

Section 5.4 - Inverse Trigonometry

In this section, we will define inverse sine, cosine and tangent functions.

RECALL – Facts about inverse functions:

A function $f(x)$ is **one-to-one** if no two different inputs produce the same output (or: passes the horizontal line test)

Example: $f(x) = x^2$ is NOT one-to-one. $g(x) = x^3$ is one-to-one.

A function $f(x)$ is **invertible** if it is one-to-one.

The inverse function is notated as: $f^{-1}(x)$.

$f : A \rightarrow B$	$f^{-1} : B \rightarrow A$
Domain: A	Domain: B
Range: B	Range: A

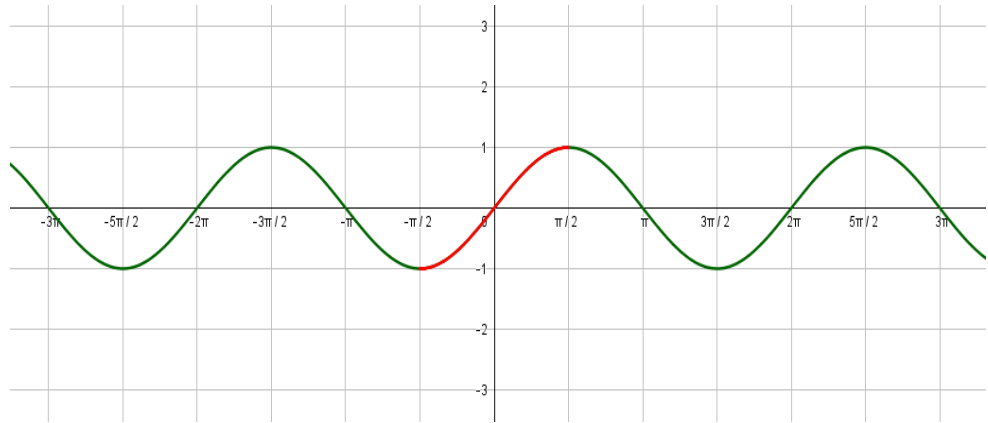
Important: $f(a) = b$ if and only if $f^{-1}(b) = a$.

INVERSE SINE FUNCTION

Here's the graph of $f(x) = \sin(x)$.

Domain: $(-\infty, \infty)$

Range: $[-1, 1]$



Is it one to one?

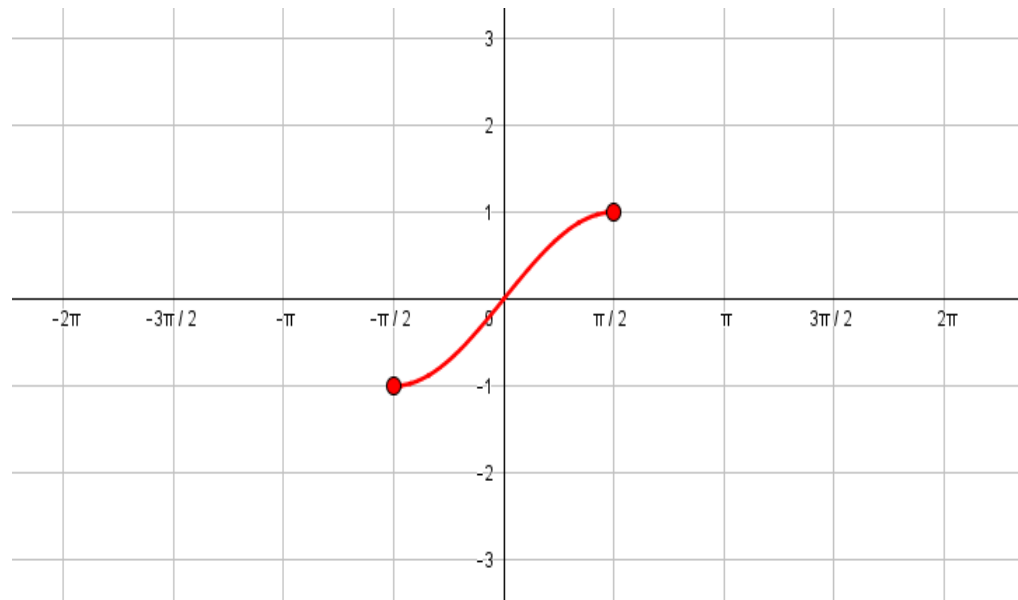
If the function is not one-to-one, we run into problems when we consider the inverse of the function. What we want to do with the sine function is to restrict the values for sine. When we make a careful restriction, we can get something that IS one-to-one.

