BIRKHOFF SUM CONVERGENCE OF FRÉCHET OBSERVABLES TO STABLE LAWS FOR GIBBS-MARKOV SYSTEMS AND APPLICATIONS.

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Abstract. We use a Poisson point process approach to prove distributional convergence to a stable law for non square-integrable observables $\varphi : [0,1] \to \mathbb{R}$, mostly of the form $\varphi(x) = d(x, x_0)^{-\frac{1}{\alpha}}$,$0 < \alpha \leq 2$, on Gibbs-Markov maps. A key result is to verify a standard mixing condition, which ensures that large values of the observable dominate the time-series, in the range $1 < \alpha \leq 2$. Stable limit laws for observables on dynamical systems have been established in two settings: "good observables" (typically Hölder) on slowly mixing non-uniformly hyperbolic systems and "bad" observables (unbounded with fat tails) on fast mixing dynamical systems. As an application we investigate the interplay between these two effects in a class of intermittent-type maps.

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1. Introduction

In this paper we consider distributional convergence to stable laws for non square-integrable observables $\varphi : [0,1] \to \mathbb{R}$ of form $\varphi(x) = d(x,x_0)^{-\frac{1}{\alpha}}$,$0 < \alpha \leq 2$, on Gibbs-Markov maps of the unit interval $[0,1]$ ($x_0 \in [0,1]$). Our results imply distributional convergence, in some parameter regimes, to stable laws for non square-integrable observables on certain systems modeled by first return time Young Towers in which the base map is Gibbs-Markov, in particular intermittent-type maps of the unit interval.

Most of our results consider distance-like observables $\varphi(x) = d(x,x_0)^{-\frac{1}{\alpha}}$, where $\alpha \in (0,2)$ and $x_0 \in [0,1]$. But our result on mixing conditions, Theorem 8.1, extends to observables φ which are regularly varying with stable index α and for which, for sufficiently large t, $\|\varphi \mathbf{1}_{\{|\varphi|<t\}}\|_{BV} \leq Kt$ for some constant K.

Stable limit laws for observables on dynamical systems have been established in two somewhat distinct settings: "good observables" (typically Hölder) on slowly mixing non-uniformly hyperbolic systems and "bad" observables (unbounded with fat tails) on fast mixing dynamical systems.

For results on the first type we refer to the influential papers [Gou04, Gou07] and [MZ15]. In the setting of "good observables" (typically Hölder) on slowly mixing non-uniformly hyperbolic systems the technique of inducing on a subset of phase space and constructing a Young Tower has been used with some success. "Good" observables lift to well-behaved observables lying in a suitable Banach space on the Young Tower. This is not the case in general with unbounded observables with fat tails, though in [Gou04] the induction technique permits analysis of an observable which is unbounded at the fixed point $x = 0$ in a family of intermittent maps. As $x = 0$ is not in the Young Tower the observable lifts to a function on the Tower which is bounded on each column of the Tower and with sufficient regularity for spectral techniques to apply.

For general results on distributional and functional stable laws for non-square integrable observables using a Poisson point process approach we refer to the papers of Marta Tyran-Kaminska [TK10a, TK10b]. Tyran-Kaminska considers convergence of Birkhoff sums to stable laws and corresponding functional convergence in the J_1 topology to Lévy processes. She uses a point process approach but her work explicitly excludes clustering behavior, and so is not applicable to observables $\varphi(x) = d(x,x_0)^{-\frac{1}{\alpha}}$ maximized at a periodic point x_0 (for which clustering of extremes is expected). In the setting of Gibbs-Markov maps Tyran-Kaminska shows, among other results, that functions which are measurable with respect to the Gibbs-Markov partition and in the domain of attraction of a stable law with index α converge (under the appropriate scaling) in the J_1 topology to a Lévy process of index

α [TK10b, Theorem 3.3, Corollaries 4.1 and 4.2]. Her result is not applicable in our setting as $\varphi(x) = d(x, x_0)^{-\frac{1}{\alpha}}$ is not measurable with respect to the Gibbs-Markov partition in the settings we consider. However in the setting of weakly-mixing AFU maps she showed [TK10b, Theorem 4.4] that if an observable h is in the domain of attraction of a stable law, piecewise monotonic with finitely many branches, and the return time to the set $(|h| > \varepsilon b_n)$ diverges then the associated point process N_n converges to that of a Lévy process of index α, N_α. This implies convergence to a stable law in the case of $0 < \alpha < 1$. As a corollary, in [TK10b, Example 4.4], Tyran-Kaminska shows that for the specific observable $h(y) = |y - y_0|^{-1/\alpha}$ in the setting of AFU maps one obtains stable laws in the case $0 < \alpha < 1$ if the distinguished point y_0 is such that the return times of shrinking balls centered at y_0 diverges to infinity.

It is interesting to note though that in the case of a slowly-mixing intermittent map with an indifferent fixed point at $x = 0$ and a Hölder observable φ, $\varphi(0) \neq 0$, the constant $\varphi(0)$ may be induced as a measurable function on the Gibbs-Markov base of the usual first return tower representation. This approach is used by Melbourne and Zweimuller [MZ15] to prove convergence to stable laws for Hölder functions on slowly-mixing systems modeled by a Young Tower.

Marta Tyran-Kaminska's work is based on a Poisson point process approach described by Durrett and Resnick [Res86, Res87, DR78]. This paper follows a similar approach to the scheme laid out by Tyran-Kaminska in that we require convergence of a counting process to a Poisson process and a form of decay of correlations estimate for a truncation of the observable that ensures the Birkhoff sum of small values of φ do not contribute too much and it is the large values that dominate. We stress that we do not prove functional convergence in the J_1 topology but rather distributional convergence. In fact, as Tyran-Kaminska shows in [TK10b, Theorem 1.1, Example 1.1] in situations where the counting process exhibits clustering convergence in the J_1 topology does not hold. Recent work has shown that in some settings where J_1 convergence does not hold that convergence is possible in the weaker M_1 topology [MZ15] and in the F' topology [FFT24]. We refer to these papers for helpful discussions of the relevant topologies and related results.

In the cases where we obtain distributional rather than functional convergence, we need only validate the weaker conditions of Davis and Tsing [DH95, Theorem 3.1] rather than the stronger condition of [TK10b, Theorem 1.1 Condition (1.5)]. This allows us to extend the results of [TK10b].

In the setting of Gibbs-Markov maps (or more generally Rychlik maps) Freitas, Freitas and Magalhaes [FFMa20] have proved that observables of the type $d(x, x_0) = d(x, x_0)^{-\frac{1}{\alpha}}$, $x_0 \neq 0$, have counting processes that converge to a simple Poisson point process if x_0 is not periodic and a "clustered' point process if x_0 is periodic. The convergence to a simple point process in the non-periodic case is a consequence of [TK10b, Theorem 4.4]. Furthermore if $0 < \alpha < 1$ then [FFT20] have shown functional convergence of the rescaled time-series for this observable in the F' topology, which implies convergence of the scaled Birkhoff sum to a stable law. One contribution of this paper is Theorem 8.1 which verifies a mixing condition in the case $1 < \alpha < 2$ and extends the stable law convergence to the parameter range $1 < \alpha < 2$.

One question that arose in our investigation (that was not satisfactorily resolved) can be stated simply. Suppose $(T_\gamma, [0,1], \mu_\gamma)$ is a LSV [LSV99] map of the unit interval (see Section 6) and μ_γ is the Lebesgue equivalent invariant measure for T_γ. Suppose φ has support in $[1/2, 1]$, $\int \varphi \, d\mu_\gamma = 0$, and locally, near $x_0 \in [1/2, 1]$ is of form $d(x, x_0)^{-\frac{1}{\alpha}}$ (elsewhere Hölder). We are able to show that the Birkhoff sum of the induced map on $[1/2, 1]$ converges in distribution to a stable law with index α. In certain parameter regions for $1 \leq \alpha \leq 2$ and $0 < \gamma < 1$, namely $\frac{1}{\gamma} \leq \alpha \leq 1 + \frac{1}{\gamma^2} - \frac{1}{\gamma}$, we are able to show that the stable law with index α lifts from that of the induced observable to give a stable law for the original observable φ. Does a stable law of index α lift for all $\alpha < \frac{1}{\gamma}$ for all parameter ranges of $1 \leq \alpha \leq 2$ and $0 < \gamma < 1$ if $\int \varphi \, d\mu_\gamma = 0$ and φ has support in $[1/2, 1]$?

Our main results are given in the section 8. We first give some background.

2. Probabilistic tools

In this section, we review some topics from Probability Theory.

2.1. Regularly varying functions and domains of attraction.

We refer to Feller [Fel71] or Bingham, Goldie and Teugels [BGT87] for the relations between domains of attraction of stable laws and regularly varying functions. For φ regularly varying we define scaling constants b_n (related to the index) and c_n (centering) by

Definition 2.1. *Given a regularly varying function φ of index $\alpha \in (0, 2)$ on a probability space (X, μ), introduce:*

– *a scaling sequence $(b_n)_{n\geq 1}$ by*

$$\lim_{n\to\infty} n\mu(|\varphi| > b_n) = 1. \tag{2.1}$$

– *a centering sequence $(c_n)_{n\geq 1}$ by*

$$c_n = \begin{cases} 0 & \text{if } \alpha \in (0,1) \\ n\mathbb{E}[\varphi] & \text{if } \alpha \in (1,2) \end{cases}. \tag{2.2}$$

The constants p, q are defined by

$$p = \lim_{t\to\infty} \frac{\mu(\varphi > t)}{\mu(|\varphi| > t)} \text{ and } q = 1 - p.$$

Note that if $\varphi = d(x, x_0)^{-\frac{1}{\alpha}}$ is an observable on the unit interval $[0, 1]$ equipped with a Lebesgue equivalent measure and $x_0 \in [0, 1]$ then $b_n \sim n^{1/\alpha}$, where $\sim$ means there exists C_1, $C_2 > 0$ with $C_1 n^{1/\alpha} \leq b_n \leq C_2 n^{1/\alpha}$. Note also that $p = 1$ as $\varphi > 0$. As we did above, we will sometimes write $\mathbb{E}[\varphi]$ for the expectation of an observable when the measure is clear from context.

