MATH 2331
PROPERTIES OF THE DETERMINANT

Adapted from *Introduction to Linear Algebra* by Gilbert Strang, 4^{th} edition.

1. Defining Properties

(1) The determinant of the $n \times n$ identity matrix is 1.
(2) If A and B are $n \times n$ matrices, and B is obtained from A by exchanging two rows of A, then $\det B = -\det A$.
(3) (a) If B is obtained from A by multiplying row j of A by c, then $\det B = c \det A$.
(b) If

$$A = \begin{bmatrix} \mathbf{u}+\mathbf{v} \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix}, \; A' = \begin{bmatrix} \mathbf{u} \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix}, \; A'' = \begin{bmatrix} \mathbf{v} \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{bmatrix}$$

then $\det A = \det A' + \det A''$.

2. Properties that are proved from the defining properties

(4) If two rows of A are equal, then $\det A = 0$.
(5) If B is obtained from A by subtracting a multiple of one row from another row , then $\det B = \det A$.
(6) If A has a row of zeros, $\det A = 0$.
(7) If A is upper triangular or lower triangular, then $\det A = a_{11}a_{22}\dots a_{nn} =$ product of the diagonal.
(8) If A is singular then $\det A = 0$. If A is invertible then $\det A \neq 0$.
(9) $\det AB = \det A \det B$, if A and B are $n \times n$ matrices.
(10) $\det A^T = \det A$.

3. Cofactor formula

$$\det A = \sum_{j=1}^{n} a_{ij} \, (-1)^{i+j} \det A_{ij} = \sum_{i=1}^{n} a_{ij} \, (-1)^{i+j} \det A_{ij}$$

The first sum is a sum across row i, where i is a fixed integer between 1 an n. The second sum is taken over column j, $1 \leq j \leq n$. A_{ij} is the matrix that remains after row i and column j are deleted.

The ij-th *cofactor* of A is $C_{ij} = (-1)^{i+j} \det A_{ij}$. One can show from the cofactor formula that $C = \left(A^T\right)^{-1} \det A$. This implies *Cramer's rule* (section 3.3).

4. The Big Formula

Let S_n be the set of all $n \times n$ permutation matrices. S_n is actually a group, but we don't need that. If $P \in S_n$ define $P(j) = i$ if and only if $P_{ij} = 1$. Because P has only one "1" in each column, for each j there is only one such i. Then

$$\det A = \sum_{P \in S_n} a_{1P(1)} a_{2P(2)} \cdots a_{nP(n)} \det P.$$

4.1. **Example.** Let

$$A = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & d & 1 \end{bmatrix}$$

Then

$$\det A = c \det I + b \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} + a \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} + d \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = c.$$

We could express this in terms of permutation matrices like this:

$$\det A = c \det I + b \cdot 0 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} + a \cdot 0 \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} + d \cdot 0 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = c.$$

4.2. **Example.** Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then

$$\det A = 0 \det I + 1 \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = 1.$$

All other terms in the expansion of the determinant of A are zero.

The cofactor formula would also work well for these examples.

5. Two proofs of property 10: $\det A^T = \det A$

5.1. **First proof.** One proof of this property involves the "LU" factorization that comes from Gaussian Elimination. The basic elimination step is subtraction of a multiple l_{ij} of row i from row j, with $i < j$. This is the same as multiplying A on the left by a matrix E_{ij} which is $I - l_{ij} e_i e_j^T$, or I with $-lij$ in row i, column j. Every such E_{ij} is lower triangular with 1's down the diagonal. If we do three such steps on a 3×3 A, we obtain:

$$E_{32} E_{31} E_{21} A = U,$$

where U is lower triangular. If $E = E_{32} E_{31} E_{21}$ then E is lower triangular with 1's down the diagonal.

The inverse of E_{ij} is L_{ij} where L_{ij} is E_{ij} with the sign of l_{ij} reversed, to add l_{ij} times row i back to row j. Then

$$E^{-1} = L_{21}L_{31}L_{32} = L.$$

L is lower triangular with 1's down the diagonal. As an interesting note, for the 3×3 example,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}.$$

Then $A = LU$, so $\det A = \det L \det U = \det U = u_{11}u_{22}\cdots u_{nn}$. If row exchanges were performed, then for some permutation P we have $PA = LU$, and $\det A = \det P^{-1} \det U = \det P \det U$. ($P^{-1} = P^T$ so $\det P \det P^T = 1$ and both are ± 1 by row exchanges, so $\det P = \det P^T$.)

Now $A^T = (LU)^T = U^T L^T$, so that $\det A^T = \det\left(U^T\right) \det\left(L^T\right)$, where U^T is lower triangular and L^T is upper triangular with 1's down the diagonal. So $\det A^T = u_{11}\cdots u_{nn} = \det A$. If row exchanges were performed then $A^T = U^T L^T P^T$ and $\det A^T = \det U \det P = \det A$.

Whew!

5.2. **Second proof.** This proof uses the Big Formula. Let $A^T = B$, with $a_{ij} = b_{ji}$.

$$\begin{aligned} \det A &= \sum_{P \in S_n} a_{1P(1)}a_{2P(2)}\cdots a_{nP(n)} \det P, \\ &= \sum_{P \in S_n} a_{P^{-1}(1)1}a_{P^{-1}(2)2}\cdots a_{P^{-1}(n)n} \det P \\ &= \sum_{P \in S_n,\ T = P^{-1}} a_{T(1)1}a_{T(2)2}\cdots a_{T(n)n} \det P \end{aligned}$$

Note again that $\det T = \det P^{-1} = \det P$. So

$$\begin{aligned} \det A &= \sum_{T \in S_n} a_{T(1)1}a_{T(2)2}\cdots a_{T(n)n} \det T \\ &= \sum_{T \in S_n} b_{1T(1)}b_{2T(2)}\cdots b_{nT(n)} \det T = \det B = \det A^T. \end{aligned}$$