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Livsic's Theorem for Semisimple Lie groups

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Abstract

In this note we show that strong generalizations of the measurable Livsic theorem for cocycles taking values in connected noncompact linear semisimple Lie groups, a canonical example being $SL(2, \mathbf{R})$, can be deduced from an elegant approach of Brin and Pesin to the dynamics of partially hyperbolic systems.

1 Introduction

Let $T : M \to M$ be a transitive Anosov diffeomorphism of a compact Riemannian manifold and let μ be a T-invariant probability measure which is equivalent to Lebesgue measure. Let $g : M \to G$ be a C^1 function taking values in a finite dimensional connected matrix Lie group G with norm $||.||$ and metric d. If G is a matrix group then we define the norm of $A \in G$ by $||A||^2 = Trace(AA^t)$ for $A \in G$.

In this note we are interested in measurable solutions $h : M \to G$ to the equation

$$h(Tx) = g(x)h(x) \quad \mu \text{ a.e.} \tag{1.1}$$

In this context we say that h is a coboundary for g. In particular we are interested in conditions under which a measurable coboundary h must have a Hölder continuous version h' such that $h = h'$ μ a.e.

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Livsic [8] studied the corresponding equation for a continuous real valued cocycle satisfying a Dini property over a transitive Anosov diffeomorphism preserving a smooth measure. He showed that a measurable solution h to equation 1.1 has a continuous version.

Quas [16] proved analogous results for essentially bounded, real valued coboundaries of continuous cocycles g where the base dynamics was a locally eventually onto continuous map or a minimal system, such as an irrational rotation of S^1.

Nicol and Pollicott extended this result to continuous cocycles taking values in Lie groups with nonpositive sectional curvature (same assumptions on the base dynamics). They also showed that essentially bounded measurable cocycles taking values in subgroups of $GL(n, \mathbf{R})$ (with base dynamics a transitive C^∞ Anosov diffeomorphism equipped with a measure equivalent to Lebesgue) are continuous. This paper (using a very different technique) removes the essentially bounded assumption- if a pinching condition on g and a smooth measure in the base are assumed.

Hypothesis 1 Assume that G satisfies the condition that there exist two cocompact lattices Γ, Γ' (i.e. G/Γ and G/Γ' are compact homogeneous spaces) such that $\Gamma \cap \Gamma' = \{e\}$.

This property holds for $G = \mathbf{R}^n, SL(2, \mathbf{R})$, and, more generally, connected noncompact linear semisimple Lie groups. The proof of this fact, suggested to us by S. Mozes and S. G. Dani, is given as an appendix. It is false, however, in the case of the Heisenberg group.

We recall that a diffeomorphism f of a compact Riemannian manifold M is Anosov if there exists a Df-invariant splitting $TM = E^u \oplus E^s$, $C > 0$ and $0 < \lambda < 1$ such that

$$||Df^n(v)|| \leq C\lambda^n ||v||, \text{ for } v \in E^s$$

and

$$||Df^{-n}(v)|| \leq C\lambda^n ||v||, \text{ for } v \in E^u,$$

where $n \geq 0$. We assume that T is Anosov, transitive and has an invariant measure μ which is equivalent to Lebesgue.

Recall that $Ad \cdot g : LG \to LG$ is the automorphism of the Lie algebra LG defined by $Ad \cdot g(v) = gvg^{-1}$.

Hypothesis 2 We assume that $g : M \to G$ satisfies the pinching condition: $\lambda < ||Ad \cdot g(x)|| < \frac{1}{\lambda}$ for all $x \in M$.

This property ensures that the group extension is a partially hyperbolic dynamical system (see definition 2.1).

Our main result is the following.

Theorem 1.1 *If G is a connected noncompact semisimple Lie group and g satisfies hypothesis 2 then any measurable solution $h : M \to G$ to the equation 1.1 must be equal to a Hölder continuous function a.e.*

Corollary 1.2 *If $g : M \to SL(2, \mathbf{R})$ satisfies hypothesis 2, then any measurable solution $h : M \to SL(2, \mathbf{R})$ to the equation 1.1 must be equal to a Hölder continuous function μ a.e.*

The following result of Nitica and Torok [10, Theorem 4, page 9] strengthens the regularity of $h(x)$ in equation 1.1.

