Section 4.4
Trigonometric Expressions and Identities

In this section, you'll learn to simplify trig expressions using identities and using basic algebraic operations. You can add, subtract, multiply, divide and factor trig expressions, in much the same manner that you can with algebraic expressions.

Examples:

$$2\sin x + 3\sin x = 5\sin x$$

$2\sin x + 4\cos x$

$$6\cos x - 10\cos x = -4\cos x$$

$$\tan(x)\cdot\tan(x) = \left(\tan(x)\right)^2 = \tan^2(x)$$

Notation:

$$\sin(x)\cdot\sin(x) = \left(\sin(x)\right)^2 = \sin^2(x)$$

$$\left(\cos(x)\right)^3 = \cos^3(x) = \cos x \cdot \cos x \cdot \cos x$$

$\sin(x) = \sin x$

Be careful: $\sin^2(x) \neq \sin(x^2)$

$\sin x + 2 \neq \sin(x+2)$

$\sin(x+1)$

Example 1: Perform the following operation and simplify:

$(\cos(\theta)+5)(\cos(\theta)-7)$ $\qquad$ $(y+5)\cdot(y-7)$ FOIL

$= \cos^2(\theta) \underbrace{-7\cos(\theta)} + \underbrace{5\cos(\theta)} - 35$

$= \cos^2(\theta) - 2\cos(\theta) - 35$

Example 2: Factor: $\sin^2 x - \sin x - 2$

Let $y = \sin x$

$$y^2 - y - 2 = (y-2)\cdot(y+1)$$

$$\sin^2 x - \sin x - 2 = (\sin x - 2)\cdot(\sin x + 1)$$

Example 3: Factor: $\sec^2(\theta) - 4\sec(\theta) - 12$ Let $y = \sec(\theta)$

$$y^2 - 4y - 12 = (y-6)\cdot(y+2)$$

$$\sec^2(\theta) - 4\sec(\theta) - 12 = (\sec(\theta) - 6)\cdot(\sec(\theta) + 2)$$

$$a^2 - 1 = (a-1).(a+1)$$

RECALL: $a^2 - b^2 = (a-b)(a+b)$

Example 4: Factor completely: $\underbrace{\tan^4 x} - \underbrace{\tan^2 x}$

$$= \tan^2 x \left(\tan^2 x - 1 \right)$$

$$= \tan^2 x \,(\tan x - 1).(\tan x + 1)$$

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Question#1: Which of the following is a factor of: $\cos^2 x + 2\cos x - 3$ **?**

Simplifying Trig Expressions

Sometimes, you can use trig identities to help you simplify trig expressions. Here is a list of trig identities we have already met. Note that NONE of these identities will be provided on the tests. You must know all of these identities.

$\tan(t) = \dfrac{\sin(t)}{\cos(t)}$ $\cot(t) = \dfrac{\cos(t)}{\sin(t)}$	**Reciprocal Identities:** $\csc(t) = \dfrac{1}{\sin(t)}$, $\sin(t) \neq 0$ $\Rightarrow$ $\csc(t) \cdot \sin(t) = 1$ $\sec(t) = \dfrac{1}{\cos(t)}$, $\cos(t) \neq 0$ $\Rightarrow$ $\sec(t) \cdot \cos(t) = 1$ $\cot(t) = \dfrac{1}{\tan(t)}$, $\tan(t) \neq 0$ $\Rightarrow$ $\cot(t) \cdot \tan(t) = 1$
Pythagorean Identities ★ $\sin^2 x + \cos^2 x = 1$ $1 - \sin^2 x = \cos^2 x$ or $1 - \cos^2 x = \sin^2 x$ ★ $1 + \tan^2 x = \sec^2 x$ reason: $\dfrac{1}{1} + \dfrac{\sin^2 x}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$ ★ $1 + \cot^2 x = \csc^2 x$ $1 + \dfrac{\cos^2 x}{\sin^2 x} = \dfrac{\sin^2 x + \cos^2 x}{\sin^2 x} = \dfrac{1}{\sin^2 x} = \csc^2 x$	

We'll use these in the next several examples.