Remark 2.2. *When $\alpha \in (0,1)$ then φ is not integrable and one can choose the centering sequence (c_n) to be identically 0. When $\alpha = 1$, it might happen that φ is not integrable, and it is then necessary to truncate. We do not consider the case $\alpha = 1$. In the literature if centering is needed it is often specified as $c_n = n\mathbb{E}(\varphi \mathbf{1}_{\{|\varphi| \le b_n\}})$ but we have opted for a simpler centering. By* [DH95, Remark 3.1]*, for $1 < \alpha \le 2$, if φ is a regularly varying function of index α then $n\mathbb{E}(\varphi)$ may be used in centering rather than the truncation $\varphi \mathbf{1}_{\{|\varphi|<b_n\}}$ above. The same limiting distribution S is obtained though shifted by the constant $(p-q)\frac{\alpha}{\alpha-1}$. More precisely*

$$\frac{1}{b_n}\left(\sum_{j=1}^{n}[\varphi \circ T^j - \mu(\varphi)]\right) \to_d S - (p-q)\frac{\alpha}{\alpha-1}$$

where $q = 1-p$. This is a consequence of

$$\frac{n}{b_n}[\mathbb{E}(\varphi) - \mathbb{E}(\varphi \mathbf{1}_{\{|\varphi|<b_n\}})] = \frac{n}{b_n}\mathbb{E}[\varphi \mathbf{1}_{\{(b_n,\infty)\}}(|\varphi|)] \to (p-q)\frac{\alpha}{\alpha-1}$$

using Karamata (see Proposition 2.3), so by convergence of types

$$\frac{1}{b_n}\left(\sum_{j=1}^{n}\varphi \circ T^j - c_n\right) \to_d S - (p-q)\frac{\alpha}{\alpha-1}$$

We will use the following asymptotics for truncated moments, which can be deduced from Karamata's results concerning the tail behavior of regularly varying functions. Recall that $p = \lim_{x\to\infty}\frac{\nu(\varphi>x)}{\nu(|\varphi|>x)}$.

Proposition 2.3 (Karamata, [BGT87])**.** *Let φ be regularly varying with index $\alpha \in (0,2)$.*

The following hold for all $\varepsilon > 0$:

(a) $\lim\limits_{n\to\infty} n\mu(|\varphi| > \varepsilon b_n) = \varepsilon^{-\alpha}$ (from the definition of b_n and the regular variation of φ)

(b) If $k > \alpha$ then

$$\mathbb{E}(|\varphi|^k \mathbf{1}_{\{|\varphi|\le u\}}) \sim \frac{\alpha}{k-\alpha}u^k\mu(|\varphi| > u) \text{ as } u \to \infty$$

In particular:

(c) if $\alpha \in (0,2)$ then

$$\mathbb{E}(|\varphi|^2 \mathbf{1}_{\{|\varphi|\le\varepsilon b_n\}}) \sim \frac{\alpha}{2-\alpha}(\varepsilon b_n)^2\mu(|\varphi| > \varepsilon b_n)$$

(d) if $\alpha \in (0,1)$ then

$$\mathbb{E}(|\varphi| \mathbf{1}_{\{|\varphi|\le\varepsilon b_n\}}) \sim \frac{\alpha}{1-\alpha}\varepsilon b_n\mu(|\varphi| > \varepsilon b_n)$$

(e) if $\alpha \in (1,2)$ then

$$\lim_{n\to\infty}\frac{n}{b_n}E(\varphi \mathbf{1}_{\{|\varphi|>\varepsilon b_n\}}) = \varepsilon^{1-\alpha}(2p-1)\alpha/(\alpha-1)$$

2.2. **Lévy α-stable processes.** A more detailed discussion of Lévy processes is given in [TK10a, TK10b].

$X(t)$ is a Lévy stable process if $X(0) = 0$, X has stationary independent increments and $X(1)$ has an α-stable distribution. Recall that the distribution F of a random variable X is called α-stable if there are constants γ_n such that for each n, if X_i are iid with distribution F then

$$\sum_{j=1}^{n} X_j + \gamma_n \sim n^{\frac{1}{\alpha}} X_1$$

The Lévy-Khintchine representation for the characteristic function of an α-stable random variable $X_{\alpha,\beta}$ with index $\alpha \in (0,2)$ and parameter $\beta \in [-1,1]$ has the form

$$\mathbb{E}[e^{itX}] = \exp\left[ita_\alpha + \int (e^{itx} - 1 - itx\mathbf{1}_{[-1,1]}(x))\Pi_\alpha(dx)\right]$$

where

- $a_\alpha = \begin{cases} \beta\frac{\alpha}{1-\alpha} & \alpha \neq 1 \\ 0 & \alpha = 1 \end{cases}$,
- Π_α is a Lévy measure given by

$$d\Pi_\alpha = \alpha(p\mathbf{1}_{(0,\infty)}(x) + (1-p)\mathbf{1}_{(-\infty,0)}(x))|x|^{-\alpha-1}dx$$

- $p = \dfrac{\beta+1}{2}$.

Note that p and β may equally serve as parameters for $X_{\alpha,\beta}$. We will drop the β from $X_{\alpha,\beta}$, as is common in the literature, for simplicity of notation and when it plays no essential role.

3. Stable law convergence

Let T be a measure preserving transformation of a probability space $(X, \mu, \mathcal{B})$.

Given $\varphi : X \to \mathbb{R}$ measurable, we define the scaled Birkhoff sum by

$$S_n := \frac{1}{b_n}[\sum_{j=0}^{n-1} \varphi \circ T^j - c_n], \tag{3.1}$$

for some real constants $b_n > 0$, c_n.

We say S_n converges to a stable law of index α if

$$S_n \xrightarrow{d} X_\alpha$$

for some random variable X_α with an α-stable distribution.

4. Poisson point processes

Suppose φ is an observable on a dynamical system (T, X, μ) with stable index α and scaling constants b_n and c_n. Let $B \subset (0, \infty) \times \mathbb{R} \setminus \{0\}$.

Define the counting process

$$N_n = \#\{(j, (\varphi \circ T^{j-1} - c_n)/b_n \in B\}$$

For each $x \in (X, \mu)$, $N_n(x)$ is an integer valued counting process on $(0, \infty) \times \mathbb{R} \setminus \{0\}$.

In our setting of Gibbs-Markov maps, Freitas, Freitas and Magalhaes [FFMa20] have proved convergence of the counting measure N_n (for (T, X, μ) a Gibbs-Markov map and $\varphi(x) = d(x, x_0)^{-1/\alpha}$) to a Poisson process which has the general form of [DH95, Corollary 2.4], namely

$$N = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \delta_{P_i Q_{ij}}$$

where $\sum_{j=1}^{\infty} \delta_{P_i}$ is a Poisson process with intensity measure Π_α and $\sum_{j=1}^{\infty} \delta_{Q_{ij}}$ are point processes taking values in $[-1, 1]$ distributed according to a measure ν. All point processes are mutually independent. In a dynamical setting, to which this Poisson point process is well suited, the Q_{ij}'s represent the "clustering" around an exceedance P_i (which is chosen to be the largest value in the cluster).

5. Gibbs-Markov Maps.

We consider the following class of ergodic maps of $X = [0, 1]$. Let m denote Lebesgue measure and let μ be a Lebesgue equivalent measure with density bounded above and away from zero below. Let $\mathcal{P}$ be a countable partition of $[0, 1]$ (mod m) into open intervals.

We suppose that all partition elements $A_i \in \mathcal{P}$ have $m(A_i) > 0$. A μ measure-preserving transformation T on X is a Gibbs-Markov map if

(1) $\mathcal{B}$ is the smallest σ-algebra which contains $\bigvee_{n \geq 0} T^{-n}\mathcal{P}$ which is complete with respect to m;

(2) Markov property: for all $A_i \in \mathcal{P}$, TA_i consists of a union of partition elements and there exists $C > 0$ such that $m(TA_i) > C$ for all i. If $T : A_i \to X$ is onto X mod m for all i, we say that T has "full branches".

(3) Local invertibility: for all $A_i \in \mathcal{P}$, $T : A_i \to TA_i$ is invertible.

(4) Expansitivity: There exists $C > 1$ such that $|T^{'}(x)| > C$ for all x where defined.

(5) Bounded Distortion: There exist constants $C > 0$ and $\lambda \in (0, 1)$ such that for all $A \in \bigvee_{j=0}^{n} T^{-j}\mathcal{P}$ and all $x, y \in A$,

$$\left| \frac{DT(x)}{DT(y)} - 1 \right| \leq C\lambda^n$$

A Gibbs-Markov map T has exponential decay in $\mathrm{BV}(X)$, meaning that there are $\lambda \in (0,1)$, $C>0$ such that the transfer operator $P: L^1(\mu) \to L^1(\mu)$ defined by

$$\int_X f\circ T \cdot g \, d\,\mu = \int_X f \cdot P(g)\, d\,\mu, \qquad \text{for all } f \in L^\infty(\mu), g \in L^1(\mu)$$

satisfies

$$\|P^k(g)\|_{\mathrm{BV}} \le C\lambda^k \|g\|_{\mathrm{BV}}, \qquad \text{for } g \in \mathrm{BV}(X) \text{ with } \int_X g\, d\,\mu = 0, \text{ and } k \ge 0 \tag{5.1}$$

6. Intermittent Maps.

Here we consider a simple class of intermittent type maps $T_\gamma : [0,1] \to [0,1]$, which we will call LSV maps as defined by [LSV99], given by

$$T_\gamma(x) := \begin{cases} (2^\gamma x^\gamma + 1)x & \text{if } 0 \le x < \frac{1}{2}; \\ 2x-1 & \text{if } \frac{1}{2} \le x \le 1. \end{cases} \tag{6.1}$$

For $\gamma \in [0,1)$, there is a unique absolutely continuous ergodic invariant probability measure μ_γ with density h_γ bounded away from zero and satisfying $h_\gamma(x) \sim Cx^{-\gamma}$ for x near zero. The existence of stable laws, and moreover the existence of functional limit theorems or weak invariance principles for Hölder functions on this class of intermittent maps has been thoroughly examined in [Gou04]. For instance, when $\gamma \in [0,1/2)$, $n^{-1/2}\sum_{i=0}^{n-1} \varphi \circ T_\gamma^j$ follows the CLT ; when $\gamma = 1/2$ and $\varphi(0) \neq 0$, $(n \log n)^{-1/2} \sum_{i=0}^{n-1} \varphi \circ T_\gamma^j$ follows the CLT; when $\gamma \in (1/2,1)$ and $\varphi(0) \neq 0$, $n^{-\gamma}\sum_{i=0}^{n-1} \varphi \circ T_\gamma^j$ follows a stable law where the index is γ^{-1}. Gouëzel [Gou04, Theorem 1.3] gives the characteristic function of the stable law for normalized Hölder φ as

$$\exp\left(-c|t|^{\frac{1}{\gamma}}(1 - i\beta \mathrm{sign}(t)\tan(\pi/2\gamma))\right)$$

where $\beta = \mathrm{sign}(\varphi(0))$ and

$$c = \frac{h_\gamma(1/2)}{4\gamma^{\gamma^{-1}}} \varphi(0)^{\gamma^{-1}} \Gamma(1 - \frac{1}{\gamma}) \cos(\pi/2\gamma)$$

The dependence of the characteristic function on only $\varphi(0)$ and $h_\gamma(1/2)$ is explained by the fact that the stable law for φ may be obtained by inducing (and then lifting) the constant function $\varphi(0)$ on the usual Young Tower for T_γ with base $[1/2,1]$.