If we limit the function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the graph will look like this:

Restricted Sine function

Domain: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

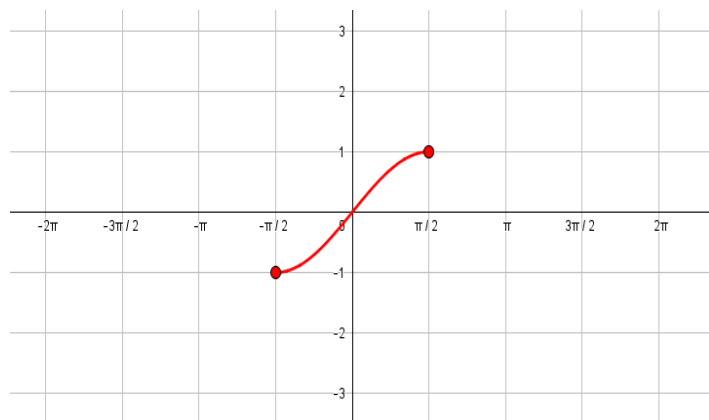
Range: $[-1, 1]$



On this limited interval, we have a one-to-one function.

INVERSE SINE FUNCTION

Here's the graph of restricted sine function:



Restricted Sine function	Inverse Sine Function:
$f(x) = \sin(x)$	$\sin^{-1}(x)$ or $\arcsin(x)$
Domain: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	Domain: $[-1, 1]$
Range: $[-1, 1]$	Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (quadrants 1 and 4)

Example: $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \rightarrow \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Example: $\sin^{-1}\left(\frac{1}{2}\right) = ?$

Question: What is the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $1/2$?

Example: $\sin^{-1}(1) = ?$

Question: What is the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is 1 ?

Example: $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = ?$

Question: What is the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $\frac{\sqrt{3}}{2}$?

Important: When we covered the unit circle, we saw that there were two angles that had the same value for most of our angles. With inverse trig functions, this will not happen since we start with restricted functions that are one-to-one. We'll have one quadrant in which the values are positive and one quadrant where the values are negative. The restricted graphs we looked at can help us know where these values lie. We'll only state the values that lie in these intervals (same as the intervals for our graphs):

Example: $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$;

However, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ (unique answer!) since $\frac{5\pi}{6}$ is Not in the range of inverse sine function.

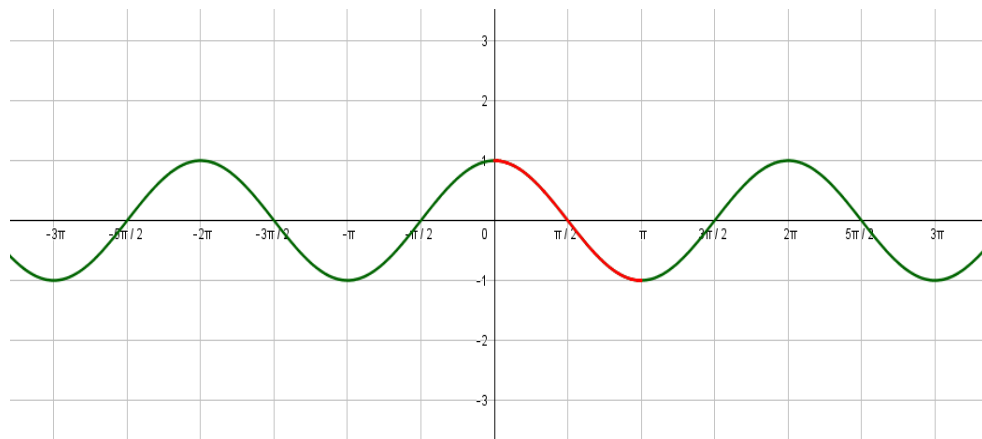
INVERSE COSINE FUNCTION

Let's do the same thing with $f(x) = \cos(x)$.

Here's the graph of $f(x) = \cos(x)$.