Note:

$\sin^2 x + \cos^2 x = 1 \rightarrow \boxed{1 - \sin^2 x = \cos^2 x}$ and $\boxed{1 - \cos^2 x = \sin^2 x}$

$1 + \tan^2 x = \sec^2 x \rightarrow \sec^2 x - 1 = \tan^2 x$ and $\sec^2 x - \tan^2 x = 1$ ★

$1 + \cot^2 x = \csc^2 x \rightarrow \csc^2 x - 1 = \cot^2 x$ and $\csc^2 x - \cot^2 x = 1$ ★

Example 5: Simplify $\underbrace{2\sin^2 x + 2\cos^2 x} + 5\tan x\cot x$

$$2\underbrace{(\sin^2 x + \cos^2 x)}_{\substack{1 \\ \text{identity}}} + 5 \cdot \underbrace{\tan x \cdot \cot x}_{\substack{1 \\ \text{identity}}}$$

$$= 2 \cdot 1 + 5 \cdot 1$$

$$= 2 + 5$$

$$= \boxed{7}$$

Example 6: Simplify $\dfrac{\sec^2(\theta)-\tan^2(\theta)}{2\tan(\theta)\cot(\theta)}$ $= \dfrac{1}{2\cdot 1} = \boxed{\dfrac{1}{2}}$

identity

identity

$$\frac{a}{a^2} = \frac{1}{a}$$

Example 7: Simplify: $\dfrac{\cos(x)}{\sin^2(x)-1}$

$$= \frac{\cos x}{-\cos^2 x}$$

$$= -\frac{1}{\cos x}$$

$$= \boxed{-\sec x}$$

$\bigstar \quad \cos^2 x + \sin^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$-1 \qquad -1$

$$\boxed{\sin^2 x - 1 = -\cos^2 x}$$

or

$\underbrace{1 - \sin^2 x}_{\cdot -1} = \underbrace{\cos^2 x}_{\cdot -1}$

$\sin^2 x - 1 = -\cos^2 x$

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Question#2: Simplify: $5\cos^2 x + 5\sin^2 x + \left(\sec^2 x - \tan^2 x\right)$

Example 8: Simplify: $5\tan(x)\csc x$

No identities

use sinx & cosx.

$5\tan(x)\cdot\csc(x)$

$= 5\cdot\frac{\sin x}{\cos x}\cdot\frac{1}{\sin x}$

$= \frac{5}{\cos x} = \boxed{5\cdot\sec x}$

tanx : + secx : +

Example 9: Let $x \in \left(0, \frac{\pi}{2}\right)$; simplify: $\dfrac{\sqrt{4+4\tan^2 x}}{\sqrt{\sec^2 x - 1}}$

$\rightarrow 1 + \tan^2 x = \sec^2 x$

$\rightarrow \sec^2 x - 1 = \tan^2 x$

$$= \frac{\sqrt{4(1+\tan^2 x)}}{\sqrt{\sec^2 x - 1}}$$

$$= \frac{\sqrt{4 \cdot \sec^2 x}}{\sqrt{\tan^2 x}}$$

$$= \frac{2 \cdot |\sec x|}{|\tan x|} = \frac{2\sec x}{\tan x} = \frac{2 \cdot \frac{1}{\cos x}}{\frac{\sin x}{\cos x}}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$$

$$= \frac{2}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{\sin x} = \frac{2}{\sin x}$$

$$= \boxed{2\csc x}$$

Example 10: Simplify: $(1-\cos x)(\csc x+\cot x)$

No identities
$\sin x$ & $\cos x$

$\boxed{(1-a)\cdot(1+a)=1-a^2}$

$$(1-\cos x)\cdot\left(\frac{1}{\sin x}+\frac{\cos x}{\sin x}\right)$$

$$=(1-\cos x)\cdot\left(\frac{1+\cos x}{\sin x}\right)$$

$$=\frac{(1-\cos x)\cdot(1+\cos x)}{\sin x}$$

identity

$$=\frac{1-\cos^2 x}{\sin x}=\frac{\sin^2 x}{\sin x}=\boxed{\sin x}$$

$$*\ (a-b)(a+b) = a^2 - b^2$$

$$\frac{1}{a+b}\cdot\frac{(a-b)}{(a-b)} + \frac{1}{a-b}\cdot\frac{(a+b)}{(a+b)}$$