Proposition 1.3 *If $g : M \to G$ is C^k and h is a continuous solution to (1.1) then h is C^{k-1}.*

Combined with Theorem 1.1 this has the following corollary.

Corollary 1.4 *If G satisfies hypothesis 1 and g is C^k and satisfies hypothesis 2 then any measurable solution $h : M \to G$ to the equation 1.1 must be equal to a C^{k-1} function μ a.e. In particular if g is C^∞, then h has a C^∞ version μ a.e.*

We shall base our proof of the theorem on Brin and Pesin's analysis of partially hyperbolic diffeomorphisms on compact manifolds [2] and Brin's work [1] originally for frame flows.

In section 2 we review these results on partially hyperbolic systems, which are a generalisation of the classical theory of Anosov systems. In section 3 we prove our main result.

2 Partially hyperbolic systems

Suppose that $T : M \to M$ is a transitive Anosov diffeomorphism of a compact Riemannian manifold preserving a smooth invariant measure μ.

There is a well-known theory concerning the properties of stable and unstable manifolds defined by

$$W^s(x) := \{y : d(T^n(x), T^n(y)) \to 0 \text{ as } n \to \infty\}$$

$$W^u(x) := \{y : d(T^n(x), T^n(y)) \to 0 \text{ as } n \to -\infty\}$$

In particular, for each $x \in M$, there are C^1 immersed submanifolds $W^u(x), W^s(x) \subset M$ with $T_x W^s(x) = E^s_x$ and $T_x W^u(x) = E^u_x$.

Definition 2.1 *We say $\hat{T} : N \to N$ is a partially hyperbolic diffeomorphism of a compact Riemannian manifold N if there is a splitting $TN = E^u \oplus E^s \oplus E^0$ and constants $C > 0$ and*

$$0 < \lambda < \rho_1 < 1 < \rho_2 < \frac{1}{\lambda}$$

such that $||D\hat{T}^n|_{E^s}|| \le C\lambda^n$, $||D\hat{T}^{-n}|_{E^u}|| \le C\lambda^n$ and $C\rho_1^n \le ||D\hat{T}^n|_{E^0}|| \le C\rho_2^n$ for $n \ge 0$ (where $||.||$ is the norm on TN).

We shall be interested in partially hyperbolic diffeomorphisms which arise as homogeneous extensions of Anosov diffeomorphisms of a compact Riemannian manifold M satisfying $||D\hat{T}^n|_{E^s}|| \leq C\lambda^n$, $||D\hat{T}^{-n}|_{E^u}|| \leq C\lambda^n$ for all $n \geq 0$. Suppose that Γ is a cocompact lattice of G. Then $H := G/\Gamma$ is a compact homogeneous space with a left invariant Haar measure ν.

We consider the compact manifold $N = M \times H$ and the diffeomorphism $\hat{T} : N \to N$ given by $\hat{T}(x, h\Gamma) = (Tx, g(x)h\Gamma)$, which preserves the measure $\hat{\mu} = \mu \times \nu$.

We assume that the cocycle g satisfies

$$\rho_1 \leq ||Ad \cdot g(x)|| \leq \rho_2 \tag{2.2}$$

for μ a.e. $x \in M$ where $\lambda < \rho_1 < 1 < \rho_2 < \frac{1}{\lambda}$ (where $||.||$ is, with abuse of notation, the norm on LG). With these assumptions $\hat{T} : N \to N$ is a C^1 partially hyperbolic diffeomorphism.