Domain: $(-\infty, \infty)$

Range: $[-1, 1]$



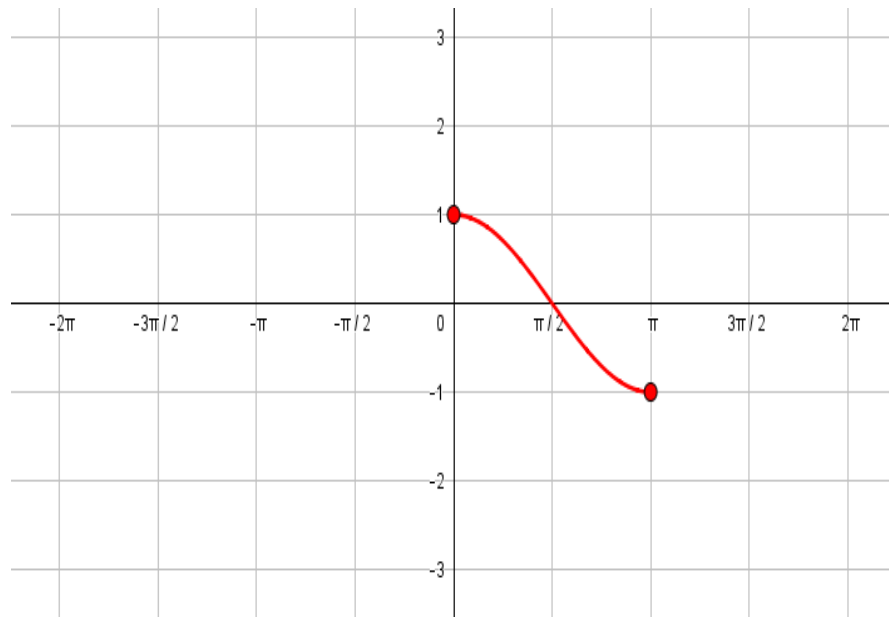
It's not one-to-one. If we limit the function to the interval $[0, \pi]$, however, the function IS one-to-one.

Here's the graph of the restricted cosine function.

Restricted Cosine function

Domain: $[0, \pi]$

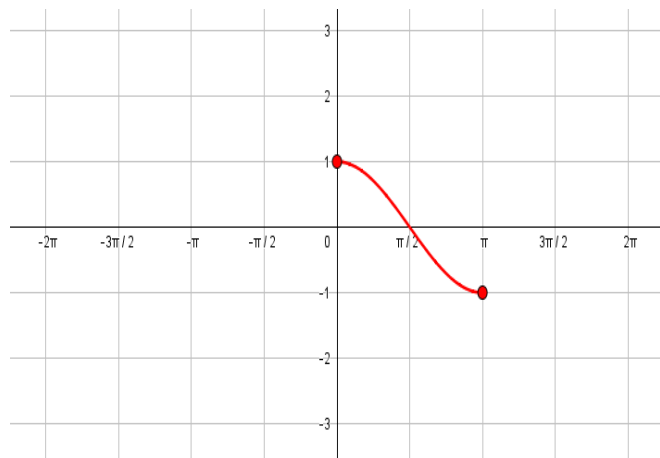
Range: $[-1, 1]$



INVERSE COSINE FUNCTION

Now, let's work on defining the inverse of cosine function.

Here's the graph of restricted cosine function:



Restricted Cosine function	Inverse Cosine Function:
$f(x) = \cos(x)$	$\cos^{-1}(x)$ or $\arccos(x)$
Domain: $[0, \pi]$	Domain: $[-1, 1]$
Range: $[-1, 1]$	Range: $[0, \pi]$ (quadrants 1 and 2)

Example: $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \rightarrow \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Example: $\cos^{-1}\left(\frac{1}{2}\right) = ?$

Question: What is the angle in $[0, \pi]$ whose cosine is $\frac{1}{2}$?

Example: $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = ?$

Question: What is the angle in $[0, \pi]$ whose cosine is $\frac{\sqrt{2}}{2}$?

