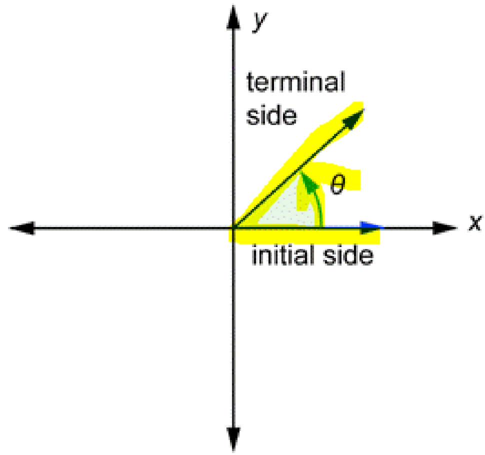
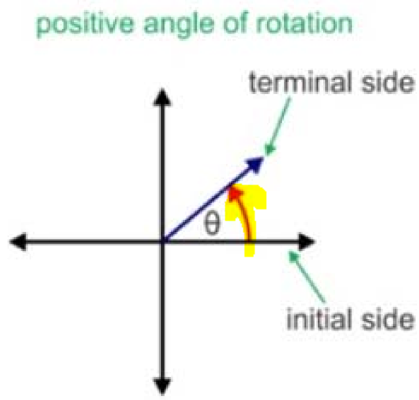
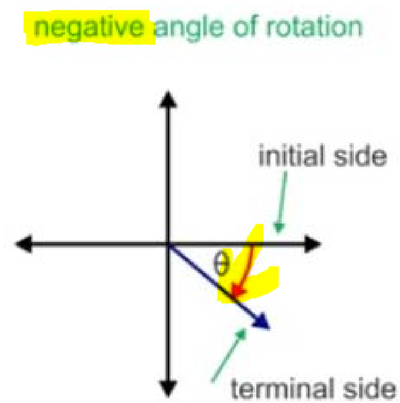
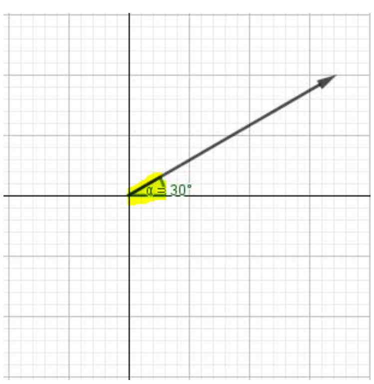
Math 1330 - Section 4.3
Unit Circle Trigonometry

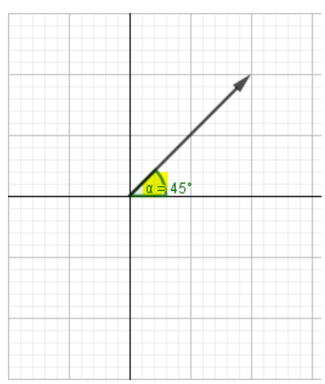
An angle is **in standard position** if its vertex is at the origin and its initial side is along the positive x axis. **Positive angles** are measured **counterclockwise** from the initial side. **Negative angles** are measured **clockwise**. We will typically use the Greek letter θ to denote an angle.

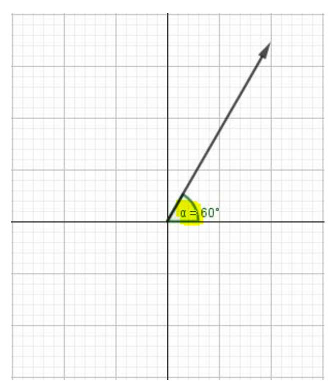


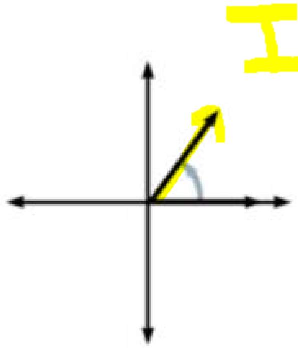


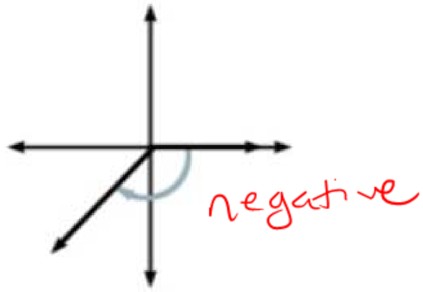


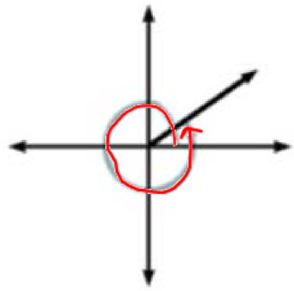







I

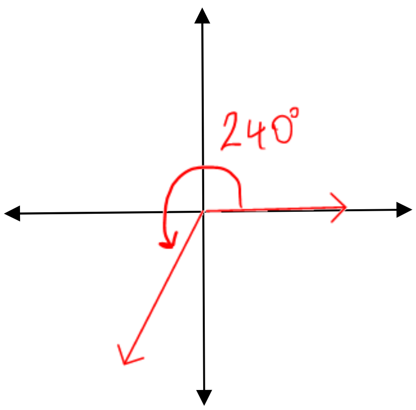

negative



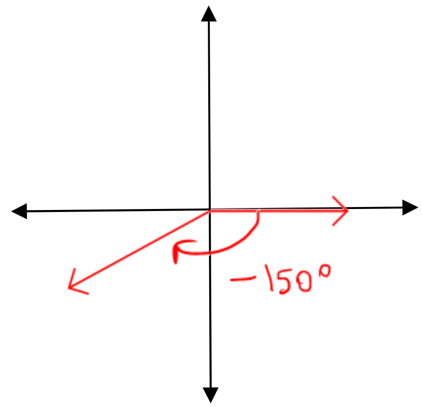
$30° = \frac{\pi}{6}$ $60° = \frac{\pi}{3}$ $45° = \frac{\pi}{4}$ rad.

Example 1: Sketch each angle in standard position.

