Problem 1.

(a) Find the domain and range of each of the following functions:

(i) $f(x,y) = \ln\sqrt{1+x^2+y^2}$ (ii) $F(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2-1}}$

(b) Identify the level curves/surfaces of each of the following functions:

(i) $f(x,y) = e^{-4x^2-y^2}$ (ii) $F(x,y,z) = 2x+3y+6z$

(c) An open rectangular container (i.e., no top) is to have a volume of 12 cubic feet. The cost of the material for the sides is \$3 per square foot and the cost for the base is \$5 per square foot. Express the total cost C of the container as a function of its length x and width y.

Problem 2. Let $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$

(a) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ if $(x,y)\to(0,0)$ along the x-axis.

(b) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ if $(x,y)\to(0,0)$ along the y-axis.

(c) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ if $(x,y)\to(0,0)$ along the line $y=mx$.

(d) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ if $(x,y)\to(0,0)$ along the parabola $y=\lambda x^2,\ \lambda>0$.

(e) Does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist ?

Problem 3.

(a) Let $f(x,y) = y^2e^{xy} + \dfrac{x}{y}$. Calculate f_{xx} and f_{yx}.

(b) Let $z = \ln\sqrt{x^2+y^2}$. Show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 1$

(c) Let $u = x^2-2y^2+z^3$ where $x=\sin t,\ y=e^{2t},\ z=3t$. Calculate $\dfrac{du}{dt}$ and express your answer in terms of t.

(d) Let $z = e^{2x}\ln y$ where $x=u^2-2v$ and $y=v^2-2u$. Calculate $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$

Problem 4. Let $f(x,y) = x\tan^{-1}\left(\dfrac{y}{x}\right)$ and $F(x,y,z) = x^2+3yz+4xy$.

(a) (i) Find the gradient of F.

(ii) Determine the direction in which f decreases most rapidly at the point $(2,2)$. At what rate is f decreasing?

(b) Find the directional derivative of F at the point $(1, 1, -5)$ in the direction of the vector $\mathbf{a} = 2\,\mathbf{i} + 3\,\mathbf{j} - \sqrt{3}\,\mathbf{k}$.

(c) Find an equation for the tangent plane to the level surface $F(x, y, z) = 3$ at the point $(3, -1, -2)$.

(d) Find an equation for the tangent plane and scalar parametric equations for the normal line to the surface $z = f(x, y)$ at the point $(2, -2, -\pi/2)$.

Problem 5. In each of the following, determine whether $\mathbf{F}$ is the gradient of a function f. If it is, find all such functions f.

(a) $\mathbf{F}(x, y) = \big(3x^2y^2 + 3y + x\big)\ \mathbf{i} + \big(2x^3y + 3xy - \sqrt{y}\big)\ \mathbf{j}$.

(b) $\mathbf{F}(x, y) = \big(2x\,e^y + 4xy + e^{2x}\big)\ \mathbf{i} + \big(x^2\,e^y + 2x^2 + \cos 2y - 1\big)\ \mathbf{j}$.

Problem 6.

(a) Find the stationary points of $f(x, y) = x^2 + 2y^2 - x^2y$.

(b) For each stationary point P found in (a), determine whether f has a local maximum, a local minimum, or a saddle point at P.

Problem 7.

(a) Find the absolute maximum and absolute minimum values of $f(x, y) = x^2 + 2y^2 - x$ on the closed disk $D : x^2 + y^2 \leq 1$.

(b) Find the absolute maximum and absolute minimum values of $f(x, y) = 2 + 2x + 2y - x^2 - y^2$ on the closed triangular region bounded by the lines $x = 0,\ y = 0,\ x + y = 9$.

Problem 8.

(a) According to U.S. Postal Service regulations, the length plus the girth (perimeter of a cross-section) of a package cannot exceed 108 inches. What are the dimensions of the rectangular box of maximum volume that is acceptable for mailing? What is the maximum volume?

(b) A rectangular box without a top is to have a volume of 12 cubic feet. The materials used to construct the box cost \$3 per square foot for the sides and \$4 per square foot for the bottom. What dimensions will yield the minimum cost?