In this paper in the setting of LSV maps we consider "bad" observables, for example $\varphi(x) = d(x,x_0)^{-\frac{1}{\alpha}}$. Our result in this setting is Corollary 8.4. For an observable φ which behaves locally as $d(x,x_0)^{-\frac{1}{\alpha}}$ close to a point $x_0 \neq 0$ and is Hölder elsewhere one expects a competition between the stable law coming from the slow-mixing property of the LSV map if $\gamma \in (1/2,1)$ and the stable law arising from the tail of the unbounded observable φ. One technical issue that arises immediately is to prove the convergence to a stable law for φ in a slowly mixing system. A natural technique to try is to induce, prove that the induced system satisfies a stable law and then lift. If $\frac{1}{\alpha} \ge \gamma$ this approach works in a straightforward manner. Furthermore if $\gamma > \frac{1}{\alpha}$ and $\varphi(0) - \mathbb{E}[\varphi] \neq 0$ then a stable law of index $\frac{1}{\gamma}$ holds for a restriction of the observable in the neighborhood of the indifferent fixed point. This effect dominates and in fact we obtain the same stable law with index $\frac{1}{\gamma}$ we would obtain if φ were Hölder with $\varphi(0) - \mathbb{E}[\varphi] \neq 0$ i.e. with the same formula for β and c above with $\varphi(0)$ replaced by $\varphi(0) - \mathbb{E}[\varphi]$.

However suppose $1 < \alpha < 2$ and φ is locally of form $d(x, x_0)^{-\frac{1}{\alpha}}$, Hölder elsewhere, with $\varphi(0) - \mathbb{E}[\varphi] = 0$, for example with $\mathbb{E}[\varphi] = 0$ and with support bounded away from the indifferent fixed point. In this setting if $\gamma > \frac{1}{\alpha}$ we are only able to prove we may lift in the parameter range $\alpha < 1 + \frac{1}{\gamma^2} - \frac{1}{\gamma}$. If this condition holds we show that the stable law of index α dominates and we obtain Birkhoff sum convergence to a stable law of index α. This latter results relies on a form of the law of the iterated logarithm valid for this parameter range [DGM12].

Finally we note that the case of $\varphi(x) = d(x, 0)^{-\frac{1}{\alpha}}$ has been clarified by Gouëzel, and here the two effects combine so that a stable law holds with scaling constants $b_n = n^{\gamma + \frac{1}{\alpha}}$ if $1/2 < \gamma + \frac{1}{\alpha} < 1$.

7. Stable limits for Birkhoff sums of dependent variables.

Our results are based upon the investigations and results of R. Davis [Dav83] and R. Davis and T. Hsing [DH95] into the partial sum convergence of dependent random variables with infinite variance.

We paraphrase [DH95, Theorem 3.1] below.

Proposition 7.1 ([DH95, Theorem 3.1]). *Let $\{X_j\}$ be a stationary sequence of random variables on a probability space (X, μ) such that:*

(i)

$$n\mu\left(\frac{X_1}{b_n} \in \cdot\right) \to_v \nu(\cdot)$$

where

$$\nu(dx) = [p\alpha x^{-\alpha-1}\mathbf{1}_{(0,\infty)} + (1-p)\alpha(-x)^{-\alpha-1}\mathbf{1}_{(-\infty,0)}]dx$$

and $\to_v$ denotes vague convergence on $\mathbb{R} \setminus \{0\}$; and

(ii)

$$N_n := \sum_{j=1}^{n} \delta_{X_j/b_n} \to_d N = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \delta_{P_i Q_{ij}}$$

where the convergence is in the space of random counting measures, $\sum_{i=1}^{\infty} \delta_{P_i}$ is a Poisson process with intensity measure ν, $Q_i := \sum_{j=1}^{\infty} \delta_{Q_{ij}}$, $i \geq 1$, are point processes that are iid, $Q_{ij} \in [-1, 1]/\{0\}$, and all point processes are mutually independent.

Then:

(a) *For* $0 < \alpha < 1$,
$$\frac{1}{b_n}\sum_{j=1}^{n} X_j \to_d S$$
where $S = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_i Q_{ij}$ *has a stable distribution with index* α.

(b) *If* $1 \leq \alpha < 2$ *and*
$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} P\{|\frac{1}{b_n}\sum_{j=1}^{n} X_j \mathbf{1}_{\{|X_j|<b_n\varepsilon\}} - \frac{1}{b_n}\mathbb{E}[\sum_{j=1}^{n} X_j \mathbf{1}_{\{|X_j|<b_n\varepsilon\}}]| > \delta\} = 0 \text{ for all } \delta > 0, \tag{7.1}$$
then
$$\frac{1}{b_n}\left(\sum_{j=1}^{n} X_j - \mathbb{E}[\sum_{j=1}^{n} X_j \mathbf{1}_{\{|X_j|<b_n\}}]\right) \to_d S$$
where S *is the distributional limit of*
$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_i Q_{ij} \mathbf{1}_{(\varepsilon,\infty)}(|P_i Q_{ij}|) - \int_{\varepsilon<|x|\leq 1} x\nu(dx)$$
as $\varepsilon \to 0$. S *has a stable distribution with index* α.

Remark 7.2. *Condition (i) above is equivalent to*
$$P(|X_1| > x) = x^{-\alpha}L(x) \tag{7.2}$$
and
$$\lim_{x\to\infty}\frac{P(X_1 > x)}{P(|X_1| > x)} = p \tag{7.3}$$
for a slowly varying function $L(x)$ *and* $0 \leq p \leq 1$. *See* [DH95, Introduction].

Remark 7.3. *By Chebyshev's inequality, Condition* (7.1) *is implied by*
$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\mathbb{E}\left(\left|\frac{1}{b_n}\sum_{j=0}^{n-1} X_j \mathbf{1}_{\{|X_j|<b_n\varepsilon\}} - \frac{1}{b_n}\mathbb{E}(\sum_{j=0}^{n-1} X_j \mathbf{1}_{\{|X_j|<b_n\varepsilon\}})\right|^2\right) = 0, \tag{7.4}$$

By [Dav83, Theorem 3], (7.4) *is implied by*
$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{n}{b_n^2}\sum_{j=1}^{n}\max\{0, \mathbb{E}(Y_1 Y_j)\} = 0, \tag{7.5}$$
where $Y_j = X_j \mathbf{1}_{\{|X_j|<b_n\varepsilon\}} - E(X_j \mathbf{1}_{\{|X_j|<b_n\varepsilon\}})$.

Remark 7.4. *Marta-Tyran Kaminska's work* [TK10b, Theorem 1.3] *has the same condition, Equation* (7.1), *in the case* $1 < \alpha \leq 2$, *but requires convergence in (ii) to a simple Poisson process i.e.* $Q_{ij} = 1$ *for* $i = j = 1$ *and* 0 *otherwise. Her condition was motivated by the goal of establishing functional limit theorems rather than distributional convergence of Birkhoff sums.*

8. Main Results

Theorem 8.1. *Suppose (T, X, μ) is a Gibbs-Markov map of the unit interval $X = [0, 1]$. Let $\varphi : X \to \mathbb{R}$ be in the domain of attraction of a stable law of index $\alpha \in (1, 2)$ and suppose that*

$$\text{there exists } K > 0 \text{ such that } \|\varphi \cdot \mathbf{1}_{\{|\varphi|<t\}}\|_{\mathrm{BV}} \le Kt \text{ for } t \text{ sufficiently large.} \tag{8.1}$$

Define b_n as in Definition 2.1, by $\lim_{n\to\infty} n\mu(|\varphi| > b_n) = 1$. Then for all $\delta > 0$,

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mu\left\{ |\frac{1}{b_n}\sum_{j=1}^{n} \varphi \circ T^j \mathbf{1}_{\{|\varphi\circ T^j|<b_n\varepsilon\}} - \frac{1}{b_n} E[\sum_{j=1}^{n} \varphi \circ T^j \mathbf{1}_{\{|\varphi\circ T^j|<b_n\varepsilon\}}]| > \delta \right\} = 0 \tag{8.2}$$

Remark 8.2. *The condition (8.1) is satisfied, e.g., if φ has finitely many intervals of monotonicity. For example it holds for $\varphi(x) = 3|x - x_1|^{-2/3} - 6|x - x_2|^{-2/3}$ where $x_1, x_2 \in [0, 1]$.*

Although Theorem 8.1 holds for functions $\varphi : X \to \mathbb{R}$ that satisfy the condition (8.1), we will restrict now to observables of form $\varphi(x) = d(x, x_0)^{-\frac{1}{\alpha}}$ where $\alpha \in (0, 2]$. This is because we rely on recent work [FFMa20] which has shown that for such observables on Gibbs-Markov maps the corresponding counting process N_n converges to a Poisson point process, and this is key to verifying the conditions of [DH95, Theorem 3.1].

Combined with Point process convergence results of Freitas, Freitas, Magalhaes (2018) and Pené and Saussol (2018) we have:

Corollary 8.3. *Suppose (T, X, μ) is a Gibbs-Markov map of the unit interval $X = [0, 1]$. Let $\varphi(x) = d(x, x_0)^{-\frac{1}{\alpha}}$ where $\alpha \in (0, 1) \cup (1, 2)$, $x_0 \in (0, 1)$. Define b_n as in Definition 2.1, by $\lim_{n\to\infty} n\mu(|\varphi| > b_n) = 1$.*

(a) If $0 < \alpha < 1$ then

$$\frac{1}{b_n}\sum_{j=1}^{n} \varphi \circ T^j \to_d S$$

where $S = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_i Q_{ij}$ has a stable distribution with index α.

(b) If $1 < \alpha < 2$ then

$$\frac{1}{b_n}\left(\sum_{j=1}^{n} \varphi \circ T^j - E[\sum_{j=1}^{n} \varphi \circ T^j \mathbf{1}_{\{|\varphi\circ T^j|<b_n\}}]\right) \to_d S \tag{8.3}$$

where S is the distributional limit of

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P_i Q_{ij} \mathbf{1}_{(\varepsilon,\infty)}(|P_i Q_{ij}|) - \int_{\varepsilon<|x|\le 1} x\nu(dx)$$

as $\varepsilon \to 0$. S has a stable distribution with index α.

We now give an application to intermittent-type maps, describing the interplay between the slow-mixing parameter γ and the heavy tails parameter α.