Example 11: Simplify: $\dfrac{1}{\cot x + \csc x} + \dfrac{1}{\cot x - \csc x}$

$(\cot x - \csc x)$ $\quad$ $(\cot x + \csc x)$

$$= \frac{\cot x \boxed{\cancel{- \csc x}} + \cot x \boxed{\cancel{+ \csc x}}}{(\cot x + \csc x)\cdot(\cot x - \csc x)}$$

$$= \frac{2\cot x}{\cot^2 x - \csc^2 x}$$

identity

$\csc^2 x - \cot^2 x = 1$

$(\cdot -1)$ $\quad$ $\cot^2 x - \csc^2 x = -1$

identity

$$= \frac{2\cot x}{-1}$$

$$= \boxed{-2\cot x}$$

Example 12: Simplify: $\tan(\theta)+\dfrac{\cos(\theta)}{1+\sin(\theta)}$

$$= \frac{\sin\theta}{\cos\theta}\,(1+\sin\theta) + \frac{\cos\theta}{1+\sin\theta}\,(\cos\theta)$$

$$= \frac{\sin\theta\,(1+\sin\theta) + \cos^2\theta}{\cos\theta\,(1+\sin\theta)}$$

identity

$$= \frac{\sin\theta + \sin^2\theta + \cos^2\theta}{\cos\theta\,(1+\sin\theta)}$$

$$= \frac{\sin\theta + 1}{\cos\theta\,(1+\sin\theta)}$$

$$\frac{(y+1)}{(1+y)}$$

$$= \frac{1}{\cos\theta} = \boxed{\sec\theta}$$

$(a-1)(a+1)=a^2-1$

Example 13: Simplify: $\dfrac{\cot(x)}{\csc(x)-1}+\dfrac{\cot(x)}{\csc(x)+1}$

$\csc x+1 \qquad \csc x-1$

$$=\frac{\cot x(\csc x+1)+\cot x(\csc x-1)}{(\csc x-1)(\csc x+1)}$$

$$=\frac{\cot x(\csc x+1+\csc x-1)}{\csc^2 x-1}$$

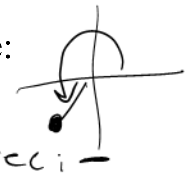
identity $\cot^2 x$

$$=\frac{\cancel{\cot x}\cdot 2\cdot\csc x}{\cot^{\cancel{2}} x}$$

$$=\frac{2\cdot\csc x}{\cot x}=\frac{2\cdot\frac{1}{\sin x}}{\left(\frac{\cos x}{\sin x}\right)}$$

$$=\frac{2}{\cancel{\sin x}}\cdot\frac{\cancel{\sin x}}{\cos x}=\frac{2}{\cos x}=\boxed{2\sec x}$$

You can also use the identities to help you solve problems like this one. (Note: you can also use a triangle to help you work this problem.)



Example 14: If $\cot\theta = \frac{5}{12}$, where $\boxed{\pi < \theta < \frac{3\pi}{2}}$, find the exact values of $\tan(\theta)$ and $\sec(\theta)$.

$$\tan\theta = \frac{1}{\cot\theta} = \frac{1}{5/12} = \frac{12}{5}$$

identity: $1 + \tan^2\theta = \sec^2\theta$

$$1 + \left(\frac{12}{5}\right)^2 = \sec^2\theta$$

$$\Rightarrow \frac{1}{1} + \frac{144}{25} = \frac{169}{25} = \sec^2\theta$$

??

$$\Rightarrow \sec\theta = \pm\sqrt{\frac{169}{25}} = \pm\frac{13}{5}$$

$$\boxed{\sec\theta = -\frac{13}{5}}$$

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Question#3: Simplify: $(\sin x + \cos x)^2 - 1$ **?**

Hint: $(a+b)^2 = a^2 + 2ab + b^2$