Example 2.2 *Suppose $M = \mathbf{R}^2/\mathbf{Z}^2$ and*

$$T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

In this case $\lambda = \frac{1}{\rho} = \frac{3+\sqrt{5}}{2}$. Suppose $G \subset M(n, \mathbf{C})$. We may choose the inner product $< A, B >:= tr(AB^t)$ on G which induces the norm $||A||^2 = tr(A\bar{A}^t)$. In this case the automorphism of $Ad \cdot g$ on the Lie algebra LG by $Ad \cdot g(X) = g(x)Xg(x)^{-1}$ has norm bounded by $||g(x)||$. If

$$g(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$$

then $||g(x)|| = \sqrt{a(x)^2 + \ldots + d(x)^2}$ and if $\frac{3+\sqrt{5}}{2} > \sqrt{a(x)^2 + \ldots + d(x)^2}$ for all $x \in T^2$ then the system is partially hyperbolic.

The following result extends the familiar results on stable and unstable manifolds for Anosov diffeomorphisms to the context of partially hyperbolic diffeomorphisms.

Proposition 2.3 *(Brin and Pesin [2], Hirsch et. al. [3])*

(i) Let $\hat{T} : N \to N$ be a C^1 partially hyperbolic diffeomorphism. For each $z \in N$ there are C^1 immersed stable and unstable submanifolds $W^s(z)$, $W^u(z)$, and a center submanifold $W^0(z) \subset M$ with $T_zW^s(z) = E^s_z$, $T_zW^u(z) = E^u_z$ and $T_zW^0(x) = E^0_z$.

(ii) The foliations $\{W^s(z)\}$, $\{W^u(z)\}$ and $\{W^0(z)\}$ are each Hölder continuous.

Remark 2.4 *In general the foliations may have only Hölder regularity, even though $\hat{T}$ is C^∞ [6].*

Proposition 2.5 *(Brin [1], page 170) Let $\hat{T} : N \to N$ be a C^1 partially hyperbolic diffeomorphism preserving a measure $\hat{\mu}$ equivalent to Lebesgue and suppose that $A \subset N$ is $\hat{T}$ invariant. Then A consists up to a set of $\hat{\mu}$ measure zero (mod 0) of the union of stable and unstable manifolds for $\hat{T}$.*

3 Proof of Theorem 1.1

Since G satisfies hypothesis 1 we may choose discrete subgroups $\Gamma, \Gamma' < G$ satisfying the conditions of hypothesis 1. Note that by assumption $G/\Gamma, G/\Gamma'$ are compact homogeneous spaces.

Define $H := G/\Gamma$. We now consider the compact manifold $N = M \times H$ and the diffeomorphism $\hat{T} : N \to N$ given by $\hat{T}(x, g\Gamma) = (Tx, g(x)g\Gamma)$. For ease of notation we will write $(x, g\Gamma)$ as (x, g). Note that G/Γ has an induced right and left invariant Haar measure ν_Γ, an induced metric which we will write as d_H and that $\hat{T}(x, g) = (Tx, g(x)g)$ preserves $\mu \times \nu_\Gamma$. We will use the notation d_M to refer to the metric in the base M. The measurable map $h : M \to G$ induces a measurable map $h : M \to H$ (we will use the same notation for both maps when it is clear which is meant).

Given $g \in G$ define the measurable sets

$$\Lambda_g = \{(x, h(x)g) : x \in M\} \subset M \times H$$

Each set Λ_g is invariant under the skew product

$$\hat{T} : M \times H \to M \times H$$

$$\hat{T}(x, g) = (Tx, g(x)g)$$

This is clear since

$$\hat{T}(x, h(x)g) = (Tx, g(x)h(x)g) = (Tx, h(Tx)g),$$

since $h(Tx) = g(x)h(x)$ μ a.e.

Given small $\delta > 0$, let $U(\delta)$ be a neighbourhood of the coset of the identity element in H of diameter equal to δ in the induced metric.

Consider the set $\Lambda_{U(\delta)} = \{\Lambda_g : g \in U(\delta)\}$. Note that $(\mu \times \nu_\Gamma)(\Lambda_{U(\delta)}) > 0$ and $\Lambda_{U(\delta)}$ is measurable with respect to the ergodic decomposition of $\hat{T}$. Thus by [1, page 170] the set $\Lambda_{U(\delta)}$ consists up to a set of measure zero of entire stable and unstable manifolds for $\hat{T}$. Without loss of generality, by adding or removing a set of zero measure, we can assume that $\Lambda_{U(\delta)}$ is simply a union of stable and unstable manifolds.