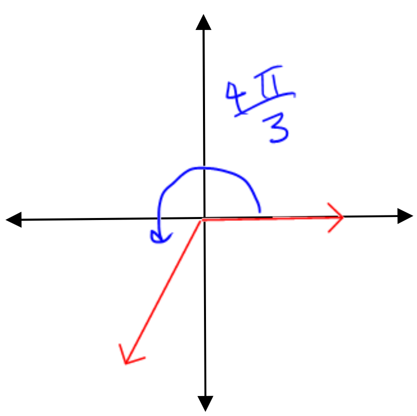
a. 240°



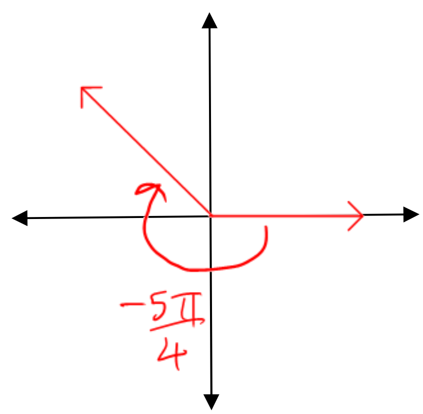
b. -150°



c. $\frac{4\pi}{3}$

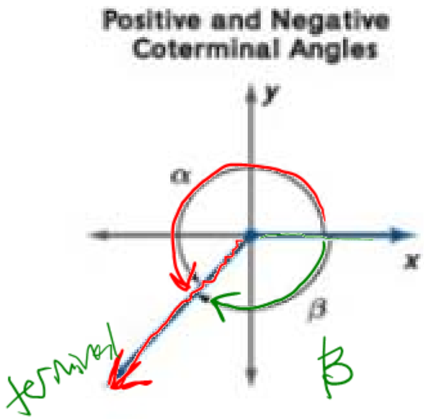


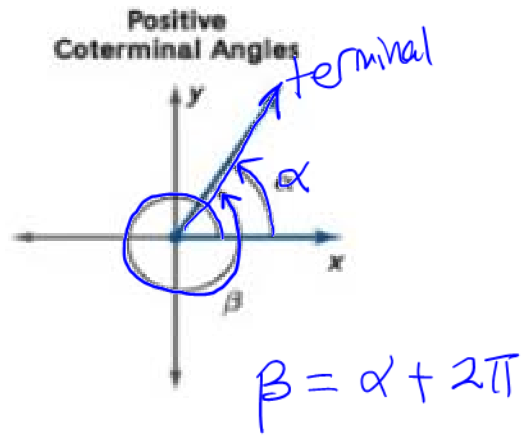
d. $\frac{-5\pi}{4}$



COTERMINAL ANGLES

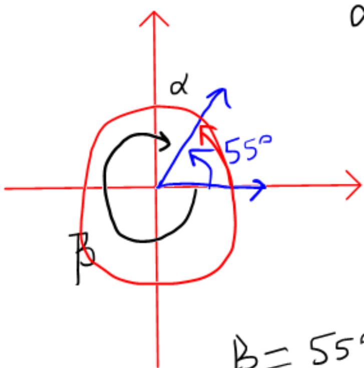
Angles that have the same terminal side are called **coterminal angles**. Measures of coterminal angles differ by a multiple of 360° if measured in degrees or by a multiple of 2π if measured in radians.



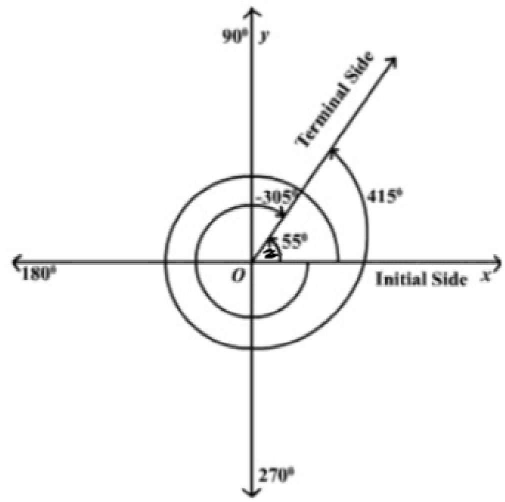


Example 2: Find two angles, one positive and one negative that are coterminal with each angle.

a. 55°



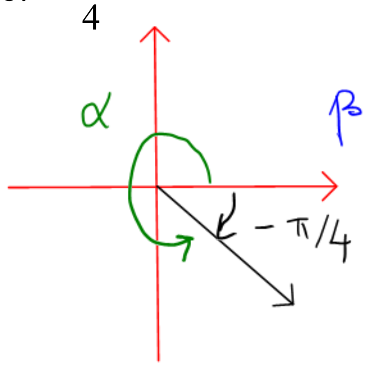
$\alpha = 55° \underbrace{+ 360°}_{\text{add!}} = 415°$



$\beta = 55° \underbrace{- 360°}_{\text{subtract!}} = -305°$

b. $-\dfrac{\pi}{4}$

$\alpha = -\dfrac{\pi}{4} + 2\pi = \dfrac{7\pi}{4}$



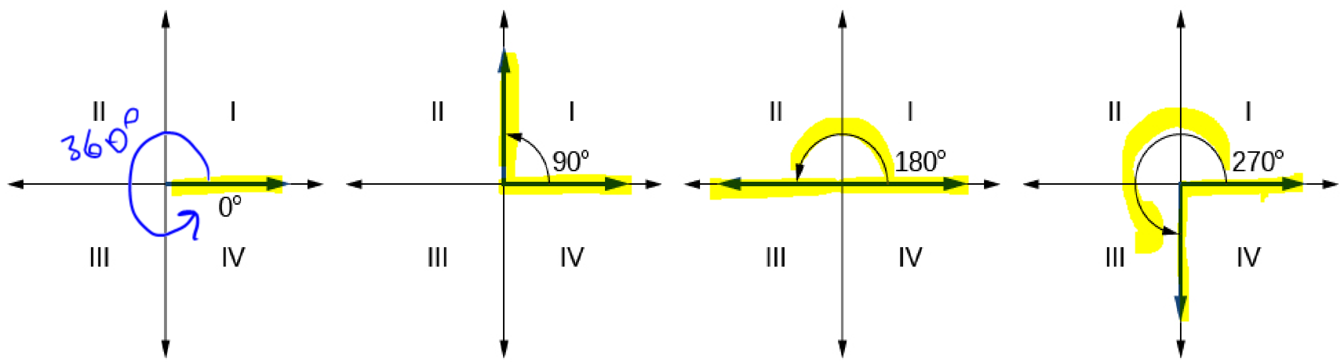
$\beta = -\dfrac{\pi}{4} - 2\pi = -\dfrac{\pi}{4} - \dfrac{8\pi}{4} = -\dfrac{9\pi}{4}$

POPPER for SECTION 4.3

Question#1: Which of the following is a coterminal angle with $\frac{\pi}{3}$?

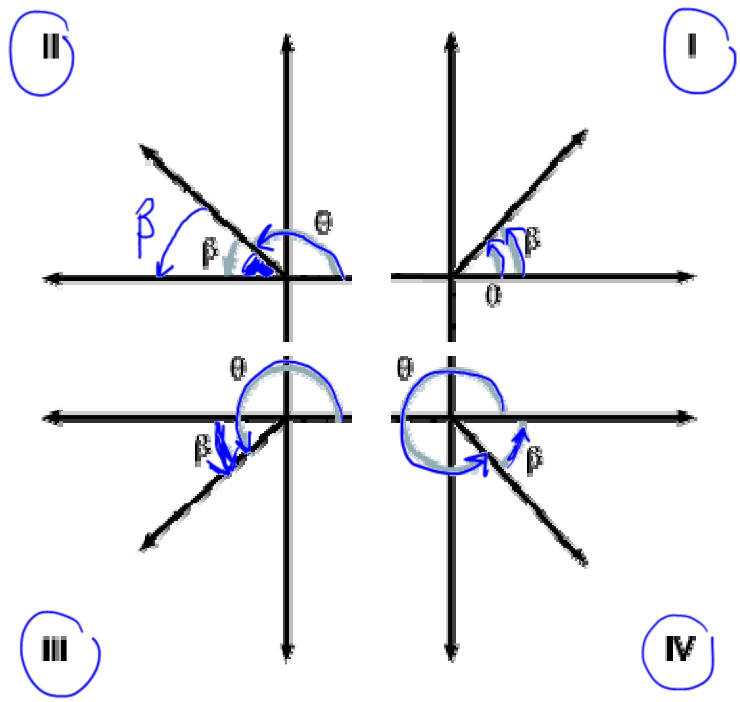
QUADRANTAL ANGLES

If an angle is in standard position and its terminal side lies along the *x* or *y* axis, then we call the angle a **quadrantal angle**.