Example: $\cos^{-1}(1) = ?$

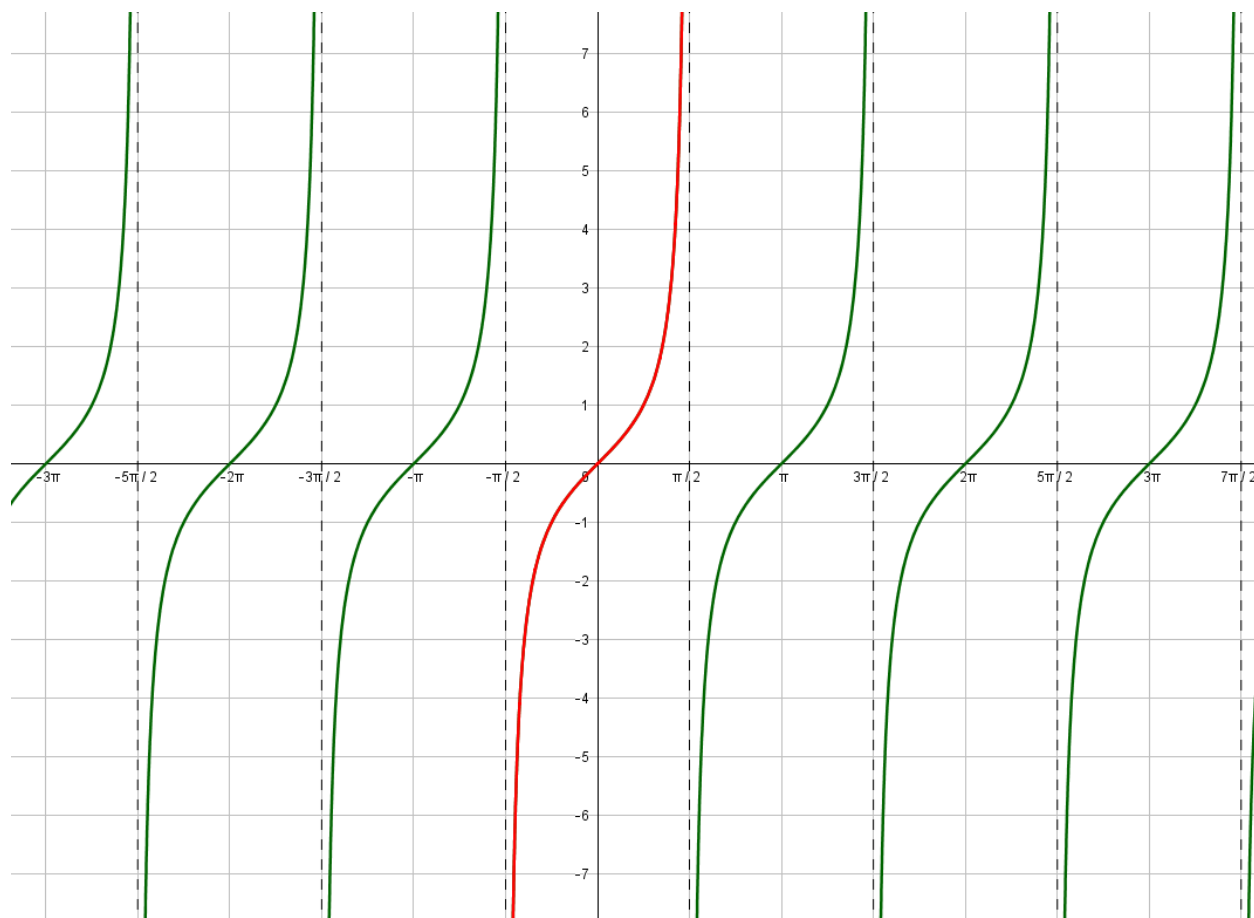
Question: What is the angle in $[0, \pi]$ whose cosine is 1?

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Question#1: What is the RANGE of $g(x) = \cos^{-1}(x)$?

INVERSE TANGENT FUNCTION

Here's the graph of $f(x) = \tan(x)$. Is it one-to-one?

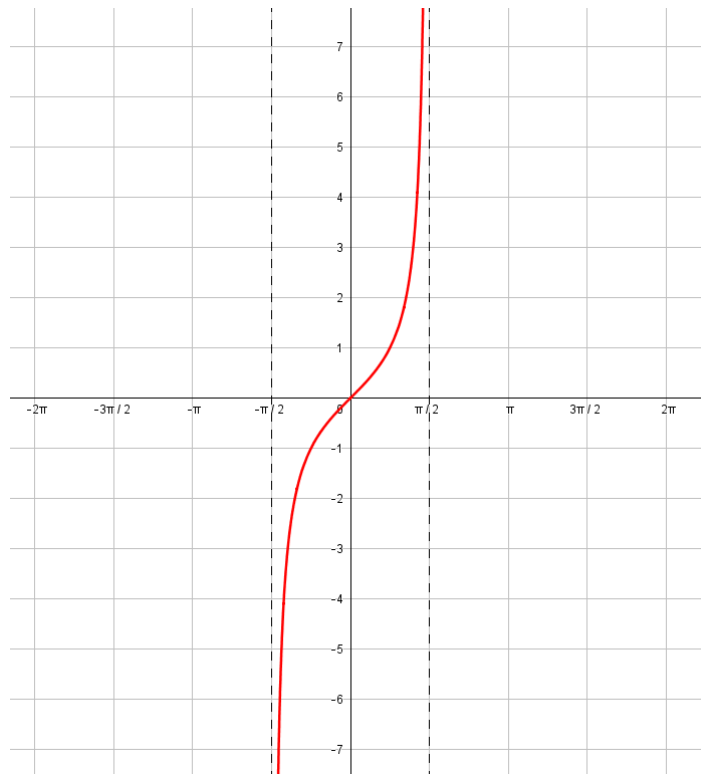


If we restrict the function to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then the restricted function IS one-to-one.