By Proposition 2.3(ii) the foliations of $M \times H$ into stable $\{W^s(x, \gamma)\}$ and unstable $\{W^u(x, \gamma)\}$ manifolds are Hölder continuous for some Hölder exponent (α say) and Hölder constant (C say). This means that for any $(x, g) \in M \times H$ there is a Hölder continuous map $\Pi_x^s : W^s(x) \to W^s(x, g)$, and similarly there is a Hölder continuous map $\Pi_x^u : W^u(x) \to W^u(x, g)$.

It is a standard result [4] that there exists $\delta' > 0$ such that if $d_M(x, y) < \delta'$ then there exists a unique point $x' \in M$ with $W^s(x) \cap W^u(y) = \{x'\}$ i.e. there is a "local product structure".

Consider those $x, y \in M$ for which $h(x)$ and $h(y)$ are well-defined and suppose that $d_M(x, y) = \delta < \delta'$. Since we know that $W^s(x, h(x)\gamma) \subset \Lambda_{U(\delta)}$ the above result on the Hölder nature of the foliations show that there exists $\gamma' = \gamma'(x') \in H$ such that $(x', h(x')\gamma') \in W^s(x, h(x)\gamma)$. Similarly there exists $\gamma'' = \gamma''(y) \in H$ such that $(y, h(y)\gamma'') \in W^u(x', h(x')\gamma')$.

Since the foliations $W^s(x,\gamma)$ and $W^u(x,\gamma)$ are C^1 immersed submanifolds which are Hölder continuous in (x,γ), we can use the triangle inequality to estimate

$$d((x, h(x)\gamma), (y, h(y)\gamma'')) < C d_M(x,y)^\alpha < C\delta^\alpha.$$

By the triangle inequality

$$d_H(h(x), h(y)) \leq d_H(h(x), h(x)\gamma) + d_H(h(x)\gamma, h(y)\gamma'') + d_H(h(y)\gamma'', h(y))$$

Moreover, since the map $\gamma \mapsto f(\gamma) := \sup_{g \in H} d_H(g, g\gamma)$ is continuous, for any $\epsilon > 0$ we can assume δ was chosen sufficiently small that $f(\gamma) < \epsilon$ if $\gamma \in U(\delta)$. In particular, we conclude $d_H(h(x), h(y)) \leq 2\epsilon + C\delta^\alpha$. Since ϵ is arbitrary, we see that $x \mapsto h(x)$ is continuous (in fact Hölder continuous).

We wish to conclude the continuity of $h : M \to G$, rather than $h : M \to H$. We will denote by $\bar{h}(x)$ the value of $h : M \to G$ and $h_H(x)$ the possible values (in G) of $h : M \to H$ i.e. $h_H(x) = \{\bar{h}(x)\gamma : \gamma \in \Gamma\}$ is a multivalued function with continuous branchs and $h(x) \in h_H(x)$. It remains to show that $h(x)$ which, a priori, could change branchs in a measurable way does, in fact, involve a single constant choice.

Now we can use hypothesis 2 and the second cocompact lattice Γ'. Define $H' := G/\Gamma'$ and repeat the construction given above.

Note that by hypothesis 1 the sets $h_H(x) = \{\bar{h}(x)\gamma : \gamma \in \Gamma\}$ and $h_{H'}(x) = \{\bar{h}(x)\gamma' : \gamma' \in \Gamma'\}$ have intersection only when $\gamma = \gamma' = e$. Thus there is a well-defined solution $h(x)$ which, by local considerations, is Hölder continuous.

4 Related results

It is interesting to ask whether the generalization of this result holds for the identity

$$h(Tx)f(x) = h(x)g(x) \quad \mu a.e. \tag{4.3}$$

In this case we say that f and g are cohomologous via h. In fact h induces a conjugacy between extensions with cocycles f and g respectively which preserves the fibre G.