REFERENCE ANGLE

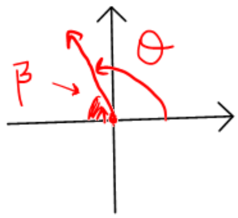
You will need to be able to work with reference angles. Suppose θ is an angle in standard position and θ is not a quadrantal angle. The **reference angle for** θ is the **acute angle of positive measure** that is formed by the terminal side of the angle and the x axis.



Example 3: Find the reference angle for each of these angles:

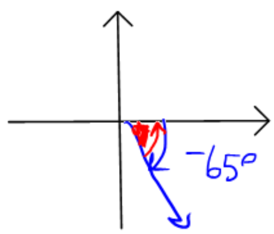
a. $120°$

$\beta = 180° - 120° = 60°$

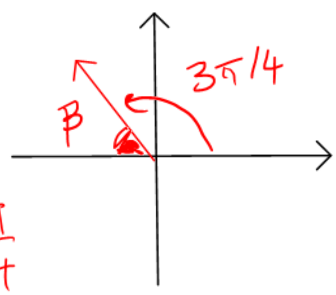


b. $-65°$

ref. angle: $65°$



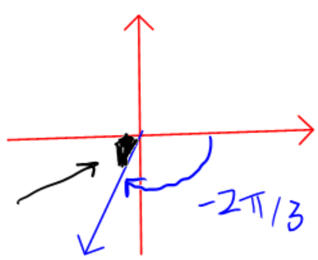
c. $\dfrac{3\pi}{4}$



$\beta = \pi - \dfrac{3\pi}{4} = \dfrac{\pi}{4}$

d. $\dfrac{-2\pi}{3}$

$\beta = \dfrac{\pi}{3}$

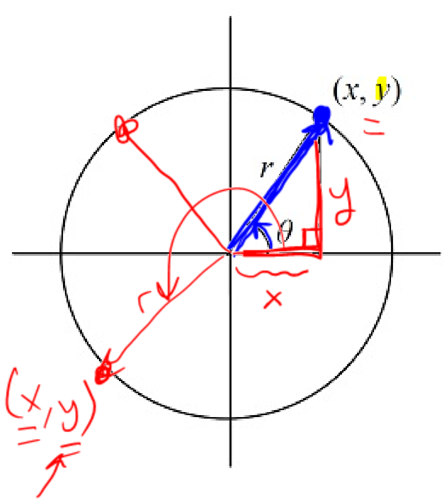


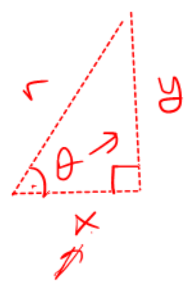
POPPER for SECTION 4.3

Question#2: Which of the following is the reference angle for $\frac{2\pi}{3}$?

TRIGONOMETRIC FUNCTIONS OF AN AGLE

We previously defined the six trigonometric functions of an angle as ratios of the lengths of the sides of a right triangle. Now we will look at them using a circle centered at the origin in the coordinate plane. This circle will have the equation $x^2 + y^2 = r^2$. If we select a point $P(x, y)$ on the circle and draw a ray from the origin through the point, we have created an angle in standard position. The length of the radius will be *r*.





The six trig functions of θ are defined as follows, using the circle above:

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y},\ y \neq 0$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x},\ x \neq 0$$

$$\tan\theta = \frac{y}{x},\ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$

If θ is a first quadrant angle, these definitions are consistent with the definitions given in Section 4.1.

An **identity** is a statement that is true for all values of the variable. Here are some identities that follow from the definitions above.

Know

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

$\frac{\sin\theta}{\cos\theta} = \frac{y/r}{x/r} = \frac{y}{x}$

$= \tan\theta$

$\sin\theta$ =

cosine θ =

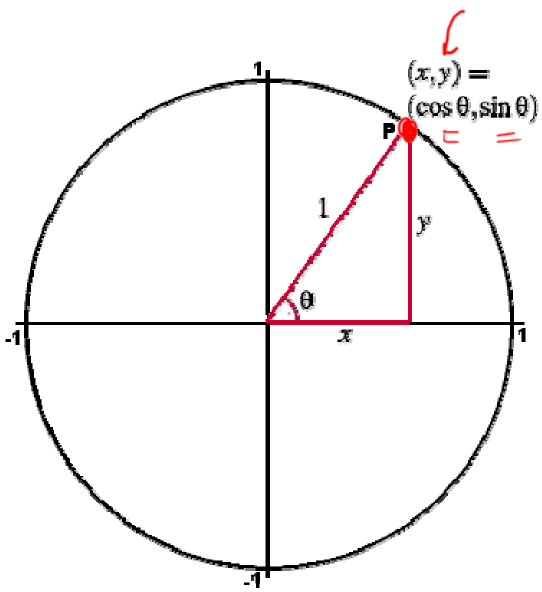
UNIT CIRCLE TRIGONOMETRY

We will work most often with a **unit circle**, that is, a circle with radius 1.

UNIT CIRCLE is the circle with center at (0,0) and radius 1.

Equation: $x^2 + y^2 = 1$

$\sin\theta = \frac{y}{r} = \frac{y}{1} = y$



$\cos\theta = \frac{x}{r} = x$

In this case, each value of r is 1. This adjusts the definitions of the trig functions as follows:

$$\boxed{\begin{array}{l}\sin\theta = y \\ \cos\theta = x\end{array}}$$

$$\sin\theta = y \rightarrow \csc\theta = \frac{1}{y},\ y \neq 0$$

$$\cos\theta = x \rightarrow \sec\theta = \frac{1}{x},\ x \neq 0$$

$$\rightarrow \tan\theta = \frac{y}{x},\ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$

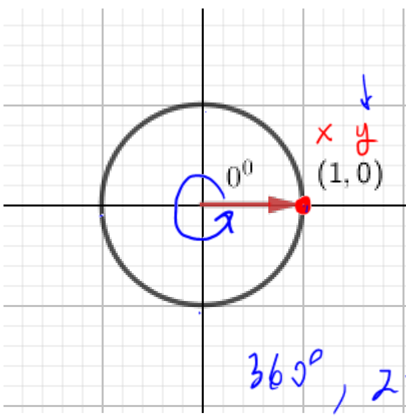
Trigonometric Functions of Quadrantal Angles and Special Angles

You will need to be able to find the trig functions of quadrantal angles and of angles measuring $30°, 45°$ or $60°$ without using a calculator.

