\frac{1}{2}}\|\|L^{-\frac{1}{2}}B^T M_p^{-\frac{1}{2}}\| + \|M_p^{-\frac{1}{2}}CM_p^{-\frac{1}{2}}\| \leq \gamma_3\alpha_2 + \gamma_2.$$

Hence the low bound in (3.15) holds with the h-independent constant $c_4 = \frac{1}{(\gamma_3\alpha_2+\gamma_2)}$.

Finally, we find an upper bound for $\|S_3 M_p^{-1}\|_*$:

$$\|S_3 M_p^{-1}\|_*^{-1} = \|M_p^{-\frac{1}{2}} S_3 M_p^{-\frac{1}{2}}\|^{-1} \tag{3.19}$$

$$= \inf_{q\neq 0}\sup_{p\neq 0} \frac{\langle M_p^{\frac{1}{2}} S_3^{-1} M_p^{\frac{1}{2}} q, p\rangle}{\|q\|\|p\|} \geq \inf_{q\neq 0} \frac{\langle M_p^{\frac{1}{2}} S_3^{-1} M_p^{\frac{1}{2}} q, q\rangle}{\|q\|^2}$$

$$= \inf_{q\neq 0} \frac{\langle AL^{-1}B^T M_p^{-\frac{1}{2}}q, L^{-1}B^T M_p^{-\frac{1}{2}}q\rangle + \langle CM_p^{-\frac{1}{2}}q, M_p^{-\frac{1}{2}}q\rangle}{\|q\|^2}.$$

Denote $u = L^{-\frac{1}{2}}B^T M_p^{-\frac{1}{2}}q$. The condition (3.12) yields $\langle AL^{-\frac{1}{2}}u, L^{-\frac{1}{2}}u\rangle \geq \alpha_1\|u\|^2$. Therefore, we continue with (3.19):

$$\|S_3 M_p^{-1}\|_*^{-1} \geq \inf_{q\neq 0} \frac{\alpha_1\langle L^{-\frac{1}{2}}B^T M_p^{-\frac{1}{2}}q, L^{-\frac{1}{2}}B^T M_p^{-\frac{1}{2}}q\rangle + \langle CM_p^{-\frac{1}{2}}q, M_p^{-\frac{1}{2}}q\rangle}{\|q\|^2}$$

$$= \inf_{q\neq 0} \frac{\langle(\alpha_1 BL^{-1}B^T + C)q, q\rangle}{\|M_p^{\frac{1}{2}}q\|^2} \geq \min\{\alpha_1, 1\}\gamma_1^2.$$

Thus the upper bound in (3.15) is proved with the h-independent constant $C_4 = (1 + \alpha_1^{-1})\gamma_1^{-2}$. □

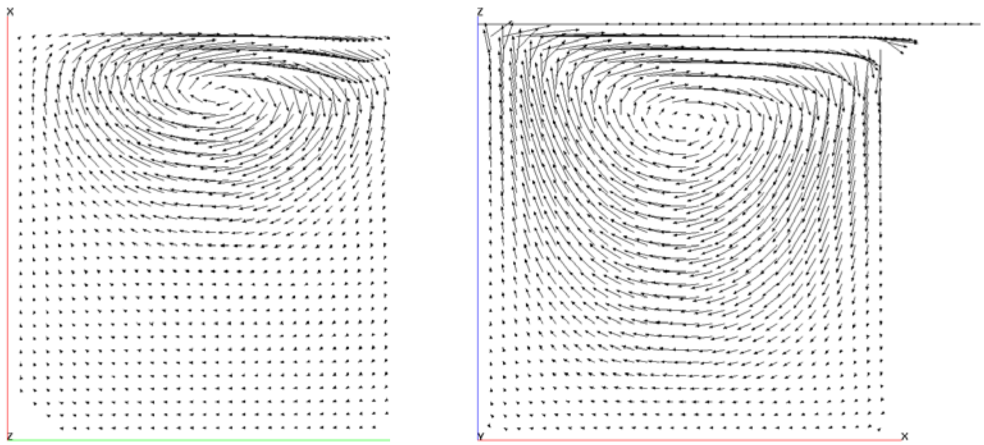

FIG. 4.1. *Left picture: The 2D velocity field* (4.2). *Right picture: The central cross-section by the OXZ-plane of the 3D velocity field.*