A positive answer to this question would imply that a measurable conjugacy (which preserves the fibre) between extensions with cocycles f and g respectively is necessarily continuous.

However, Walkden[19] has shown that if G is non-connected and noncompact then there exist smooth cocycles f, g which are cohomologous via a measurable cobounding function h which has no continuous version.

In the context of cohomologous cocycles f, g we may arrange that h is C^0 but not C^α. In the case of a coboundary $g(x) = h(Tx)h(x)^{-1}$ then h has a C^{k-1} version if g is C^k.

Consider a linear hyperbolic toral automorphism $T : \mathbf{R}^4/\mathbf{Z}^4 \to \mathbf{R}^4/\mathbf{Z}^4$ defined by a matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in SL(4, \mathbf{Z})$$

where $A, B \in SL(2, \mathbf{Z})$.

By structural stability, we know that for any nearby diffeomorphism $S : \mathbf{R}^4/\mathbf{Z}^4 \to \mathbf{R}^4/\mathbf{Z}^4$ there exists a unique conjugating homeomorphism $\pi : \mathbf{R}^4/\mathbf{Z}^4 \to \mathbf{R}^4/\mathbf{Z}^4$ (i.e. $T \circ \pi = \pi \circ S$).

We can assume that S is of the form $S(x, y) = (Ax, By + v\phi(x))$, where $Bv = \lambda v$ where v is the maximal eigenvector and $\phi : \mathbf{R}^4/\mathbf{Z}^4 \to \mathbf{R}$ is some C^∞ function, sufficiently small in the C^0 topology to ensure the existence of a unique conjugating homeomorphism. The conjugacy takes the form $\pi(x, y) = (x, y + \psi(x)v)$. In particular, $\psi(x)$ satisfies

$$\lambda\psi(x) - \psi(Ax) = \phi(x)$$

An argument based on Fourier series similar to that given by de la Llave [6, Theorem 6.3] shows that given α we may choose A and B so that ψ is C^1 but not $C^{1+\alpha}$.

Thus the conjugating homeomorphism is differentiable (with derivative $D\pi$) and we can clearly differentiate the conjugating identity to write

$$(DT) \circ \pi \cdot (D\pi) = (D\pi) \circ S \cdot (DS).$$

Since T is linear, we have that DT is constant on $\mathbf{R}^4/\mathbf{Z}^4$ and thus $(DT) \circ \pi = DT$, and so if we define

$$\begin{aligned} g &= DT : \mathbf{R}^4/\mathbf{Z}^4 \to SL(4, \mathbf{R}) \\ h &= D\pi : \mathbf{R}^4/\mathbf{Z}^4 \to SL(4, \mathbf{R}) \text{ and} \\ f &= DT : \mathbf{R}^4/\mathbf{Z}^4 \to GL(4, \mathbf{R}) \end{aligned}$$

then

$$g(x)h(x) = h(Sx)f(x)$$

where $h(x)$ is continuous but not C^α.

We can hope to replace the hypothesis that there exists a measurable solution $h(x)$ by an intrinsic property of f. Let G have metric d. Let (T, X, μ) be a mixing subshift of finite type. We call d a Lipschitz metric if G is complete with respect to d and if there exists for every $g \in G$ a constant α_g such that the following condition holds

$$\begin{aligned} \frac{1}{\alpha_g} &\le \min\{d(gh, gh'), d(hg, h'g)\} \\ &\le \max\{d(gh, gh'), d(hg, h'g)\} \le \alpha_g d(h, h') \end{aligned}$$

for all $h, h' \in G$. We call a group Lipschitz if there exists a Lipschitz metric on G. If (G, d) is a Lipschitz group then we define the distortion of g by $a_g := \inf\{\alpha_g \ge 1 : \alpha_g$ satisfies the condition above for all $h, h' \in G\}$.

We say that a cocycle f has *bounded distortion* if there exists a constant $c \geq 1$ with $\alpha_\delta(f(T^n x)...f(x)) \leq c$ for μ-a.e. $x \in X$ and every $n \in \mathbf{Z}$.