Since $\sin\theta = y$ and $\cos\theta = x$, each ordered pair on the unit circle corresponds to $(\cos\theta, \sin\theta)$ of some angle θ.

We'll show the values for sine and cosine of the quadrantal angles on this graph. We'll also indicate where the trig functions are positive and where they are negative.

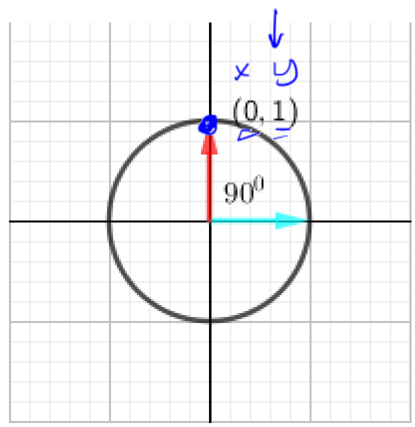
$0°$ or 0 rad.



$\sin(0^0) = 0$

$\cos(0^0) = 1$

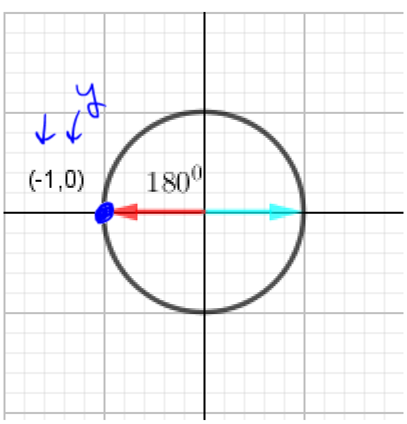
$90°$ or $\frac{\pi}{2}$ rad



$\sin(90^0) = 1$

$\cos(90^0) = 0$

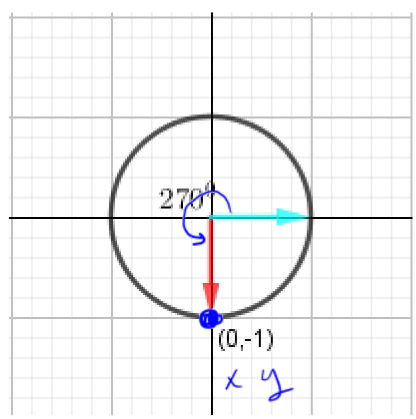
$180°$, π rad



$\sin(180^0) = 0$

$\cos(180^0) = -1$

$270°$ or $\frac{3\pi}{2}$ rad



$\sin(270^0) = -1$

$\cos(270^0) = 0$

Using the identities given above, you can find the other four trig functions of an angle, given just sine and cosine. Note that some values are not defined for quadrantal angles.

know

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

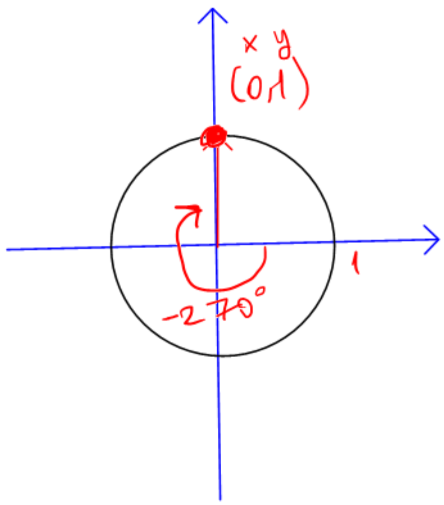
$$\sec\theta = \frac{1}{\cos\theta}$$

Values of Trigonometric Functions for Quadrantal Angles

1/0 1/0

	θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0	0°	0	1	0	Undefined	1	Undefined
$\pi/2$	90°	1	0	Undefined	0	Undefined	1
π	180°	0	−1	0	Undefined	−1	Undefined
$3\pi/2$	270°	−1	0	Undefined	0	Undefined	−1
2π	360°	0	1	0	Undefined	1	Undefined

Example 4: Sketch an angle measuring -270° in the coordinate plane. Then give the six trigonometric functions of the angle. Note that some of the functions may be undefined.



① $\sin(-270°) = 1$

② $\cos(-270°) = 0$

③ $\tan(-270°) = \dfrac{\sin(-270°)}{\cos(-270°)} = \dfrac{1}{0}$

: undefined.

④ $\cot(-270°) = \dfrac{\cos(-270°)}{\sin(-270°)} = \dfrac{0}{1} = 0$

⑤ $\sec(-270°) = \dfrac{1}{\cos(-270°)} = \dfrac{1}{0}$

: undefined

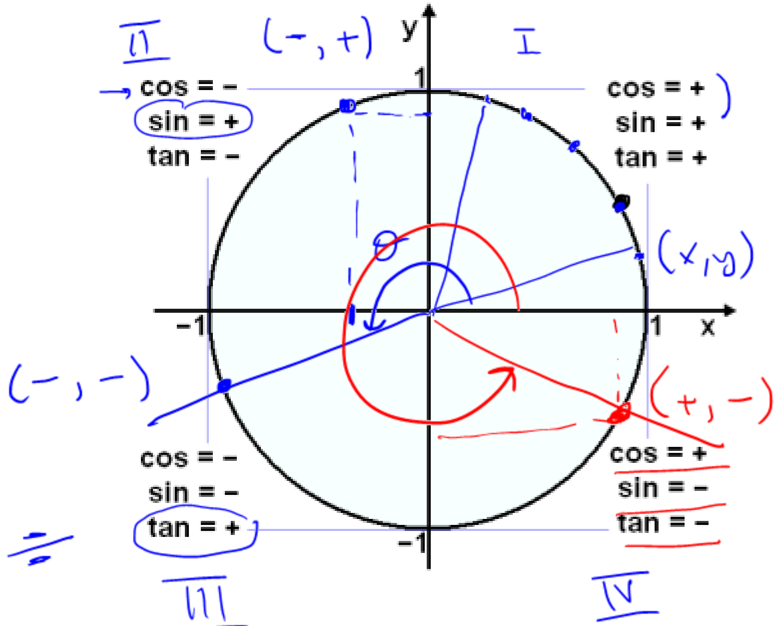
⑥ $\csc(-270°) = \dfrac{1}{\sin(-270°)} = \dfrac{1}{1} = 1$

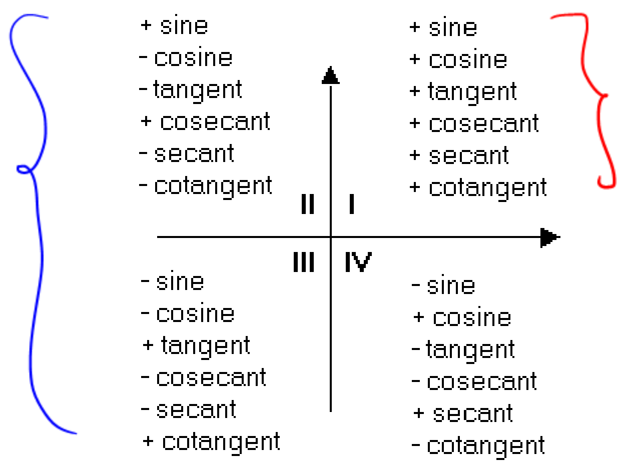
POPPER for SECTION 4.3

Question#3: Evaluate: $\sin\left(90^0\right)+\cos\left(0^0\right)+\sin\left(270^0\right)$

SIGNS OF TRIG FUNCTIONS

Recall the signs of the points in each quadrant. Remember, that each point on the unit circle corresponds to an ordered pair, (cosine, sine).