Theorem 8.4. *Suppose (T_γ, X, μ) is a LSV map of the unit interval and $0 \le \gamma < 1$. Suppose $\varphi(x) = d(x, x_0)^{-\frac{1}{\alpha}}$ where $x_0 \in (0, 1]$. If $\alpha \in [1, 2)$ and φ is integrable then we define $c_n = \mathbb{E}[\varphi] = \int d(x, x_0)^{-\frac{1}{\alpha}} d\mu_\gamma$, otherwise $c_n = 0$.*

(i) *If $\frac{1}{\alpha} \ge \gamma$ then*

$$\frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} [\varphi \circ T^j - c_n] \to_d S$$

where $S = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P_i Q_{ij}$ has a stable distribution of index α;

(ii) *if $\frac{1}{\alpha} < \gamma$ and $\varphi(0) - \mathbb{E}[\varphi] \ne 0$ then*

$$\frac{1}{n^{\gamma}} \sum_{j=1}^{n} (\varphi \circ T^j - E[\varphi]) \to_d S$$

where S has a stable distribution of index γ.

(iii) *if $\varphi(0) - \mathbb{E}[\varphi] = 0$ and $\frac{1}{\gamma} < \alpha < 1 + \frac{1}{\gamma^2} - \frac{1}{\gamma}$ then*

$$\frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} (\varphi \circ T^j - \mathbb{E}[\varphi]) \to_d S$$

where S has a stable distribution with index α.

Remark 8.5. *To satisfy $\gamma > \frac{1}{\alpha}$ in case (ii) and case (iii) above it is necessary that $\alpha \in (1, 2)$. The extra condition, $\frac{1}{\gamma} < \alpha < 1 + \frac{1}{\gamma^2} - \frac{1}{\gamma}$, in case (iii) occurs because we rely on a result of Dedecker, Gouëzel and Merlevède* [DGM12] *which is given in the section A.2. Our standard 'lifting' argument fails in this case and we rely on a law of the iterated logarithm result in* [DGM12] *which is known to hold in this parameter regime.*

Remark 8.6. *In* [FFT20, Section 2.2.1] *it is shown that (i) holds for $\gamma \in (0, 0.289)$ and $0 < \alpha < 1$ (actually they prove a stronger functional convergence in the F' topology which implies a stable law) and it is conjectured that convergence in F' holds for $0 < \alpha < 1$ and all $\gamma < \frac{1}{2}$. In* [FFT24, Section 4] *this is extended for certain observables to $\gamma \in (0, 1)$.*

Remark 8.7. *The case where φ is a function of the distance to the origin 0 has been clarified by Gouëzel* [Gou04]*. In the set-up of the LSV maps where $0 \le \gamma < 1$ if $\varphi(x) = x^{-\frac{1}{\alpha}}$, (so that $x_0 = 0$) and $1 > \frac{1}{\alpha} + \gamma > \frac{1}{2}$ then φ converges to a stable law in distribution and the corresponding scaling constant is $n^{\gamma + \frac{1}{\alpha}}$. If $\frac{1}{\alpha} + \gamma < \frac{1}{2}$ then we have a CLT.*

9. Proof of Theorem 8.1.

Recall the Karamata estimates of Proposition 2.3 for regularly varying functions.

Remark 9.1. *Although we consider the case of a Gibbs-Markov map $T : X \to X$ and $\varphi(x) := d(x, x_0)^{-1/\alpha}$, we are using only the following (e.g., no need for the Markov property):*

- *for the map $T : X \to X$, $X \subset \mathbb{R}$:*
  - *big images w.r.t. the invariant measure*
  - *uniform expansion: there is $\theta \in (0, 1)$ such that $|T'(x)| \geq \theta^{-1}$ for each x where the derivative exists*
  - *exponential decay on* BV *of the transfer operator P of T w.r.t. the invariant measure μ*
  - *bounded distortion*
  - *invariant measure comparable to Lebesgue: density bounded above, and away from zero*
- *for the observation φ:* (8.1) *holds, that is, there is a constant $K > 0$ such that $\|\varphi \cdot \mathbf{1}_{\{|\varphi|<t\}}\|_{\text{BV}} \leq Kt$ for t sufficiently large.*

This allows to consider, e.g., β-transformation with $\beta > 1$, not necessarily integer. Condition (8.1) *is satisfied, e.g., if φ has finitely many intervals of monotonicity.*

Proof of Theorem 8.1. Let $([0, 1], \mathcal{B}, \mu, T, \mathcal{P})$ be an expanding Gibbs-Markov system as in Section 5. We will check the hypotheses of Theorem 7.1.

Condition (i) is satisfied since φ is in the domain of attraction of a stable law of index α (see Remark 7.2).

Condition (ii) holds by [FFMa20]. Recall that by (2.1) and (7.2)

$$n \sim 1/\mu(|\varphi| > b_n) = b_n^\alpha L(b_n)^{-1}$$

Since L grows slower than any power (see Lemma A.1), we will sometimes abuse notation and consider that

$$b_n \sim n^{1/\alpha} \tag{9.1}$$

Consider now the case of $\alpha \in (1, 2)$.

We need to establish (7.1). By Remark 7.3, condition (7.1) is implied by

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{n}{b_n^2} \sum_{j=1}^{n} \max\left\{0, \int \Phi_n \cdot \Phi_n \circ T^j d\mu\right\} = 0, \tag{9.2}$$

where, for a fixed $\varepsilon > 0$, we denote

$$\varphi_n := \varphi \cdot \mathbf{1}_{\{|\varphi|\leq\varepsilon b_n\}} \text{ and } \Phi_n := \varphi_n - E(\varphi_n).$$

To obtain (9.2), by the exponential decay of correlations (5.1), we need only show that

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{n}{b_n^2} \sum_{j=1}^{\lfloor k \log n \rfloor} \max\left\{0, \int \Phi_n \Phi_n \circ T^j d\mu\right\} = 0, \tag{9.3}$$

where k is independent of n and ε. Indeed, using (8.1):

$$\frac{n}{b_n^2}\sum_{j\geq\lfloor k\log n\rfloor}\left|\int \Phi_n\Phi_n\circ T^j d\mu\right| \leq \frac{n}{b_n^2}\sum_{j\geq\lfloor k\log n\rfloor} C\lambda^j\|\Phi_n\|_{\mathrm{BV}}\|\Phi_n\|_\infty$$
$$\leq \frac{n}{b_n^2}\sum_{j\geq\lfloor k\log n\rfloor} C'\lambda^j K(\varepsilon b_n)^2 \leq \frac{n}{b_n^2}C''\lambda^{k\log n}(\varepsilon b_n)^2 = \frac{n}{b_n^2}C''n^{k\log\lambda}(\varepsilon b_n)^2$$

and take k large enough that $k\log\lambda < -1$.

Since μ is T-invariant, can rewrite the covariance $\int \Phi_n\Phi_n\circ T^j d\mu$ as $\mathbb{E}(\varphi_n\cdot\varphi_n\circ T^j) - [E(\varphi_n)]^2$. Because $\varphi\in L^1(\mu)$, one can neglect the $[\mathbb{E}(\varphi_n)]^2$ terms in (9.3) as their contributions is of order

$$\frac{n}{b_n^2}(\mathbb{E}(\varphi_n))^2\log n \leq \frac{n}{b_n^2}(\mathbb{E}(|\varphi|))^2\log n \sim (\mathbb{E}(|\varphi|))^2 n^{1-\frac{2}{\alpha}}\log n$$

and $\alpha < 2$.

Thus, it suffices to show

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{n}{b_n^2}\sum_{j=1}^{\lfloor k\log n\rfloor}\int |\varphi_n|\cdot|\varphi_n|\circ T^j d\mu = 0. \tag{9.4}$$

Introduce

$$\frac{1}{2} < r < 1, \quad u_n := b_n^r, \quad U_n := \{|\varphi|\geq u_n\}.$$

Since φ is in the domain of attraction of a stable law with index α (see (7.2) in Remark 7.2),

$$\mu(U_n) = u_n^{-\alpha}L(u_n).$$

From Karamata's Theorem 2.3, (7.2) and that $u_n = b_n^r$, we have

$$\int {\varphi_n}^2 d\mu = \int \varphi^2\cdot\mathbf{1}_{\{|\varphi|\leq\varepsilon b_n\}}d\mu \sim \frac{\alpha}{2-\alpha}(\varepsilon b_n)^2\mu(|\varphi|>\varepsilon b_n) = C_\alpha\varepsilon^2 b_n^2(\varepsilon b_n)^{-\alpha}L(\varepsilon b_n) \tag{9.5}$$

$$\int_{U_n^c}\varphi^2 d\mu \sim \frac{\alpha}{2-\alpha}u_n^2\mu(|\varphi|>u_n) = C_\alpha b_n^{2r}b_n^{-\alpha r}L(u_n) \tag{9.6}$$

We decompose the sum of integrals in (9.4) as (I) + (II) + (III), where

$$(\mathrm{I}) = \sum_{j=1}^{\lfloor k\log n\rfloor}\int_{U_n\cap T^{-j}U_n}|\varphi_n|\cdot|\varphi_n|\circ T^j d\mu,$$

$$(\mathrm{II}) = \sum_{j=1}^{\lfloor k\log n\rfloor}\int_{U_n\cap T^{-j}U_n^c}|\varphi_n|\cdot|\varphi_n|\circ T^j d\mu$$

and

$$(\mathrm{III}) = \sum_{j=1}^{\lfloor k\log n\rfloor}\int_{U_n^c}|\varphi_n|\cdot|\varphi_n|\circ T^j d\mu.$$

Consider (II) and (III) first.