Restricted Tangent function

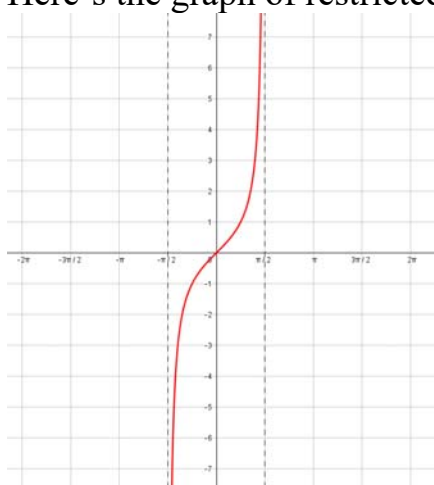
Domain: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Range: $(-\infty, \infty)$



Inverse Tangent Function

Here's the graph of restricted tangent function:



Restricted Tangent function	Inverse Tangent Function:
$f(x) = \tan(x)$	$\tan^{-1}(x)$ or $\arctan(x)$
Domain: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$	Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (quadrants 1 and 4)

Example: $\tan\left(\frac{\pi}{4}\right) = 1 \rightarrow \tan^{-1}(1) = \frac{\pi}{4}$

Example: $\tan^{-1}(1) = ?$

Question: What is the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is 1?

Example: $\tan^{-1}(0) = ?$

Question: What is the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is 0?

Example: $\tan^{-1}(-1) = ?$

Question: What is the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is -1?

Note: We always give inverse trig angles in **radians**.

Example 1: Compute each of the following:

a) $\cos^{-1}(0) =$

b) $\tan^{-1}(\sqrt{3}) =$

c) $\sin^{-1}\left(-\frac{1}{2}\right) =$

d) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) =$

e) $\arccos\left(-\frac{\sqrt{2}}{2}\right)$

Example 2: Compute

$$\arcsin\left(\frac{1}{2}\right) + \arccos(0) + \arctan(-1)$$

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Question#2: Find the following sum:

$$\arcsin\left(\frac{\sqrt{2}}{2}\right) + \arccos(0)$$

Note: If you need to compute **inverse secant** or **inverse cosecant** functions:

Question: $\sec^{-1}(2) = ?$

First, call it an angle: $\sec^{-1}(2) = \theta$

Then, convert: $\sec(\theta) = 2$

Now, express this in terms of cosine: $\cos(\theta) = \frac{1}{2}$

And answer according to “inverse cosine function”: $\theta = \frac{\pi}{3}$

Final answer: $\sec^{-1}(2) = \frac{\pi}{3}$

Question: $\csc^{-1}(1) = ?$

First, call it an angle: $\csc^{-1}(1) = \theta$

Then, convert: $\csc(\theta) = 1$

Now, express this in terms of sine: $\sin(\theta) = 1$

And answer according to “inverse sine function”: $\theta = \frac{\pi}{2}$

Final answer: $\csc^{-1}(1) = \frac{\pi}{2}$

NOTE: Domains of inverse trig functions:

$$f(x) = \sin^{-1}(x); \quad [-1, 1]$$

$$f(x) = \cos^{-1}(x); \quad [-1, 1]$$

$$f(x) = \tan^{-1}(x); \quad (-\infty, \infty)$$

$$f(x) = \cot^{-1}(x); \quad (-\infty, \infty)$$

$$f(x) = \sec^{-1}(x); \quad (-\infty, -1] \cup [1, \infty)$$

$$f(x) = \csc^{-1}(x); \quad (-\infty, -1] \cup [1, \infty)$$

For example; $\sin^{-1}(2)$ or $\cos^{-1}(\sqrt{2})$ are not defined.

Composition of a trig function with its inverse:

Example 3: Find the exact value: $\sin^{-1}\left[\sin\left(\frac{7\pi}{6}\right)\right]$.