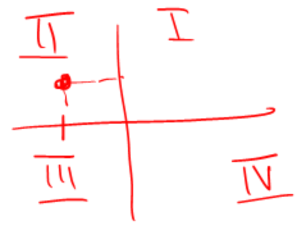




Example 5: Name the quadrant in which both conditions are true:

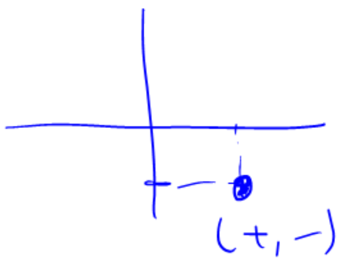
a. $\cos\theta < 0$ and $\csc\theta > 0$. $\underline{\text{II}}$

$\cos\theta : -$ $\quad\underbrace{\csc\theta : +}$ $\Rightarrow \sin\theta : +$



b. $\sin\theta < 0$ and $\tan\theta < 0$

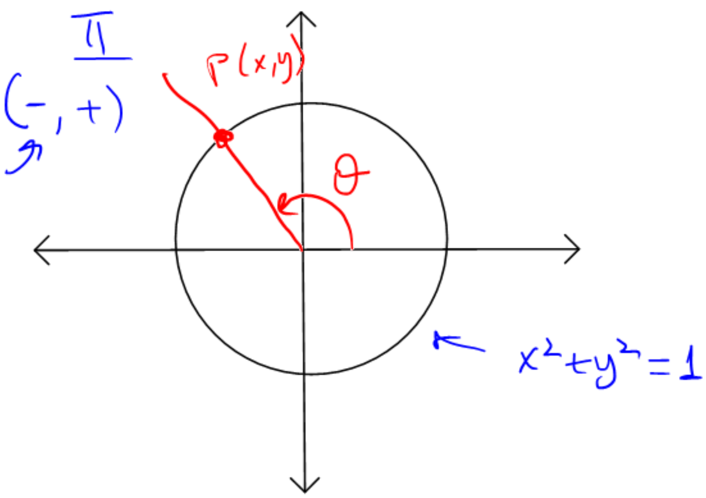
$\sin\theta : -$ & $\tan\theta : - \Rightarrow \cos\theta : +$



$\underline{\text{IV}}$

This is a very typical type of problem you'll need to be able to work.

Example 6: Let $P(x, y)$ denote the point where the terminal side of an angle θ intersects the unit circle. If P is in quadrant II and $y = \frac{5}{13}$, find the six trig functions of angle θ.



$P\left(x, \frac{5}{13}\right)$

$x^2 + y^2 = 1$

$x^2 + \left(\frac{5}{13}\right)^2 = 1$

$x^2 = 1 - \frac{25}{169}$

$x^2 = \frac{144}{169}$

$x = \pm\sqrt{\frac{144}{169}}$

$\text{II} \Rightarrow x = -\frac{12}{13}$

$P\left(-\frac{12}{13}, \frac{5}{13}\right)$

1) $\sin\theta = \frac{5}{13} \Rightarrow$ 6) $\csc\theta = \frac{13}{5}$

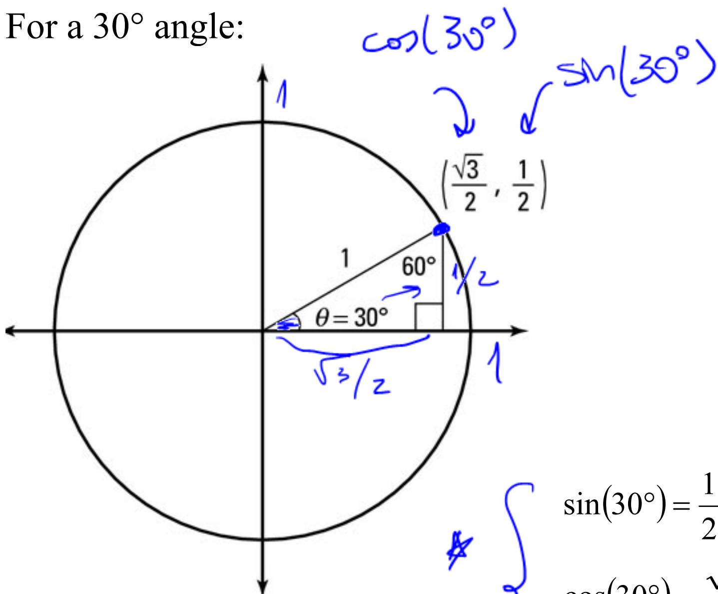
2) $\cos\theta = \frac{-12}{13} \Rightarrow$ 5) $\sec\theta = -\frac{13}{12}$

3) $\tan\theta = \frac{y}{x} = \frac{5/13}{-12/13} = \frac{-5}{12} \Rightarrow$ 4) $\cot\theta = -\frac{12}{5}$

Trigonometric Functions of Special Angles

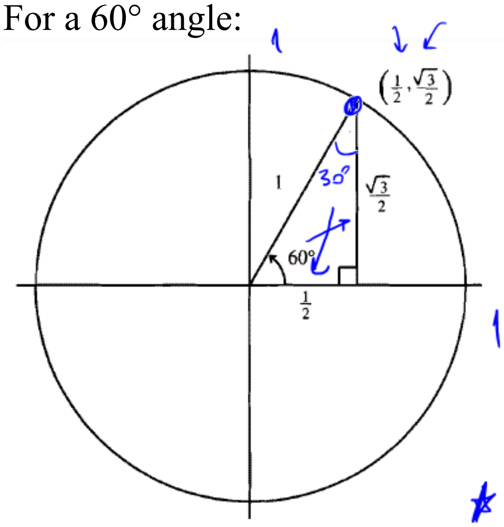
You'll also need to be able to find the six trig functions of $30°, 60°$ and $45°$ angles. YOU MUST KNOW THESE!!!!!