4. Numerical experiments. In this section we present numerical results for two model problems in $[0,1]^d$ for $d = 2, 3$. In the first problem the wind is parallel to the x-axis and constant:

$$\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{4.1}$$

The second problem is the linearized "driven cavity" problem. In the 2D case the velocity function is suggested in [7]:

$$\mathbf{w} = \begin{pmatrix} \frac{r_2}{2\pi}\frac{e^{r_2 y}}{(e^{r_2}-1)} \sin\left(\frac{2\pi(e^{r_2 y}-1)}{e^{r_2}-1}\right)\left(1-\cos\left(\frac{2\pi(e^{r_1 x}-1)}{e^{r_1}-1}\right)\right) \\ -\frac{r_1}{2\pi}\frac{e^{r_1 x}}{(e^{r_1}-1)} \sin\left(\frac{2\pi(e^{r_1 x}-1)}{e^{r_1}-1}\right)\left(1-\cos\left(\frac{2\pi(e^{r_2 y}-1)}{e^{r_2}-1}\right)\right) \end{pmatrix}, \tag{4.2}$$

where $r_1 = 4$, $r_2 = 0.1$. This type of convection simulates a rotating vortex, whose center has coordinates (x_0, y_0), $x_0 \approxeq 0.831$, $y_0 \approxeq 0.512$, and $\max_\Omega |\mathbf{w}| \approxeq 1$ (Figure 4.1, left). In the 3D case the convection velocity field $\mathbf{w}$ is the solution of the "driven cavity" Stokes problem (Figure 4.1, right).

For the discretization method we use isoP_2-P_0 and isoP_2-P_1 finite elements defined on uniform triangulation (tetrahedrization) of a square (cubic) mesh in $[0,1]^d$. The velocity triangulation is built by connecting the midpoints on the edges of triangles or tetrahedra. In all the cases the convection term is stabilized by using the SUPG stabilization [36]. In the tables below h denotes the size for the pressure triangulations.

We use the block triangular matrix (1.9) as a right preconditioner in the Krylov subspace method for solving system (1.8). Some details of experiments may differ in the 2D and 3D cases (cf. below), since we used two different FE software packages to treat 2D and 3D problems, respectively. In the 2D experiments we used the BiCGStab and for 3D ones the full GMRES method was applied as an outer iterative solver. Note that the expense of one BiCGStab iteration approaches the cost of two GMRES iterations. The stopping criterion is the 10^{-6} decrease of the Euclidean norm of the residual. The approximate inverses involved in the application of the

Table 4.1

Number of the preconditioned iterations with the PCD preconditioner; isoP_2-P_1 FE; Neumann/Dirichlet boundary conditions in A_p.

Meshsize h	Viscosity ν			
	0.1	0.01	10^{-3}	10^{-4}
w is 2D cavity vortex, #BiCGStab				
1/32	12 / 12	24 / 24	57 / 67	248 / 769
1/64	12 / 12	22 / 22	64 / 81	619 / 1267
w is constant 2D wind, #BiCGStab				
1/32	12 / 13	27 / 21	124 / 28	1027 / 48
1/64	11 / 13	23 / 24	116 / 33	1425 / 51
w is constant 3D wind, #GMRES				
1/8	28 / 36	69 / 58	368 / 201	548 / 285
1/16	28 / 42	45 / 39	232 / 162	1108 / 430

preconditioner (1.9) were computed as follows. The application of $\hat{A}^{-1}$ to a vector is achieved via 3 multigrid W(1,1)-cycles with the alternating Gauss–Seidel method as a smoother. In the 2D case a W-cycle of geometric multigrid was used, whereas in the 3D case a V-cycle of algebraic multigrid was adopted. Both choices give a fairly good approximation to A^{-1} for all values of h and ν under consideration. Application of $(B\hat{M}_u^{-1}B^T)^{-1}$ and L_p^{-1} was evaluated using 10 V(4,4)-cycles in the 2D case and exact sparse factorization in the 3D case. We note that although the latter method is applicable to general meshes, it is asymptotically not optimal. If N is the number of the pressure degrees of freedom, then the method needs about $O(N^{2.2})$ flops for the computation of $O(N^{1.5})$ nonzero entries in triangular factors. This limits the size of L_p to several tens of thousands for the PC we used for running numerical tests. At the same time, in the case of isoP_2-P_1 elements, N is less than the size of matrix A by a factor of 24, and the practical limit for N is not very restrictive. L^{-1} was evaluated using interior iterations to provide a very good approximation of the inversion. Thus all the inverses involved in all Schur complement preconditioners were evaluated with pretty high accuracy.