In this context we mention the following proposition due to Schmidt [18].

Proposition 4.1 *Let $f: X \longmapsto G = \mathrm{GL}(n, \mathbf{C})$ be a Borel map such that $\mathbf{f}: \mathbf{Z} \times \mathbf{X} \longmapsto \mathbf{G}$ has bounded distortion with respect to the natural metric. Then f is cohomologous to a Borel map $g: X \longmapsto \mathrm{U}(n)$.*

There are many open questions to consider. Is it possible to have a cocycle g which doesn't satisfy the pinching condition taking values in a semisimple Lie group which satisfies $g(x) = h(Tx)h(x)^{-1}$ for a measurable but not continuous cobounding function h?

One may also ask if Theorem 1.1 holds if the base transformation is an Axiom A diffeomorphism equipped with a Gibbs measure. If Proposition 2.5 can be extended to Axiom A base transformations with Gibbs measures then our main theorem does as well.

5 Appendix: Some groups which satisfy hypothesis 1

We begin by recalling a result of A. Borel which guarantees the existence of cocompact lattices.

Proposition 5.1 *(Borel) [Ra, Chapter 14] Let $\tilde{M}$ be a noncompact symmetric space, then it admits a lattice Γ with compact quotient.*

S Mozes and S. G. Dani have suggested the following approach to checking which groups satisfy hypothesis 1. Recall that any proper subgroup of G has dimension strictly less than G.

Using Borel's theorem with $G = \tilde{M}$, choose any cocompact lattice Γ. Suppose $\Gamma = \{\gamma_1, ..., \gamma_n, ...\}$ is an enumeration of this lattice. Let $A_{i,j} = \{\gamma \in G : \gamma\gamma_i\gamma^{-1} = \gamma_j\}$ and let $C_G(\gamma_i)$ denote the centralizer of γ_i (in the group G). Note that the dimension of $C_G(\gamma_i)$ is strictly less than that of G, for each i, since $C_G(\gamma_i)$ is a proper subgroup of G.

We claim that $A_{i,j} = t_{i,j}^{-1} C_G(\gamma_i)$ for some $t_{i,j} \in G$. Suppose that $A_{i,j} \neq \emptyset$ and observe that $A_{j,i} = \{\gamma^{-1} : \gamma \in A_{i,j}\} \neq \emptyset$. Fix an element $t_{i,j} \in A_{j,i}$ and let $\gamma \in A_{i,j}$. Note that $(t_{i,j}\gamma)\gamma_i(t_{i,j}\gamma)^{-1} = t_{i,j}\gamma_j t_{i,j}^{-1} = \gamma_i$ and hence $t_{ij}\gamma \in C(\gamma_i)$. Hence we see that $\gamma \in t_{i,j}^{-1} C(\gamma_i)$, and our claim follows.

The union $C := \cup_{1 \leq i,j} A_{i,j}$ consists of all elements in G which conjugate at least one element in Γ to another. Since this is a countable union, C again has dimension strictly less than that of G.

Fix an element $g \in G - C$ then the subgroup $\Gamma' = g^{-1}\Gamma g$ is a cocompact lattice. Furthermore, by construction Γ and Γ' satisfy hypothesis 1.

S. G. Dani has made the observation that if the group has a finite centre then the above argument works for lattices containing no nontrivial central elements.

Since in a linear semisimple Lie group any lattice contains a subgroup of finite index which is torsion-free (and hence has trivial intersection with the centre) there are always cocompact lattices with trivial intersection with the centre [17, Corollary 6.13, Theorem 6.15 and Remark 6.18]. Thus all linear semisimple Lie groups satisfy hypothesis 1.

Unfortunately this argument fails in the case of the Heisenberg group. Any lattice in a connected linear semisimple Lie group contains a subgroup of finite index in the centre [17, Corollary 5.17]. Hence this will also be true of the intersection of two lattices. Thus if the centre of the group is infinite then the intersection of two lattices cannot be trivial. The Heisenberg group has an infinite centre so the intersection of two cocompact lattices in the Heisenberg group must contain more than the identity element.

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