For a 30° angle:



$$\sin(30°) = \frac{1}{2} \longrightarrow \csc(30°) = 2$$
$$\cos(30°) = \frac{\sqrt{3}}{2} \longrightarrow \sec(30°) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\tan(30°) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot(30°) = \sqrt{3}$$

For a 60° angle:



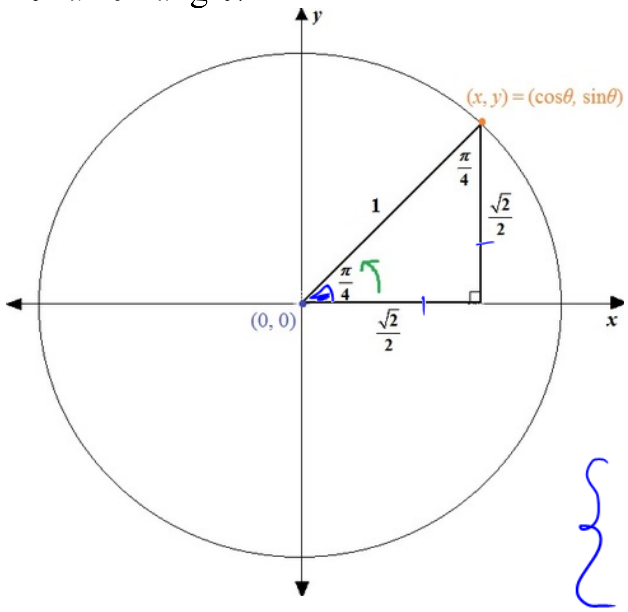
$$\sin(60°) = \frac{\sqrt{3}}{2} \rightarrow \csc(60°) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\cos(60°) = \frac{1}{2} \rightarrow \sec(60°) = 2$$
$$\tan(60°) = \sqrt{3} \rightarrow \cot(60°) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\frac{\sqrt{3}/2}{1/2} = 2\sqrt{3}$$

$\pi/4$ rad.

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

For a 45° angle:



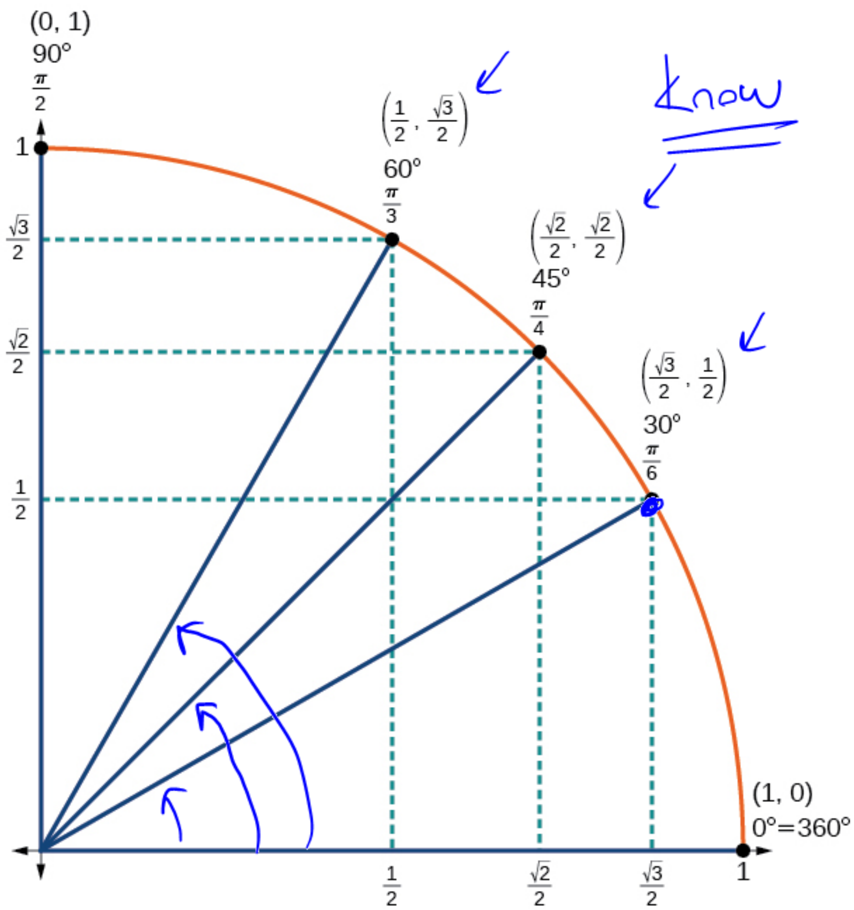
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$\sin(45°) = \frac{\sqrt{2}}{2} \rightarrow \csc(45°) = \sqrt{2}$$

$$\cos(45°) = \frac{\sqrt{2}}{2} \rightarrow \sec(45°) = \sqrt{2}$$

$$\rightarrow \tan(45°) = 1 \qquad \rightarrow \cot(45°) = 1$$

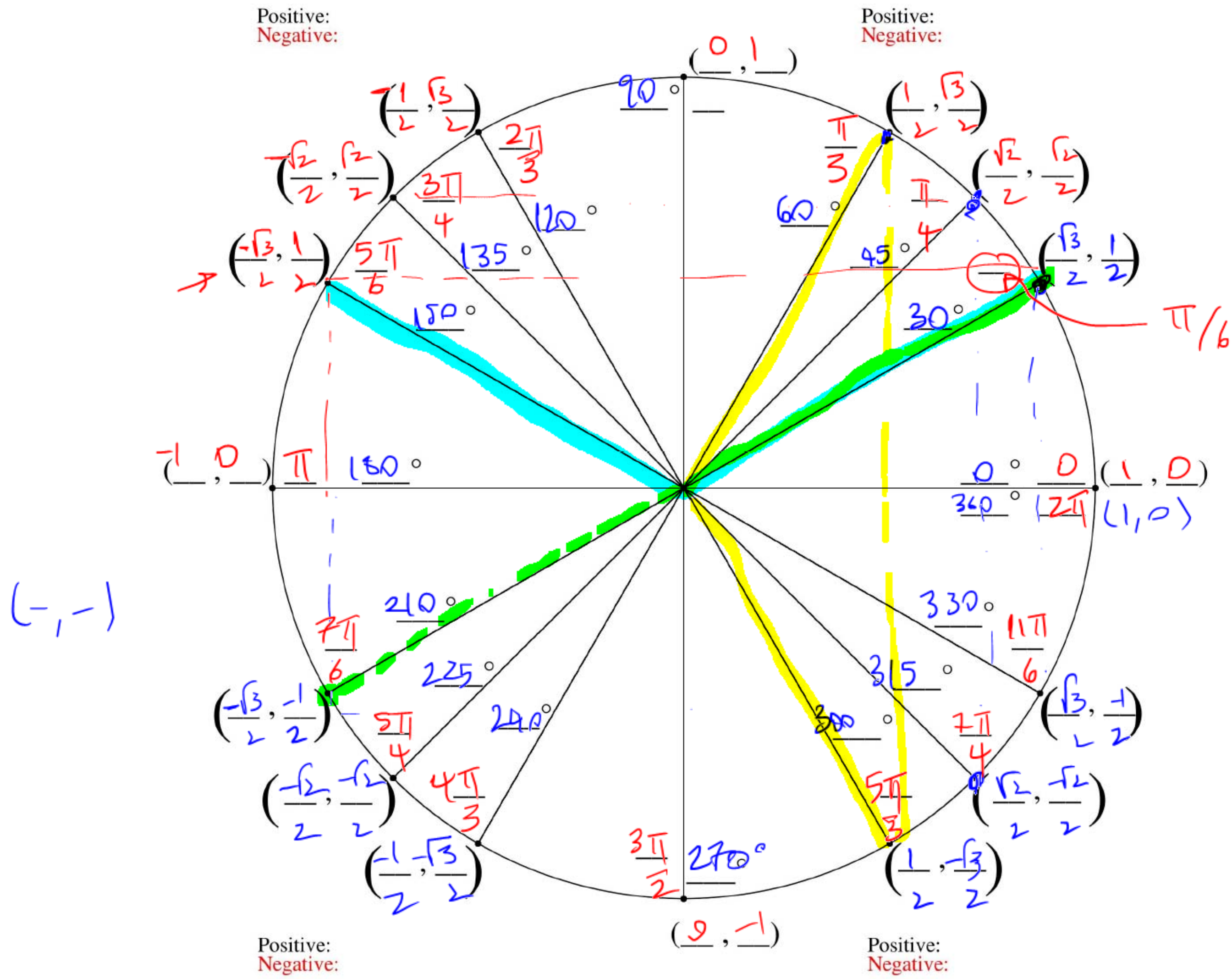
FIRST QUADRANT OF UNIT CIRCLE:



UNIT CIRCLE WITH ALL IMPORTANT ANGLES

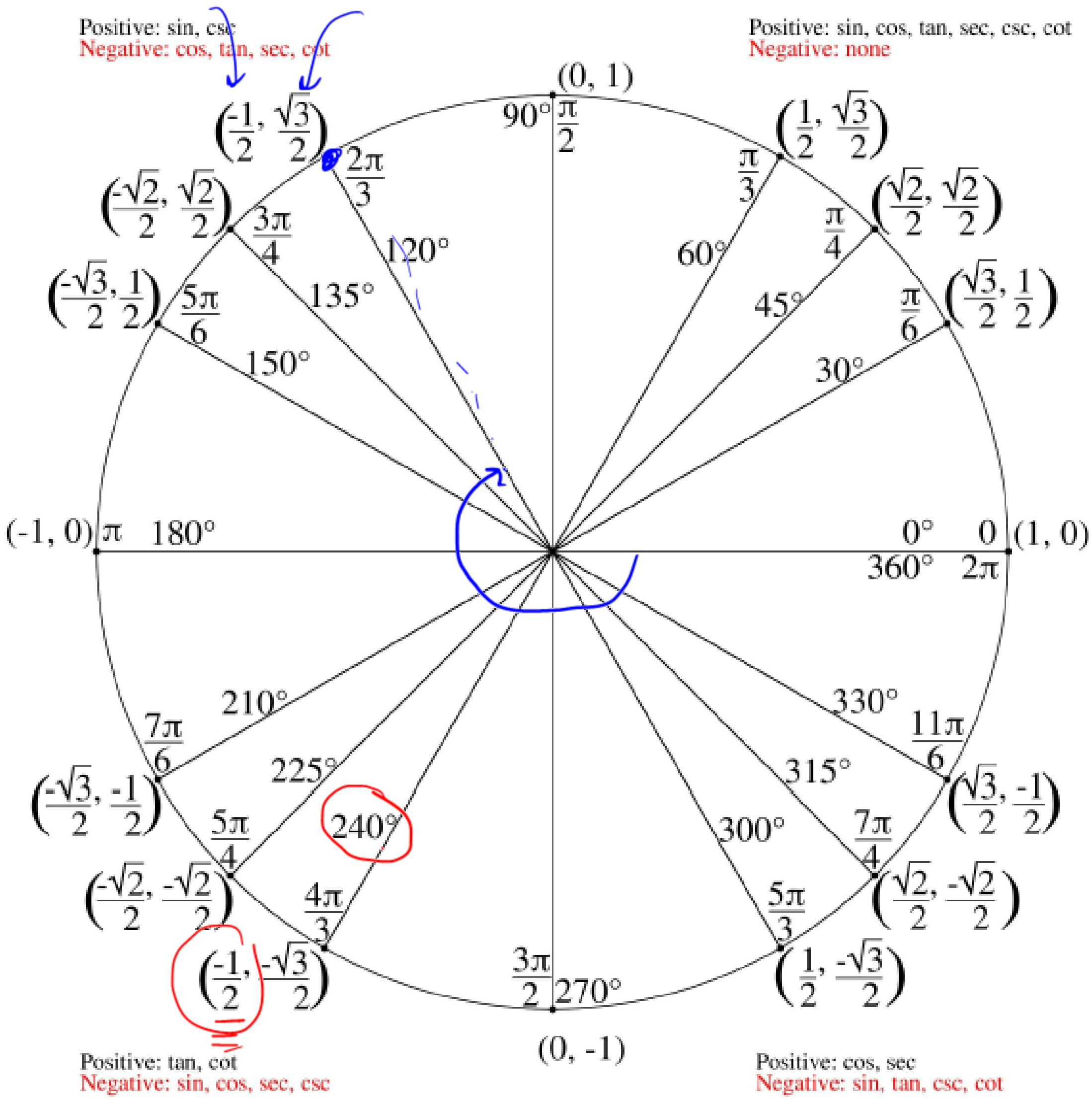
How do we find the trigonometric functions of other special angles?

Method 1: Fill them in. Learn the patterns.



Know how to fill this out!

The Unit Circle



Method 2: The Chart

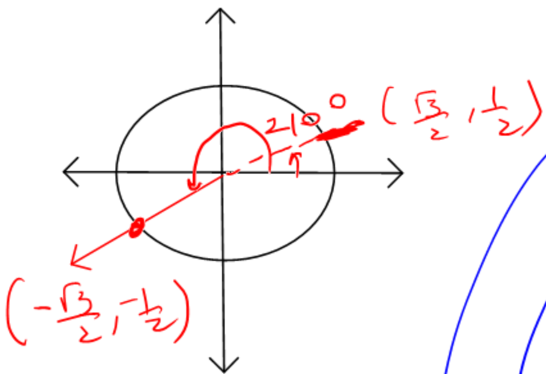
Write down the angle measures, starting with 0° and continue until you reach 90°. Under these, write down the equivalent radian measures. Under these, write down the numbers from 0 to 4. Next, take the square root of the values and simplify if possible. Divide each value by 2. This gives you the sine value of each of the angles you need. To find the cosine values, write the previous line in the reverse order.

Now you have the sine and cosine values for the quadrantal angles and the special angles. From these, you can find the rest of the trig values for these angles. Write the problem in terms of the reference angle. Then use the chart you created to find the appropriate value.