In the first experiment we illustrate the effect of different boundary conditions in A_p on the performance of the PCD preconditioner. Recall that our analysis in section 2.1 suggests that for the problem with **w** from (4.1) one should implement Dirichlet conditions at the inflow boundaries, while for the problem with **w** from (4.2) one should use Neumann conditions on the entire boundary. In Table 4.1 we compare iteration counts for the PCD preconditioner with different boundary conditions in A_p in two and three dimensions and for two types of convection flow **w**. For the case of **w** from (4.1) we test two variants of boundary conditions in A_p: one consists in setting Neumann boundary conditions on the whole boundary; alternatively, we set Dirichlet boundary conditions on the inflow and Neumann conditions on the rest of the boundary. For the case of **w** from (4.2) the problem has only characteristic boundaries; thus we test setting either Dirichlet or Neumann conditions in A_p on the whole boundary. It was mentioned in section 2.1 that Dirichlet boundary conditions may not be imposed on the boundary nodes, since these nodes contribute to the set of pressure degrees of freedom. The Dirichlet condition is imposed on fictitious boundary nodes of an h-extension of the original mesh. Therefore, the actual boundary nodes are considered as interior in the extended mesh. This may be implemented in two ways. For a rectangular mesh we simply copy matrix entries for interior nodes to

Table 4.2
Number of preconditioned iterations for isoP_2-P_1 FE. PCD/BFBt/preconditioner (2.12).

Meshsize h	Viscosity ν			
	0.1	0.01	10^{-3}	10^{-4}
w is 2D constant wind, #BiCGStab				
1/32	13 / 8 / 22	21 / 6 / 35	28 / 8 / 39	48 / 10 / 46
1/64	13 / 12 / 27	24 / 8 / 34	33 / 7 / 34	51 / 12 / 40
1/128	13 / 17 / 23	24 / 16 / 30	41 / 6 / 37	62 / 9 / 37
w is 2D cavity vortex, #BiCGStab				
1/32	12 / 9 / 27	24 / 14 / 29	56 / 30 / 70	248 / 120 / 242
1/64	12 / 13 / 27	22 / 20 / 31	64 / 45 / 76	619 / 211 / 314
1/128	13 / 17 / 29	25 / 26 / 33	77 / 61 / 69	1053 / 349 / 446
w is 3D cavity vortex, #GMRES				
1/8	35 / 71 / 129	46 / 104 / 178	112 / 233 / 512	243 / 437 / 787
1/16	38 / 82 / 143	50 / 102 / 167	114 / 415 / 781	462 / 1402 / >2000

matrix entries for actual boundary nodes. However, for a general mesh one has to generate the fictitious mesh layer by reflecting the close-to-boundary layer of cells with respect to the actual boundary.

In L_p we use Neumann conditions only. We found that changing boundary conditions in L_p does not improve convergence rates. Results of the experiments in Table 4.1 are consistent with the analysis in section 2.1. In particular, better results with Neumann conditions for the cavity vortex test confirm that this is the right choice on characteristic boundaries, for the constant wind Dirichlet boundary conditions on the inflow show an advantage for small enough ν. This behavior is also expected, since the analysis in section 2.1, predicting this choice, was done for the limit case of $\nu = 0$. The same phenomenon was observed in numerical experiments for discretizations with isoP_2-P_0 finite elements. Thus, further in the experiments we will always define A_p with Neumann conditions for rotating $\mathbf{w}$ and Dirichlet conditions at the "inflow" for $\mathbf{w}$ from (4.1).

In Table 4.2 we compare convergence results for all three preconditioners tested for the isoP_2-P_1 discretization of the Oseen problem with $\mathbf{w}$ from (4.1) and (4.2), the latter case being examined in both two and three dimensions. All preconditioners demonstrate almost h-independent results except for the case of small ν. In general, rotating flow with small viscosity turns out to be a hard problem for all preconditioners.