For (III), using that μ is T-invariant, we have

(9.7)
$$\int_{U_n^c} |\varphi_n| \cdot |\varphi_n| \circ T^j d\mu \le \left(\int_{U_n^c} \varphi^2 d\mu \right)^{\frac{1}{2}} \left(\int \varphi_n^2 \circ T^j d\mu \right)^{\frac{1}{2}} = \left(\int_{U_n^c} \varphi^2 d\mu \right)^{\frac{1}{2}} \left(\int \varphi_n^2 d\mu \right)^{\frac{1}{2}}.$$

Similarly, for (II),

(9.8)
$$\begin{aligned}\int_{U_n \cap T^{-j}U_n^c} |\varphi_n| \cdot |\varphi_n| \circ T^j d\mu &\le \left(\int \varphi_n^2 d\mu \right)^{\frac{1}{2}} \left(\int_{T^{-j}U_n^c} \varphi^2 \circ T^j d\mu \right)^{\frac{1}{2}} \\ &= \left(\int \varphi_n^2 d\mu \right)^{\frac{1}{2}} \left(\int (\varphi^2 \cdot \mathbf{1}_{U_n^c}) \circ T^j d\mu \right)^{\frac{1}{2}} = \left(\int \varphi_n^2 d\mu \right)^{\frac{1}{2}} \left(\int \varphi^2 \cdot \mathbf{1}_{U_n^c} d\mu \right)^{\frac{1}{2}} \\ &= \left(\int \varphi_n^2 d\mu \right)^{\frac{1}{2}} \left(\int_{U_n^c} \varphi^2 d\mu \right)^{\frac{1}{2}}\end{aligned}$$

By (9.5) and (9.6) we obtain

$$\left(\int \varphi_n^2 d\mu \right)^{\frac{1}{2}} \left(\int_{U_n^c} \varphi^2 d\mu \right)^{\frac{1}{2}} \le C_\alpha \varepsilon^{1-\frac{\alpha}{2}} b_n^{(1-\frac{\alpha}{2})(1+r)} L(\varepsilon b_n)^{1/2} L(b_n^r)^{1/2}$$

By (2.1) and (7.2),

$$n \sim 1/\mu(|\varphi| > b_n) = b_n^\alpha L(b_n)^{-1}$$

which gives

$$\frac{n}{b_n^2}[(\mathrm{II}) + (\mathrm{III})] \le 2C_\alpha k \varepsilon^{1-\frac{\alpha}{2}} b_n^{-(1-\frac{\alpha}{2})(1-r)} \log n \cdot \left(\frac{L(\varepsilon b_n) L(b_n^r)}{L(b_n)^2} \right)^{1/2}$$

Since L is slowly varying, it grows slower than any power (see Lemma A.1), so, because $r < 1$,

$$\limsup_{n \to \infty} \frac{n}{b_n^2}[(\mathrm{II}) + (\mathrm{III})] = 0 \tag{9.9}$$

It remains to bound (I).

Denote by $\{A_t^{(m)}\}_{t \ge 1}$ the partition induced by $\bigvee_{j=0}^{m-1} T^{-j}\mathcal{P}$.

Consider some fixed $1 \le j \le k \log n$; in order to estimate $\int_{U_n \cap T^{-j}U_n} |\varphi_n| \cdot |\varphi_n| \circ T^j d\mu$, we have the following three possibilities.

Case 1: $U_n \subset A_r^{(j)}$ for some $r \in \mathbb{N}$.

Using the Hölder inequality and the expression of the transfer operator P,

$$a_j := \int_{U_n \cap T^{-j} U_n} |\varphi_n| \cdot |\varphi_n| \circ T^j d\mu \leq \left(\int_X \varphi_n^2 d\mu \right)^{1/2} \left(\int_{U_n} \varphi_n^2 \circ T^j d\mu \right)^{1/2}$$
$$= \left(\int_X \varphi_n^2 d\mu \right)^{1/2} \left(\int_X P^j(\mathbf{1}_{U_n}) \varphi_n^2 \right)^{1/2}$$

with

$$P^j(\mathbf{1}_{U_n})|_x = \frac{h(y)}{h(x)} \cdot \frac{1}{(T^j)'(y)}$$

where $y \in U_n \subset A_r^{(j)}$ is the unique point such that $T^j(y) = x$, and h is the density of the invariant measure, $d\,\mu = h\; d$ Leb, bounded above and away from zero. Since T is piecewise expanding, we obtain that

$$\|P^j(\mathbf{1}_{U_n})\|_{L^\infty(X)} \leq C\theta^j$$

for $C > 0$ independent of j and n. Thus

$$a_j \leq C\theta^j \int \varphi_n^2 \; d\,\mu$$

Case 2: $U_n \subset A_r^{(j)} \cup A_{r+1}^{(j)}$ for some $r \in \mathbb{N}$.
Consider $U_n \cap A_r^{(j)}$ and $U_n \cap A_{r+1}^{(j)}$. They both satisfy **Case 1**, and therefore we have

$$b_j := \int_{U_n} |\varphi_n(x)||\varphi_n(T^j x)| \; d\,\mu \leq 2C\theta^j \int \varphi_n^2 \; d\,\mu$$

Case 3: $A_r^{(j)} \subset U_n$ for some $r \in \mathbb{N}$.
There exists $r_1, r_2 \in \mathbb{N}$ such that $A_{r_1}^{(j)}, A_{r_2}^{(j)}$ cover the endpoints of U_n, therefore, by **Case 1**,

$$c_j := \int_{U_n \cap (A_{r_1}^{(j)} \cup A_{r_2}^{(j)})} |\varphi_n||\varphi_n| \circ T^j \; d\,\mu \leq 2C\theta^j \int \varphi_n^2 \; d\,\mu$$

For the sets $A_r^{(j)} \subset U_n$, by the bounded distortion of Gibbs-Markov system,

$$\mu(A_r^{(j)} \cap T^{-j} U_n) \leq C\mu(A_r^{(j)})\mu(U_n)/\mu(T^j A_r^{(j)}).$$

Therefore, by the big image property,

$$\sum_{A_r^{(j)} \subset U_n} \mu(A_r^{(j)} \cap T^{-j} U_n) \leq \widetilde{C} \sum_{A_r^{(j)} \subset U_n} \mu(A_r^{(j)})\mu(U_n) \leq \widetilde{C}\mu(U_n)^2$$

and then

$$d_j := \sum_{\{r:A_r^{(j)}\subset U_n\}} \int_{A_r^{(j)}\cap(U_n\cap T^{-j}U_n)} |\varphi_n||\varphi_n|\circ T^j \, d\,\mu \leq \sum_{\{r:A_r^{(j)}\subset U_n\}} \int_{A_r^{(j)}\cap T^{-j}U_n} |\varphi_n||\varphi_n|\circ T^j \, d\,\mu$$

$$\leq \left[\sum_{\{r:A_r^{(j)}\subset U_n\}} \mu(A_r^{(j)}\cap T^{-j}U_n) \right] \|\varphi_n\|_{L^\infty}^2 \leq C\mu(U_n)^2\|\varphi_n\|_{L^\infty}^2$$

We now collect all these estimates.

Using Karamata's estimate (9.5) of $\mathbb{E}(\varphi_n^2)$, the choice of b_n given by (2.1), and the expression of $\mu(U_n) = \mu(|\varphi| > u_n)$ given by (7.2):

$$\begin{aligned}\frac{n}{b_n^2}(I) \leq & \frac{n}{b_n^2}\sum_{j=1}^{\lfloor k\log n\rfloor}[a_j+b_j+c_j+d_j] \leq C\frac{n}{b_n^2}\sum_{j=1}^{\lfloor k\log n\rfloor}\left[\theta^j\mathbb{E}(\varphi_n^2)+\mu(U_n)^2\|\varphi_n\|_{L^\infty}^2\right] \\ &\leq C\frac{n}{b_n^2}\left[\varepsilon^2 b_n^2 P(|\varphi|\geq\varepsilon b_n) + u_n^{-2\alpha}L(u_n)^2(\varepsilon b_n)^2\log n\right] \\ &= C[\varepsilon^2 nP(|\varphi|\geq\varepsilon b_n) + nb_n^{-2\alpha r}\varepsilon^2 L(b_n^r)^2\log n] \to C\varepsilon^2 \text{ as } n\to\infty\end{aligned}$$

because $r > 1/2$ and L grows slower than any power, Lemma A.1.

Together with (9.9), this shows that condition (9.4) is satisfied. □

10. Proof of Theorem 8.4

Tyran-Kaminska [TK10b, Theorem 4.4] has proved convergence to a simple Poisson process in our setting of Gibbs-Markov maps if $\lim_{n\to\infty}\tau(|\varphi| > \varepsilon b_n) = \infty$ for all $\varepsilon > 0$, where τ is the return time function. This non-recurrence condition is not satisfied if φ is maximized at a periodic point. However recently the complete convergence of N_n to a Poisson process has been established in the case of $\varphi(x) = d(x, x_0)^{-\frac{1}{\alpha}}$ if x_0 is periodic [FFMa20]. These two results cover all cases as shown by a dichotomy result in [FFMa20].

We will induce and model the system as a Young tower over a Gibbs-Markov base map. As x_0 need not be contained in $[1/2, 1]$ we may need to induce over a base larger than the usual Young Tower base of $[1/2, 1]$ used for the LSV map.

Let T_L denote the left branch of T. We consider the partition of $(0, 1]$ into sets (A_i) and (B_j). We define $A_i \subset [1/2, 1]$ to be that set of points in $[1/2, 1)$ where the first return time to $[1/2, 1]$ under T is i and then define $B_j = [T_L^{-j-1}(1/2), T_L^{-j}(1/2)] \subset (0, 1]$, $j \geq 0$. Note that the sets $\{A_i\}_i$ constitute the usual partition of the base $[1/2, 1)$ for the usual Young tower for the LSV map but we will adjoin some of the sets B_j. Since $x_0 \neq 0$ there exists a minimal M such that $x_0 \in [1/2, 1] \cup (\cup_{j=1}^M B_j)$. Define $Y := [1/2, 1] \cup (\cup_{j=1}^M B_j)$. Inducing on Y the return map to Y is a Gibbs-Markov map (though not necessarily with full branches). We take a Tower model for (T_γ, X, μ) as a tower over Y with countable partition of the base Y consisting of (A_i) and (B_j) in Y. If $R(x)$ is the first return to Y then $T^R A_i = B_M$ for all $A_i \subset [0, 1]$. If $2 \leq j \leq M$ then $T^R B_j = B_{j-1}$ and $T^R B_1 = [1/2, 1]$. The map

$F := T^R : Y \to Y$ is a Gibbs-Markov map, though not necessarily with full branches. Note that if $x_0 \in [1/2, 1)$ then we we may take $Y = [1/2, 1]$, and $F : Y \to Y$ is a full-branched Gibbs-Markov map.

The induced map $F = T^R : Y \to Y$ has an invariant probability measure μ_Y, whose density is Lipschitz and bounded away from infinity and 0.

Denote by $\mathbb{E}_Y$ the expectation on Y with respect to μ_Y; let $\bar{R} = \int_Y R d\mu_Y = \frac{1}{\mu(Y)}$, by Kac's lemma.

We denote $R_n(x) = R(x) + R(F(x)) + \ldots + R(F^{n-1}(x))$.

We begin with Case (ii).

Case (ii): $\gamma > \frac{1}{\alpha}$ and $\varphi(0) - \mathbb{E}[\varphi] \neq 0$. .

The assumption that $\gamma > \frac{1}{\alpha}$ implies that $\alpha \in (1, 2)$ and hence $\mathbb{E}[\varphi] < \infty$. Note also that the assumption that $\gamma > \frac{1}{\alpha}$ excludes the case $0 \leq \gamma \leq \frac{1}{2}$.

We decompose φ as $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1(x) := \varphi(x)\mathbf{1}_{Y^c}$ and $\varphi_2(x) := \varphi(x)\mathbf{1}_Y$. Note that $\varphi_1 - \mathbb{E}[\varphi_1]$ induces in a good way on the base Y. In fact the induced version of $\varphi_1 - \mathbb{E}[\varphi_1]$ lies in the Banach space of functions to which the results of [Gou04, Theorem 1.2] apply.