Example 4: Find the exact value: $\cos^{-1}\left[\cos\left(\frac{4\pi}{3}\right)\right]$.

Example 5: Find the exact value: $\tan^{-1}\left[\tan\left(\frac{3\pi}{4}\right)\right]$.

Note: If a trigonometric function and its inverse are composed, then we have a shortcut. However, we need to be careful about giving an answer that is in the range of the inverse trig function.

$$\begin{array}{lll} \sin^{-1}(\sin(x)) = x & \text{when} & x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \cos^{-1}(\cos(x)) = x & \text{when} & x \in [0, \pi] \\ \tan^{-1}(\tan(x)) = x & \text{when} & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{array}$$

Examples:

$$\sin^{-1}\left[\sin\left(\frac{\pi}{8}\right)\right] = \frac{\pi}{8} \quad \text{but} \quad \sin^{-1}\left[\sin\left(\frac{7\pi}{8}\right)\right] \neq \frac{7\pi}{8}$$

$$\cos^{-1}\left[\cos\left(\frac{\pi}{8}\right)\right] = \frac{\pi}{8} \quad \text{but} \quad \cos^{-1}\left[\cos\left(\frac{9\pi}{8}\right)\right] \neq \frac{9\pi}{8}$$

$$\tan^{-1}\left[\tan\left(\frac{\pi}{8}\right)\right] = \frac{\pi}{8} \quad \text{but} \quad \tan^{-1}\left[\tan\left(\frac{7\pi}{8}\right)\right] \neq \frac{7\pi}{8}$$

If the inverse trig function is the inner function, then our job is easier:

$$\begin{array}{lll} \sin(\sin^{-1}(x)) = x & \text{when} & x \in [-1, 1] \\ \cos(\cos^{-1}(x)) = x & \text{when} & x \in [-1, 1] \\ \tan(\tan^{-1}(x)) = x & \text{when} & x \in (-\infty, \infty) \end{array}$$

Examples:

$$\sin\left[\sin^{-1}\left(\frac{1}{5}\right)\right] = \frac{1}{5}$$

$$\cos\left[\cos^{-1}\left(-\frac{2}{7}\right)\right] = -\frac{2}{7}$$

$$\tan[\tan^{-1}(5)] = 5 \ .$$

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Question#3: Evaluate: $\sin\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$

Let's work with composition of different trig and inverse trig functions:

Example 6: Find the exact value: $\cos\left[\sin^{-1}\left(\frac{5}{13}\right)\right]$.

Example 7: Find the exact value: $\tan\left(\cot^{-1}\left(\frac{2}{5}\right)\right) =$

Example 8: Find the exact value: $\tan\left[\cos^{-1}\left(-\frac{4}{5}\right)\right]$.

Example 9: Find the exact value: $\sin\left(\arccos\left(-\frac{1}{4}\right)\right) =$

Example 10: Find the exact value: $\tan(\sec^{-1}(2)) =$

Example 11: Find the exact value: $\sin(\cos^{-1}(-4))=$

Example 12: Let $y = \arctan\left(\frac{x}{4}\right)$ where $x > 0$. Express $\cos(y)$ in terms of x .

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Question#4: Evaluate: $\tan\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

Graphs of Inverse Trigonometric Functions

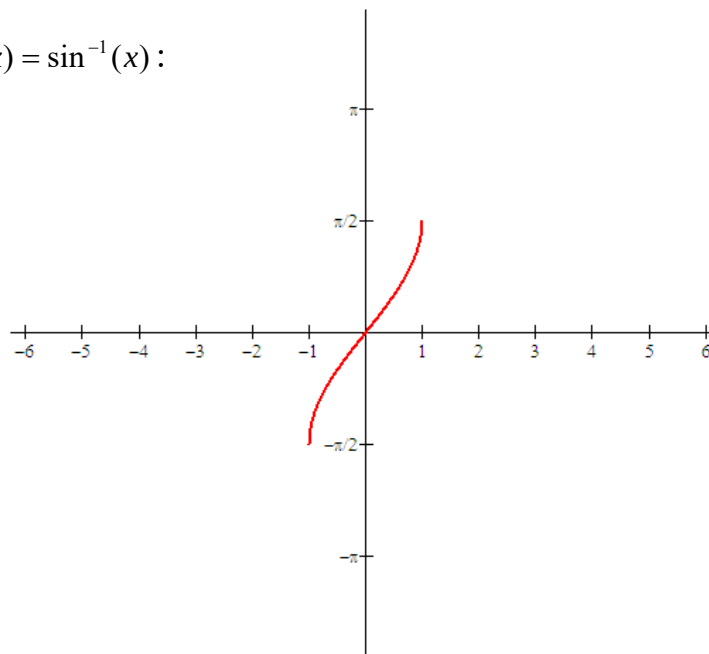
We note the inverse sine function as $f(x) = \sin^{-1}(x)$ or $f(x) = \arcsin(x)$.