	0^0	$30°$	$45°$	$60°$	90^0
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Sine	**0**	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	**1**
Cosine	**1**	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	**0**
Tangent	**0**	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	**undefined**
Cotangent	**undefined**	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	**0**
Secant	**1**	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	undefined
Cosecant	undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	**1**

EVALUATING TRIG FUNCTIONS OF AN ANGLE

Example 7: Sketch an angle measuring 210° in the coordinate plane. Give the coordinates of the point where the terminal side of the angle intersects the unit circle. Then state the six trigonometric functions of the angle.



$\sin(210°) = -\frac{1}{2}$

$\cos(210°) = \frac{-\sqrt{3}}{2}$

$\tan(210°) = \frac{-1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\cot(210°) = \sqrt{3}$

$\sec(210°) = -\frac{2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$

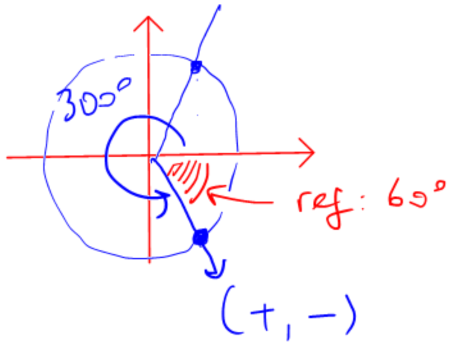
$\csc(210°) = -2$

Evaluating Trigonometric Functions Using Reference Angles

1. Determine the reference angle associated with the given angle.
2. Evaluate the given trigonometric function of the reference angle.
3. Affix the appropriate sign determined by the quadrant of the terminal side of the angle in standard position.

Example 8: Evaluate each:

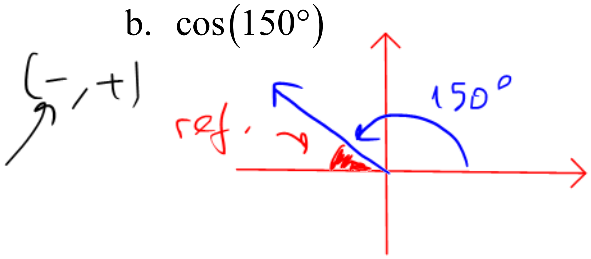
a. $\sin(300°)$



$\sin(300°) = (\pm)\sin(60°)$

$\sin(300°) = -\frac{\sqrt{3}}{2}$

b. $\cos(150°)$



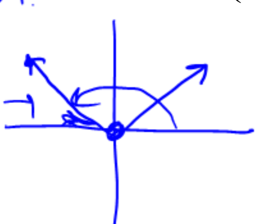
reference angle: 30°

$\cos(150°) = (\pm)\cos(30°)$

$\cos(150°) = -\frac{\sqrt{3}}{2}$

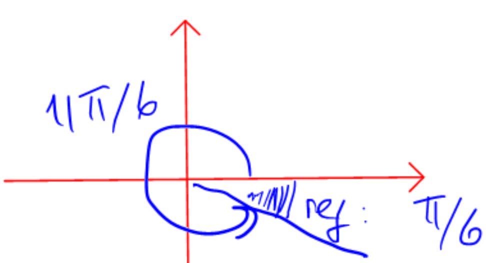
c. $\tan\left(\frac{3\pi}{4}\right)$

tan: −



ref: $\frac{\pi}{4}$ $\quad \tan\left(\frac{3\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -1$

d. $\sec\left(\frac{11\pi}{6}\right)$



sin: −

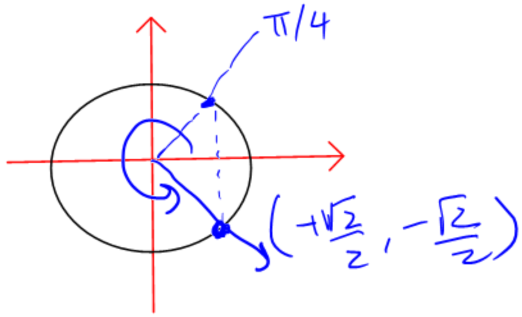
cos: +

sec: +

$\sec\left(\frac{11\pi}{6}\right) = \sec\left(\frac{\pi}{6}\right)$

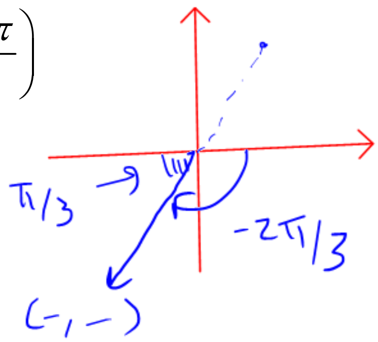
$\sec\left(\frac{11\pi}{6}\right) = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}} = \boxed{\frac{2\sqrt{3}}{3}}$

e. $\cot\left(\dfrac{7\pi}{4}\right)$



$\cot\left(\frac{7\pi}{4}\right) = -1$

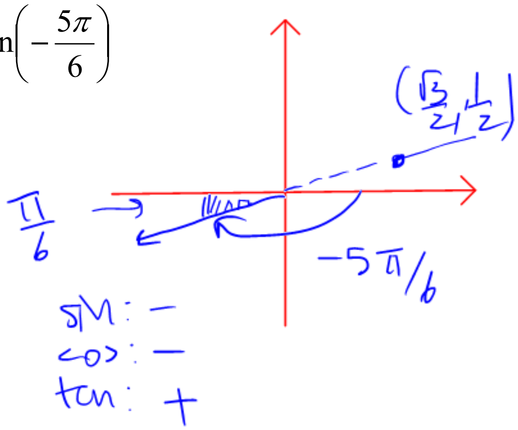
f. $\sin\left(\dfrac{-2\pi}{3}\right)$



$\sin\left(-\frac{2\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right)$

$= -\frac{\sqrt{3}}{2}$

g. $\tan\left(-\dfrac{5\pi}{6}\right)$

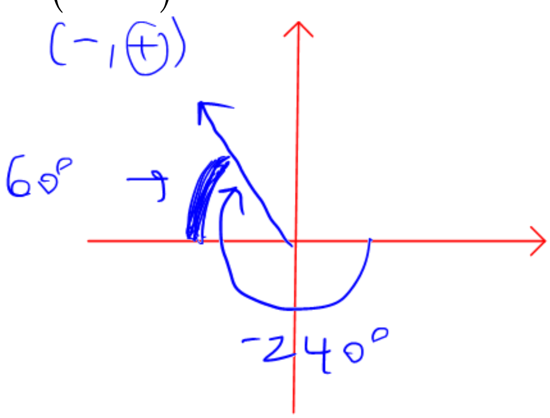


$\tan\left(-\frac{5\pi}{6}\right) = \tan\left(\frac{\pi}{6}\right)$

$= \dfrac{1/2}{\sqrt{3}/2} = \dfrac{1}{\sqrt{3}}$

$= \dfrac{\sqrt{3}}{3}$

h. $\sin\left(-240^0\right)$



$\sin(-240°) = \sin(60°)$

$= \boxed{\frac{\sqrt{3}}{2}}$

POPPER for SECTION 4.3

Question#4: Evaluate: $\sin(240°)$