Following [MZ15] we will write $\varphi_1 - \mathbb{E}[\varphi_1] = (\varphi(0) - \mathbb{E}[\varphi_1]) - \frac{1}{\mu(Y)}(\varphi(0) - \mathbb{E}[\varphi_1])\mathbf{1}_Y + \psi$ where ψ is defined by this equation. Note that $\mathbb{E}[\psi] = 0$, $\psi(0) = 0$ and ψ is piecewise Hölder. Thus ψ satisfies a CLT and so its Birkhoff sum converges to zero in distribution under any scaling $b_n = n^\kappa$, $\kappa > \frac{1}{2}$. Hence the effect of ψ is negligible, as a scaling by n^γ or $n^{1/\alpha}$ will ensure that the scaled Birkhoff sum ψ converges in distribution to zero.

Note that $g := (\varphi(0) - \mathbb{E}[\varphi_1]) - \frac{1}{\mu(Y)}(\varphi(0) - \mathbb{E}[\varphi_1])\mathbf{1}_Y$ has expectation zero, $\mathbb{E}[g] = 0$. The function g induces the function

$$\Phi_1 = (\varphi(0) - \mathbb{E}[\varphi_1])(R(x) - \bar{R})$$

on Y. For $x \in Y$,

$$\sum_{j=0}^{\bar{R}n} g \circ T^j = \sum_{j=0}^{n} \Phi_1 \circ F^j + V_n(x)$$

where

$$V_n(x) = \begin{cases} \displaystyle\sum_{R_n(x)}^{\bar{R}n} g \circ T^j(x) & \text{if } \bar{R}n \geq R_n(x) \\ \displaystyle -\sum_{\bar{R}n}^{R_n(x)} g \circ T^j(x) & \text{if } \bar{R}n < R_n(x) \end{cases}$$

Thus we have

$$\sum_{j=0}^{\bar{R}n} g\circ T^j = \sum_{j=0}^{n} \Phi_1\circ F^j + \sum_{n=0}^{\bar{R}n} \psi\circ T^j + V_n(x) \tag{10.1}$$
$$= (\varphi(0) - \mathbb{E}[\varphi_1](R_n(x) - n\bar{R}) + V_n(x) + \sum_{n=0}^{\bar{R}n} \psi\circ T^j$$

We will use this observation when considering the induced form of $\varphi_2 - \mathbb{E}[\varphi_2]$.

We induce the observable φ_2 on the Gibbs-Markov base Y by defining $\Phi_2(x) = \sum_{i=0}^{R(x)-1}\varphi_2\circ T^i(x)$ where R is the first return time to Y under T. Since φ_2 has support in Y, $\Phi_2 = 0$ on all levels of the tower except for the base level, identified with Y, and on Y we have $\varphi_2 = \Phi_2$.

Φ_2 is in the domain of attraction of a stable law of index α on the probability space (Y, μ_Y) and $\mathbb{E}_Y[\Phi_2] = \mathbb{E}[\varphi_2]/\mu(Y)$. Note that for large t, $\mu_Y(\Phi_2 > t) = \frac{1}{\mu(Y)}\mu(\varphi > t)$ and hence the b_n scaling for Φ_2 is $(n\bar{R})^{\frac{1}{\alpha}}$.

From our result on Gibbs-Markov maps Φ_2 satisfies a stable law with index α under $F := T^R$ with scaling $(n\bar{R})^{\frac{1}{\alpha}}$. By our main theorem, Corollary 8.3

$$(n\bar{R})^{-\frac{1}{\alpha}} \sum_{j=1}^{n} (\Phi_2\circ F^j - \bar{R}\mathbb{E}[\varphi_2]) \xrightarrow{d} X_\alpha$$

We write

$$\sum_{j=0}^{[\bar{R}n]} (\varphi_2\circ T^j - \mathbb{E}[\varphi_2]) = \sum_{j=0}^{R_n(x)} (\varphi_2\circ T^j - \mathbb{E}[\varphi_2]) + W_n(x)$$

where, as for $V_n(x)$,

$$W_n(x) = \sum_{R_n(x)}^{[\bar{R}n]} (\varphi_2\circ T^j(x) - \mathbb{E}[\varphi_2]) \quad \text{or} \quad W_n(x) = -\sum_{[\bar{R}n]}^{R_n(x)} (\varphi_2\circ T^j(x) - \mathbb{E}[\varphi_2]).$$

Furthermore

$$\sum_{j=0}^{R_n(x)} (\varphi_2\circ T^j - \mathbb{E}[\varphi_2]) = \sum_{j=0}^{n} (\Phi_2\circ F^j - \bar{R}\mathbb{E}[\varphi_2])$$
$$-\mathbb{E}[\varphi_2][R_n(x) - n\bar{R}]$$

Thus

$$\sum_{j=0}^{[\bar{R}n]} (\varphi_2\circ T^j - \mathbb{E}[\varphi_2]) = \sum_{j=0}^{n} (\Phi_2\circ F^j - \bar{R}\mathbb{E}[\varphi_2]) - \mathbb{E}[\varphi_2][R_n(x) - n\bar{R}] + W_n(x) \tag{10.2}$$

Adding Equations (10.1) and (10.2) we obtain the representation

$$\sum_{j=0}^{[\bar{R}n]}(\varphi \circ T^j - \mathbb{E}[\varphi]) = \sum_{j=0}^{n}(\Phi_2 \circ F^j - \bar{R}\mathbb{E}[\varphi_2]) + \sum_{j=0}^{\bar{R}n} \psi \circ T^j + (\varphi(0) - \mathbb{E}[\varphi_1] - \mathbb{E}[\varphi_2])[R_n(x) - n\bar{R}] + V_n(x) + W_n(x) \tag{10.3}$$

As noted before $n^{-\kappa}\sum_{j=0}^{\bar{R}n} \psi \circ T^j$ converges in distribution to zero for any $\kappa > \frac{1}{2}$.

We will show that

$$\frac{1}{(\bar{R}n)^\gamma} W_n(x) \xrightarrow{d} 0 \tag{10.4}$$

and

$$\frac{1}{(\bar{R}n)^\gamma} V_n(x) \xrightarrow{d} 0. \tag{10.5}$$

These imply that

$$(\bar{R}n)^{-\frac{1}{\alpha}} \sum_{j=0}^{[\bar{R}n]}(\varphi \circ T^j - \mathbb{E}[\varphi]) \xrightarrow{d} (\bar{R}n)^{-\frac{1}{\alpha}} \sum_{j=0}^{n}(\Phi_2 \circ F^j - \bar{R}\mathbb{E}[\varphi_2])$$

and hence

$$(\bar{R}n)^{-\frac{1}{\alpha}} \sum_{j=0}^{[\bar{R}n]}(\varphi \circ T^j - \mathbb{E}[\varphi]) \xrightarrow{d} X_\alpha$$

We prove next the claim (10.4) about $W_n(x)$. The proof of (10.5) for $V_n(x)$ is the same mutatis mutandis.

Proof of (10.4)***:*** We will show that

$$\frac{1}{(\bar{R}n)^\gamma} W_n(x) \xrightarrow{d} 0.$$

Since $\gamma > \frac{1}{2}$

$$\frac{R_n - n\bar{R}}{n^\gamma} \xrightarrow{d} X_{\frac{1}{\gamma}}$$

as the return time function R lies in the domain of attraction of $X_{\frac{1}{\gamma}}$ and satisfies the conditions of [Gou04, Theorem 1.2]. Thus

$$P\left(\left|\frac{R_n(x) - n\bar{R}}{n^\gamma}\right| \in [0, L^{-1}) \cup (L, \infty)\right) \to P\left(X_{\frac{1}{\gamma}} \in [0, L^{-1}) \cup (L, \infty)\right) \quad \text{as } n \to \infty$$

Therefore, given $\varepsilon > 0$, there are (large) $L = L(\varepsilon)$ and $N_1 = N_1(\varepsilon)$ such that

$$P\left(\left|\frac{R_n(x) - n\bar{R}}{n^\gamma}\right| \in [0, L^{-1}) \cup (L, \infty)\right) < \varepsilon \quad \text{for } n \geq N_1(\varepsilon) \tag{10.6}$$

Denote

$$\widetilde{\varphi}_2 = \varphi_2 - \mathbb{E}(\varphi_2)$$

and

$$\Delta_n(x) = \left|R_n(x) - n\bar{R}\right|.$$

Note that

$$\frac{1}{n^\gamma}|W_n(x)| = \frac{\left|R_n(x) - n\bar{R}\right|}{n^\gamma} \cdot \frac{\left|\sum_{j=0}^{\left|R_n(x)-n\bar{R}\right|} \widetilde{\varphi} \circ T^j(y)\right|}{\left|R_n(x) - n\bar{R}\right|} \tag{10.7}$$

where y is either $T^{R_n(x)}(x)$ or $T^{[n\bar{R}]}(x)$. The first factor is controlled by (10.6); for the second we use Wiener's Maximal Inequality, Theorem (10.1), see e.g. [Dur19, Exercise 6.2.3]

Theorem 10.1 (Wiener's Maximal Inequality)**.** *For T a measure preserving transformation on the probably space (Ω, ν), $f \in L^1(\nu)$ and $a > 0$:*

$$P\left(\sup_{n\geq 1} \frac{1}{n}\sum_{k=1}^{n} |f \circ T^k| > a\right) \leq \frac{\mathbb{E}(|f|)}{a} \tag{10.8}$$

□

Since T is ergodic, the L^2-coboundaries are L^2-dense in the set of zero-expectation functions (see e.g. [SS05, Lemma 6.5.2]). Approximating $\widetilde{\varphi}_2 \in L^1$ by L^2 functions, one has:

$$\text{for } \delta_2 > 0 \text{ there are } \xi, \psi \in L^2(\mu) \text{ such that } \psi = \xi \circ T - \xi \text{ and } \|\widetilde{\varphi} - \psi\|_{L^1} < \delta_2 \tag{10.9}$$

Applying Wiener's Maximal Inequality, Theorem 10.1,

$$P\left(\sup_{n} \frac{1}{n}\left|\sum_{j=1}^{n}(\widetilde{\varphi}_2 - \psi) \circ T^j\right| > \frac{\delta_3}{L}\right) \leq \frac{\mathbb{E}(|\widetilde{\varphi}_2 - \psi|)}{\delta_3/L} \leq \frac{L\delta_2}{\delta_3} \tag{10.10}$$