Domain: $[-1, 1]$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Key points: $\left(-1, -\frac{\pi}{2}\right), (0, 0), \left(1, \frac{\pi}{2}\right)$

Here is the graph of $f(x) = \sin^{-1}(x)$:



Inverse Cosine Function

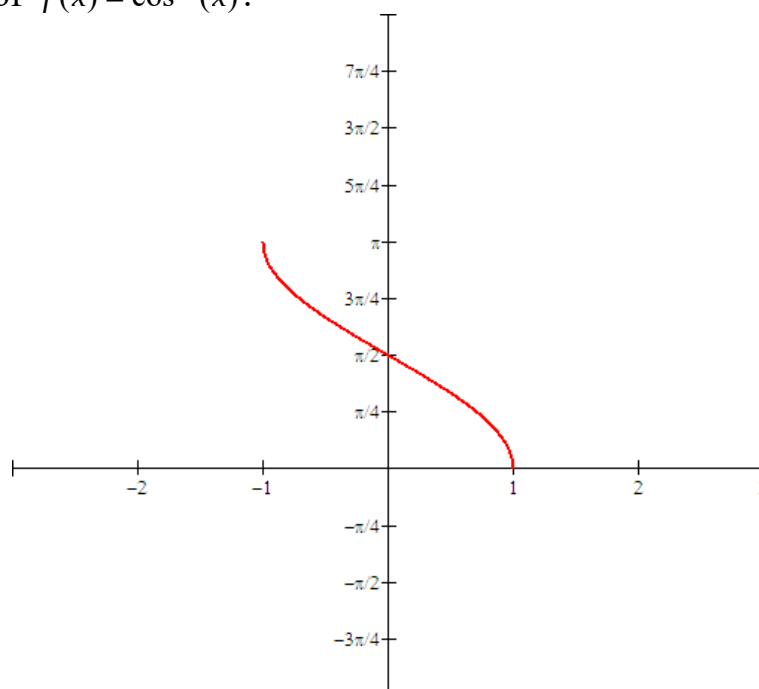
We note the inverse cosine function as $f(x) = \cos^{-1}(x)$ or $f(x) = \arccos(x)$.

Domain: $[-1, 1]$

Range: $[0, \pi]$

Key points: $(-1, \pi), \left(0, \frac{\pi}{2}\right), (1, 0)$

Here is the graph of $f(x) = \cos^{-1}(x)$:



Inverse Tangent Function:

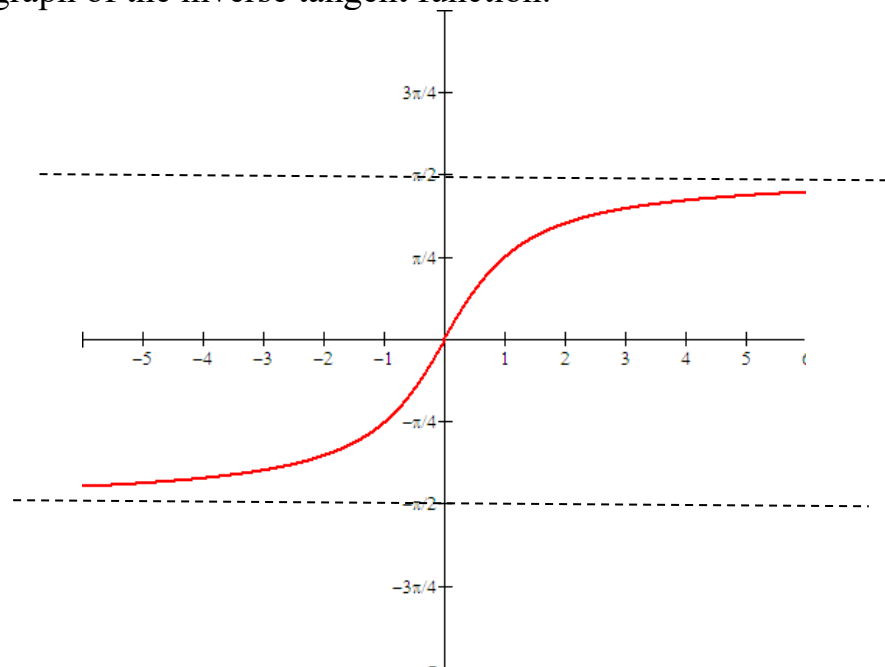
We note the function as $f(x) = \tan^{-1}(x)$ or $f(x) = \arctan(x)$.