The difficulty in treating the case of small ν may be related to the fact that Navier–Stokes flows become less stable in this case. For example, in [2] it was found that the first point of bifurcation for the 2D driven cavity problem occurs around $Re =$ 8.018. Therefore, for the case of small viscosity solving unsteady, rather then steady, Navier–Stokes equations can be more appropriate sometimes. After linearization the unsteady problem would lead to a slightly modified Oseen problem (1.1)–(1.3) with an additional positive-definite reaction term in the momentum equations. This additional term, when also included in a Schur complement preconditioner, may improve the performance of the preconditioned iterations for small values of ν [21].

Next we proceed to approximations with the isoP_2-P_0 finite element method. For this discretization with discontinuous pressure approximation the BFBt preconditioner shows a strong h-dependence result even for the simplest constant parallel flow. This is illustrated in Table 4.3.

In Table 4.4 we examine PCD and (2.12) preconditioners which demonstrate h-

Table 4.3
Number of preconditioned BiCGStab iterations for the BFBt preconditioner with isoP_2-P_0 FE.

Meshsize h	Viscosity ν			
	0.1	0.01	10^{-3}	10^{-4}
w is 2D constant wind				
1/32	39	35	52	71
1/64	80	65	93	147
1/128	174	141	154	219

Table 4.4
Number of preconditioned BiCGStab iterations for PCD/(2.12) *preconditioner with isoP_2-P_0 FE.*

Meshsize h	Viscosity ν			
	0.1	0.01	10^{-3}	10^{-4}
w is 2D constant wind				
1/32	9 / 17	11 / 24	21 / 36	68 / 58
1/64	9 / 13	12 / 23	21 / 32	57 / 66
1/128	9 / 11	14 / 23	25 / 31	50 / 59
w is 2D cavity vortex				
1/32	10 / 18	18 / 22	66 / 60	>2000 / 191
1/64	9 / 21	19 / 23	74 / 63	>2000 / 306
1/128	9 / 21	19 / 23	78 / 58	>2000 / 453

independent convergence rates at least for the simplest case (4.1). We note that in the PCD preconditioner we now use mixed approximation for the pressure Poisson problem: $L_p = (B\hat{M}_u^{-1}B^T)$. To define A_p, we set $F_p := r_{u\to p}\, A_x r_{p\to u}$, where A_x is the x-subblock of A (maybe with different boundary conditions), $r_{u\to p}$ and $r_{p\to u}$ are suitable mappings from $\mathbb{Q}_h$ to $\mathbb{V}_h$, and vice versa. The type of boundary conditions in A_p was taken the same as for isoP_2-P_1 discretizations. We observe that up to $\nu = 10^{-3}$ both methods exhibit feasible convergence rates, and for $\nu = 10^{-4}$ the PCD method fails to converge.

Finally, we remark that although Theorem 3.2 guarantees that all eigenvalues of preconditioned matrices lie in the right half of the complex plane and are bounded independent of h (at least for the PCD and (2.12) preconditioners), iteration counts in Tables 4.2 and 4.4 have some increase with $h \to 0$ in the case of small ν. A most likely explanation of this phenomenon is that the implicit dependence of the constants from bounds (3.3) and (3.4) on ν makes them not useful when $\nu \to 0$.

5. Conclusions. This paper studies a preconditioning technique for finite element discretizations of the Oseen problem arising from Picard linearizations of the steady Navier–Stokes equations. The preconditioner is block triangular and requires an approximation to the inverse of the pressure Schur complement matrix. We focus on several approaches for building the pressure Schur complement preconditioner. Two of them are well known from the literature and one is new. The preconditioners differ in implementation and performance for various discretizations and flow patterns. The paper gives an account of their properties and available theoretical results. We prove missing eigenvalue estimates and discuss some open implementation problems, such as the choice of an appropriate pressure boundary condition in the method of Kay, Loghin, and Wathen. Numerical experiments show that all the methods work satisfactorily (with mild dependence on ν) in the range of small and modest Reynolds numbers; however, they may experience serious loss of efficiency when the Reynolds number is larger.

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