Since ψ is an L^2-coboundary,

$$\lim_{n\to\infty} P\left(\frac{1}{n}\left|\sum_{j=1}^{n} \psi \circ T^j\right| > \delta_3\right) = 0, \tag{10.11}$$

so there is $N_2(\delta_2, \delta_3, \delta_3/L)$ such that

$$P\left(\frac{1}{n}\left|\sum_{j=1}^{n}(\psi \circ T^j)\right| > \frac{\delta_3}{L}\right) < \frac{\delta_2}{\delta_3} \qquad \text{for } n \geq N_2(\delta_2, \delta_3, \delta_3/L) \tag{10.12}$$

and therefore, for $n \geq N_2(\delta_2, \delta_3, \delta_3/L)$

$$\begin{aligned} P\left(\frac{1}{n}\left|\sum_{j=1}^{n} \widetilde{\varphi}_2 \circ T^j\right| > 2\frac{\delta_3}{L}\right) &\leq P\left(\frac{1}{n}\left|\sum_{j=1}^{n}(\widetilde{\varphi}_2 - \psi) \circ T^j\right| > \frac{\delta_3}{L}\right) \\ &\quad + P\left(\frac{1}{n}\left|\sum_{j=1}^{n} \psi \circ T^j\right| > \frac{\delta_3}{L}\right) \leq (L+1)\frac{\delta_2}{\delta_3} \end{aligned} \tag{10.13}$$

One has the bound

$$P\left(\frac{1}{n^\gamma}|W_n|>2\delta_3\right) \le P\left(\left(\frac{1}{n^\gamma}|W_n|>2\delta_3\right) \,\&\, \left(\left|\frac{R_n(x)-n\bar R}{n^\gamma}\right| \in [L^{-1},L]\right)\right) + P\left(\left|\frac{R_n(x)-n\bar R}{n^\gamma}\right| \in [0,L^{-1})\cup(L,\infty)\right) \tag{10.14}$$

By (10.6), the second term in (10.14) is at most ε for $n \ge N_1(\varepsilon)$. In view of (10.7), the first term in (10.14) is bounded by

$$P\left(\Delta_n \ge n^\gamma/L \text{ and } \frac{\left|\sum_{j=0}^{\Delta_n}\widetilde{\varphi}\circ T^j\right|}{\Delta_n} > 2\delta_3/L\right)$$

which, by (10.13), it is not more than

$$(L+1)\frac{\delta_2}{\delta_3}$$

for $n^\gamma/L \ge N_2(\delta_2,\delta_3,\delta_3/L)$. In conclusion (recall that $L=L(\varepsilon)$):

$$P\left(\frac{1}{n^\gamma}|W_n|>2\delta_3\right) \le (L(\varepsilon)+1)\frac{\delta_2}{\delta_3}+\varepsilon \qquad \text{if} \quad n^\gamma \ge L(\varepsilon)\cdot N_2(\delta_2,\delta_3,\delta_3/L(\varepsilon)) \text{ and } n \ge N_1(\varepsilon)$$

This shows that (10.4) holds: pick $\varepsilon>0$ arbitrary, set $\delta_3=\varepsilon/2$ and then take $\delta_2>0$ small enough that $(L(\varepsilon)+1)\delta_2/\delta_3<\varepsilon$. □

Case (i): $\frac{1}{\alpha}\ge\gamma$. We suppose $1<\alpha<2$; recall that $\mathbb{E}_Y[\Phi_2]$ is the expectation on (Y,μ_Y), so $\mathbb{E}_Y[\Phi_2]=\bar R\mathbb{E}[\varphi_2]$. The argument we give works equally well for $0<\alpha<1$ by taking $\mathbb{E}_Y[\Phi_2]=0$.

From our result on Gibbs-Markov maps

$$(\bar R n)^{-\frac{1}{\alpha}}\sum_{j=0}^{n}\left(\Phi_2\circ F^j-\mathbb{E}_Y[\Phi_2]\right)$$

converges on Y to a stable law X_α of index α. We will use [Gou08, Theorem 4.6] (see section A.1) to lift this stable law to a stable law for φ_2 under T. We verify condition (b) of Proposition A.2 to show that

$$n^{-\frac{1}{\alpha}}\sum_{j=0}^{n}(\varphi_2\circ T^j-\mathbb{E}(\varphi_2))$$

converges in distribution to X_α. In condition (b) of Proposition A.2 we take $\alpha(n)=n^\gamma$, $A_n=n\mathbb{E}_Y[\Phi_2]$ and $B_n=(n\bar R)^{\frac{1}{\alpha}}$.

The return time R is integrable on the probability space (Y,μ_Y) with expectation $\bar R$. Recall that $R_n(x)=R(x)+R(Fx)+\ldots+R(F^{n-1}x)$.

Note that R satisfies a stable law of index $\frac{1}{\gamma}$ under F (this result is well-known). Indeed, the return-time function R is constant on partition elements of Y and hence measurable with respect to the partition on Y. R is in the domain of attraction of a stable law of index $\frac{1}{\gamma}$ if $\frac{1}{2} < \gamma < 1$ or the CLT if $\gamma < \frac{1}{2}$. By [TK10b, Corollary 4.3]

$$\frac{1}{n^\gamma}\sum_{j=0}^{n}[R\circ F^j - \bar{R}]$$

converges to a stable law of index $\frac{1}{\gamma}$ on Y. Hence

$$\left\{n^{-\gamma}\sum_{j=0}^{n}[R\circ F^j - \bar{R}]\right\}_{n\geq 1}$$

is tight.

Therefore, can apply [Gou08, Theorem 4.6] (see Proposition A.2, part (b)) to conclude that φ_2 satisfies on X a stable law of index α with scaling $b_n = n^{\frac{1}{\alpha}}$ and centering $\mathbb{E}(\varphi_2)/\bar{R}$ (which is $\mathbb{E}_Y(\Phi_2)$).

Now φ_1 satisfies either a CLT (if $\varphi_1(0) - \mathbb{E}[\varphi] = 0$) or a stable law with scaling n^γ (if $\varphi_1(0) - \mathbb{E}[\varphi_1] \neq 0$), and thus the scaled Birkhoff sum $n^{-\frac{1}{\alpha}}\sum_{j=0}^{n}[\varphi_1\circ T^j - \mathbb{E}(\varphi_1)]$ converges in distribution to zero. This proves that

$$\begin{aligned} n^{-\frac{1}{\alpha}}&\left(\sum_{j=1}^{n}(\varphi_1\circ T^j - \mathbb{E}(\varphi_1)) + (\varphi_2\circ T^j - \mathbb{E}(\varphi_2))\right) \\ &= n^{-\frac{1}{\alpha}}\left(\sum_{j=1}^{n}(\varphi\circ T^j - \mathbb{E}(\varphi)\right) \xrightarrow{d} X_\alpha \end{aligned}$$

where X_α has a stable distribution with index α given by [DH95].

Case (iii): $\gamma > \frac{1}{\alpha}$, $\varphi(0) - \mathbb{E}[\varphi] = 0$ and $\alpha < 1 + \frac{1}{\gamma^2} - \frac{1}{\gamma}$. Recall the representation (10.3),

$$\begin{aligned} \sum_{j=0}^{[\bar{R}n]}(\varphi\circ T^j - \mathbb{E}[\varphi]) &= \sum_{j=0}^{n}(\Phi_2\circ F^j - \bar{R}\mathbb{E}[\varphi_2]) + \sum_{j=0}^{\bar{R}n}\psi\circ T^j \\ &+ (\varphi(0) - \mathbb{E}[\varphi_1] - \mathbb{E}[\varphi_2])[R_n(x) - n\bar{R}] + V_n(x) + W_n(x) \end{aligned}$$

As before $n^{-\frac{1}{\alpha}}\sum_{j=0}^{\bar{R}n}\psi\circ T^j$ converges to zero in distribution and under our assumption that $\varphi(0) - \mathbb{E}[\varphi] = 0$ we have

$$(\bar{R}n)^{-\frac{1}{\alpha}}\sum_{j=0}^{[\bar{R}n]}(\varphi\circ T^j - \mathbb{E}[\varphi]) - \left[(\bar{R}n)^{-\frac{1}{\alpha}}\sum_{j=0}^{n}(\Phi_2\circ F^j - \bar{R}\mathbb{E}[\varphi_2]) + (\bar{R}n)^{-\frac{1}{\alpha}}[V_n(x) + W_n(x)]\right] \xrightarrow{d} 0$$

By Corollary A.3 of a result by Dedecker, Gouëzel and Merlevède [DGM12, Theorem 1.7] we have

$$\lim_{n\to\infty} n^{-\frac{1}{\alpha}}\sum_{j=0}^{n^\gamma}(\varphi_2\circ T^j - \mathbb{E}[\varphi_2]) = 0 \ (\mu\text{-a.e. } x\in X)$$

and

$$\lim_{n\to\infty} n^{-\frac{1}{\alpha}} \sum_{j=0}^{n^\gamma} (\varphi_1 \circ T^j - \mathbb{E}[\varphi_1]) = 0 \ (\mu\text{-a.e. } x \in X)$$

in the parameter range we consider. By the Ergodic Theorem

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} [\varphi_2 \circ T^j(x) - \mathbb{E}(\varphi_2)] = 0 \ (\mu\text{-a.e. } x \in X)$$

Since $[n\bar{R} - R_n(.)]/n^\gamma$ converges in distribution to a stable law of index $\frac{1}{\gamma}$, given $\varepsilon > 0$ we may choose $\tau > 0$ and M_1 large enough that

$$\mu_Y \left\{ x \in Y : |n\bar{R} - R_n(x)| > \tau n^\gamma \right\} < \varepsilon$$

for all $n \geq M_1$.

Thus for all $n \geq M_1$ the set

$$B_n := \{x \in Y : |n\bar{R} - R_n(x)| > \tau n^\gamma\}$$

satisfies $\mu_Y(B_n) < \varepsilon$.