Domain: $(-\infty, \infty)$

Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Key points: $\left(-1, -\frac{\pi}{4}\right), (0, 0), \left(1, \frac{\pi}{4}\right)$

Here is the graph of the inverse tangent function:



Important:

Inverse tangent function has two horizontal asymptotes: $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$.

You can use graphing techniques learned in earlier lessons to graph transformations of the basic inverse trig functions.

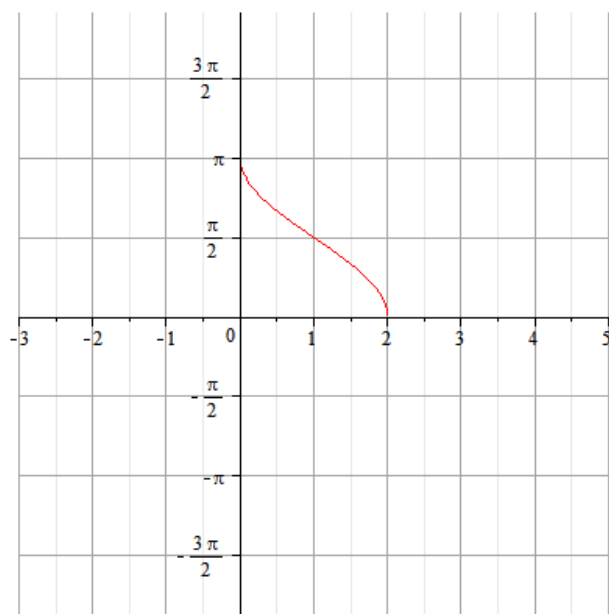
Example 1: Which of the following points is on the graph of $f(x) = \arctan(x-1)$?

A) $\left(\frac{\pi}{4}, 0\right)$

B) $\left(0, \frac{\pi}{4}\right)$

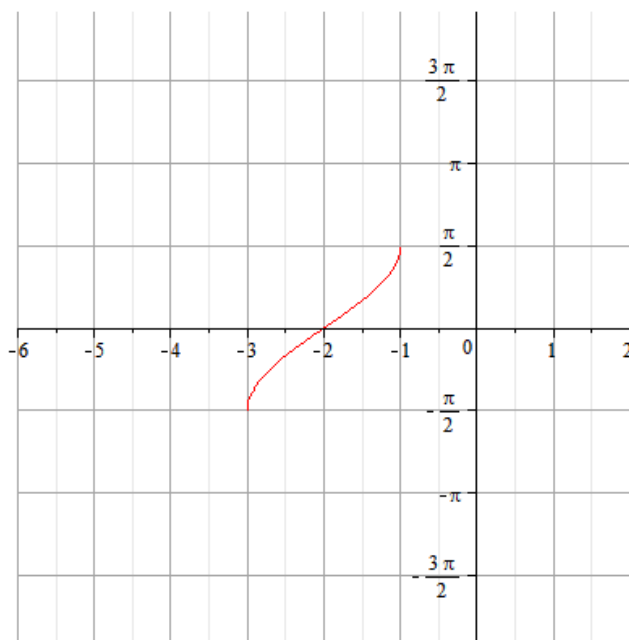
C) $\left(0, -\frac{\pi}{4}\right)$

Example 2: Which of the following can be the function whose graph is given below?



- A) $f(x) = \cos^{-1}(x - 1)$
- B) $f(x) = \sin^{-1}(x - 1)$
- C) $f(x) = \cos^{-1}(x + 1)$
- D) $f(x) = \sin^{-1}(x + 1)$
- E) $f(x) = \tan^{-1}(x - 1)$

Example 3: Which of the following can be the function whose graph is given below?



- A) $f(x) = \cos^{-1}(x - 2)$
- B) $f(x) = \sin^{-1}(x - 2)$
- C) $f(x) = \cos^{-1}(x + 2)$
- D) $f(x) = \sin^{-1}(x + 2)$
- E) $f(x) = \tan^{-1}(x + 2)$

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Question#5: Which of the following points is ON the graph of
 $f(x) = \arcsin(x + 2)$?