Choose $M_2 > M_1$ large enough that

$$\mu_Y \left\{ x \in Y : \max_{M_2 \leq k \leq \tau n^\gamma} n^{-\frac{1}{\alpha}} \left| \sum_{j=0}^{k} [\varphi_2 \circ T^j(x) - \mathbb{E}(\varphi_2)] \right| > \varepsilon \right\} < \varepsilon$$

Note that this implies that for all $n > M_2$,

$$\mu_Y \left\{ x \in Y : \text{ for all } M_2 \leq k \leq \tau n^\gamma, n^{-\frac{1}{\alpha}} \left| \sum_{j=0}^{k} [\varphi_2 \circ T^j(x) - \mathbb{E}(\varphi_2)] \right| < \varepsilon \right\} > 1 - \varepsilon$$

By measure preservation

$$\mu_Y \{x \in Y : \text{ for all } M_2 \leq k \leq \tau n^\gamma, |n^{-\frac{1}{\alpha}} \sum_{j=0}^{k} [\varphi_2 \circ T^j(T^{n\bar{R}} x) - \mathbb{E}(\varphi_2)]| < \varepsilon_1\} > 1 - \varepsilon$$

and

$$\mu_Y \{x \in Y : \text{ for all } M_2 \leq k \leq \tau n^\gamma, |n^{-\frac{1}{\alpha}} \sum_{j=n\bar{R}-n^\gamma}^{k} [\varphi_2 \circ T^j(T^{n\bar{R}-n^\gamma} x) - \mathbb{E}(\varphi_2)]| < \varepsilon\} > 1 - \varepsilon$$

Thus except for a set of points $x \in Y$ of μ_Y measure less than 2ε

$$\left| n^{-\frac{1}{\alpha}} \left(\sum_{j=0}^{n\bar{R}} [\varphi_2 \circ T^j(x) - \mathbb{E}(\varphi_2)] \right) - (n\bar{R})^{-\frac{1}{\alpha}} \sum_{j=0}^{n} \left(\Phi_2 \circ F^j(x) - \mathbb{E}_Y[\Phi_2] \right) \right| < 2\varepsilon$$

Since $(n\bar{R})^{-\frac{1}{\alpha}}\sum_{j=0}^{n}(\Phi_2\circ F^j(x)-\mathbb{E}_Y[\Phi_2])$ converges in distribution to a stable law of index α we see that

$$n^{-\frac{1}{\alpha}}\sum_{j=0}^{n\bar{R}}(\varphi_2\circ T^j(x)-\mathbb{E}[\varphi_2])$$

converges in distribution to a stable law of index α.

Thus the stable law for Φ_2 lifts to φ_2. If $\varphi(0)-\mathbb{E}(\varphi)=0$ then the scaling is $n^{\frac{1}{\alpha}}$ and if $\varphi(0)-\mathbb{E}(\varphi)\neq 0$ then the scaling is n^{γ}. □

11. Discussion

In Case (iii) of Theorem 8.4, when $\varphi(0)-\mathbb{E}[\varphi]=0$ and $\gamma>\frac{1}{\alpha}$, we require the condition $\alpha<1+\frac{1}{\gamma^2}+\frac{1}{\gamma}$ which arises from the dependence of our proof on the almost sure convergence result of [DGM12, Theorem 7.1]. A weaker distributional convergence would suffice and we conjecture that $\alpha<1+\frac{1}{\gamma^2}+\frac{1}{\gamma}$ is not necessary.

In Theorem 8.1 we show that small jumps are "negligible" for a wide class of heavy-tailed functions on Gibbs-Markov maps. This result is used to investigate the interplay between the effects of heavy-tails and slow-mixing in a common model of intermittency for observables of form $\varphi(x)=d(x,x_0)^{-\frac{1}{\alpha}}$. Our results are for stable laws but suggest that convergence in stronger topologies may hold for all $\alpha>\frac{1}{\gamma}$, $0<\gamma<1$.

Appendix A.

Lemma A.1. *A slowly varying function L grows slower than any power.*

Proof. Let $\delta>0$ be arbitrary. Using the Representation Theorem (see e.g. [BGT87, Theorem 1.3.1]):

$$\frac{L(x)}{x^\delta}\sim\frac{c(x)\exp\left(\int_1^x\frac{\varepsilon(s)}{s}ds\right)}{\exp(\delta\int_1^x\frac{1}{s}\,d\,s)}=c(x)\exp\left(\int_1^x\frac{\varepsilon(s)-\delta}{s}ds\right)$$

with $c(x)\to c\in(0,\infty)$ and $\varepsilon(x)\to 0$ as $x\to\infty$. □

A.1. **A result of Gouëzel.** We use the following result of Gouëzel [Gou08, Theorem 4.6]:

Proposition A.2. *Let (T,X,μ) be an ergodic probability preserving map, let $\alpha(n)$ and and B_n be two sequences of integers which are regularly varying with positive indices. Let $A_n\in\mathbb{R}$ and let $Y\subset X$ be a subset with positive measure. We will denote by $\mu_Y(.):=\frac{\mu|_Y}{\mu(Y)}$ the induced probability measure. Let $R:Y\to\mathbb{N}$ be the return time of T to Y and $F=T^R:Y\to Y$ be the the induced map. Define $\bar{R}=\int_Y R d\mu=\frac{1}{\mu(Y)}$. Consider a measurable function $\varphi:X\to\mathbb{R}$ and define $\Phi:Y\to\mathbb{R}$ by $\Phi(y)=\sum_{j=0}^{R(y)-1}\varphi\circ T^j$. Define $S_n(\Phi)=\sum_{j=0}^{n-1}\Phi\circ F^j$. Assume that*

$$\frac{S_n(\Phi)-A_n}{B_n}$$

converges in distribution (with respect to μ_Y) to a random variable S.

Additionally assume that either:

(a) $\frac{\sum_{j=0}^{n} R\circ F^j - n\bar{R}}{\alpha(n)}$ *tends in probability to zero and* $\max_{0\le k\le \alpha(n)} \frac{|S_k(\Phi)|}{B_n}$ *is tight*

or

(b) $\frac{\sum_{j=0}^{n} R\circ F^j - n\bar{R}}{\alpha(n)}$ *is tight and* $\max_{0\le k\le \alpha(n)} \frac{|S_k(\Phi)|}{B_n}$ *tends in probability to zero.*

Then

$$\Big(\sum_{j=0}^{n-1} \varphi\circ T^j - A_{\lfloor n\mu(Y)\rfloor}\Big)/B_{\lfloor n\mu(Y)\rfloor}$$

converges in distribution (with respect to μ*) to* S.

A.2. **A result of Dedecker, Gouëzel and Merlevède.** We paraphrase the results of Dedecker, Gouëzel and Merlevède [DGM12, Theorem 1.7] that we use for the benefit of the reader. They define a class of functions $\mathcal{F}(H,\mu)$ which are observables on intermittent-type maps. Let μ be a probability measure on $\mathbb{R}$ and H a tail function. Let $\mathrm{Mon}(H,\mu)$ denote the set of functions $f:\mathbb{R}\to\mathbb{R}$ which are monotonic on some open interval and null elsewhere such that $\mu(|f|>t)\le H(t)$. We define $\mathcal{F}(H,\mu)$ to be the closure in $L^1(\mu)$ of the set of functions that can be written as $\sum_{j=0}^{l} a_j f_j$ where $\sum_{j=0}^{l}|a_j|\le 1$ and $f_j\in\mathrm{Mon}(H,\mu)$.

Suppose the LSV map has parameter $0<\gamma<1$, φ is an observable which lies $\mathcal{F}(H,\mu)$ where $H(t)\sim t^{-\alpha}$ and $\mu(|\varphi|>t)\le t^{-\alpha}$. As a consequence of Dedecker, Gouëzel and Merlevède [DGM12, Theorem 1.7] if:

(i) $1<p\le 2$ and $0<\gamma<\frac{1}{p}$
(ii) $H(t)^{(1-p\gamma)/(1-\gamma)}\le Ct^{-p}$

then for any $b>\frac{1}{p}$

$$n^{-\frac{1}{p}}(\ln(n))^{-b}\sum_{j=0}^{n-1}[\varphi\circ T^j-\mu(\varphi)]\to 0 \quad \mu\text{-a.e.}$$

Corollary A.3. *Suppose* $\varphi=|x-x_0|^{-\frac{1}{\alpha}}$, $x_0\neq 0$, *is an observable on a LSV map given by Equation 6.1. Define* $\varphi_1=\varphi\mathbf{1}_{Y^c}$ *and* $\varphi_2=\varphi\mathbf{1}_Y$ *where* Y *is an interval containing* x_0. *If* $\gamma>\frac{1}{\alpha}$ *and* $\alpha<1+\frac{1}{\gamma^2}+\frac{1}{\gamma}$ *then for* $i=1,2$

$$\lim_{n\to\infty} n^{-\frac{1}{\alpha\gamma}}\sum_{j=0}(\varphi_i\circ T^j-\mathbb{E}[\varphi_i])=0$$

μ *a.e. and hence*

$$\lim_{n\to\infty} n^{-\frac{1}{\alpha}}\sum_{j=0}^{n^\gamma}(\varphi_i\circ T^j-\mathbb{E}[\varphi_i])=0$$

μ *a.e.*

Proof of Corollary. In the expression

$$n^{-\frac{1}{\alpha}}\sum_{j=0}^{n^\gamma}(\varphi\circ T^j-\mathbb{E}_\mu[\varphi])$$

let $m=n^\gamma$ (leaving out integer part notation) then we may rewrite the expression above as

$$m^{-\frac{1}{\alpha\gamma}}\sum_{j=0}^{m}(\varphi\circ T^j-\mathbb{E}_\mu[\varphi])$$

For any sufficiently small $\delta>0$ we will show that we may take $p=\alpha\gamma+\delta$ in Dedecker, Gouëzel and Merlevède [DGM12, Theorem 1.7] and conclude

$$\lim_{m\to\infty} m^{-\frac{1}{\alpha\gamma}}\sum_{j=0}^{m}(\varphi\circ T^j-\mathbb{E}_\mu[\varphi])=0$$

μ a.e.

We need to check the conditions of [DGM12, Theorem 1.7]. Our functions φ_i, $i=1,2$ fall in their class of functions $\mathcal{F}(H,\mu)$ where the tail function is $H(t)\sim t^{-\alpha}$ as $t\to\infty$. It can be seen that the result for φ_1, which is bounded, follows immediately but estimates are required for φ_2 since $\mu(\varphi_2>t)\sim t^{-\alpha}$. For small $\delta>0$ the condition $1<p\leq 2$ and $\gamma<\frac{1}{p}$ are satisfied if $\gamma^2<\frac{1}{\alpha}$ as $p=\alpha\gamma+\delta$. Now we consider condition (1.7)

$$H(t)^{(1-p\gamma)/(1-\gamma)}\leq Ct^{-p}$$

This condition is satisfied if $p<\frac{\alpha}{\alpha\gamma+1-\gamma}$. Taking $\delta>0$ small this condition follows if $\gamma<\frac{1}{\alpha\gamma+1-\gamma}$, which is equivalent to $\gamma-\gamma^2<1-\alpha\gamma^2$. The condition $\gamma-\gamma^2<1-\alpha\gamma^2$ imposes more restrictions than $\gamma^2<\frac{1}{\alpha}$. Thus the conditions of Dedecker, Gouëzel and Merlevède [DGM12, Theorem 1.7] are satisfied in our setting for both φ_1 and φ_2 if $\frac{1}{\gamma}<\alpha<1+\frac{1}{\gamma^2}-\frac{1}{\gamma}$. In this case we may take $p=\alpha\gamma$. As an illustrative example, if $\gamma=\frac{2}{3}$ we require $\frac{3}{2}<\alpha<\frac{7}{4}$.

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