Section 6.1 - Sum and Difference Formulas

In this section, we will learn some formulas about finding the trig values for the sum and difference of angles.

For example; we know that $\sin\left(30^0\right)=\frac{1}{2}$ and $\sin\left(45^0\right)=\frac{\sqrt{2}}{2}$.

What if we need to compute $\sin\left(30^0+45^0\right)=\sin\left(75^0\right)$?

75^0 is not one of the angles we covered on the unit circle. Can we compute its trig functions using known angles from the unit circle?

Important: $\sin(A+B)\neq\sin(A)+\sin(B)$
$\cos(A+B)\neq\cos(A)+\cos(B)$

To see this:

$$\sin\left(30^0+60^0\right)=\sin\left(90^0\right)=1$$

$$\sin\left(30^0\right)+\sin\left(60^0\right)=\frac{1}{2}+\frac{\sqrt{3}}{2}=\frac{1+\sqrt{3}}{2}$$

$$\sin\left(30^0+60^0\right)\neq\sin\left(30^0\right)+\sin\left(60^0\right)$$

Sum and Difference Formulas for Sine, Cosine and Tangent

$$\sin(A+B)=\sin A\cos B+\sin B\cos A$$

$$\sin(A-B)=\sin A\cos B-\sin B\cos A$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

opp.

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

opp.

$\sin(A+B) = \sin A \cdot \cos B + \sin B \cdot \cos A$

Example 1: Evaluate $\sin\left(75^0\right)$ using sum and difference formulas.

Hint: $75^0 = 30^0 + 45^0$

$$\sin(75^\circ) = \sin(30^\circ + 45^\circ)$$

$$= \sin(30^\circ) \cdot \cos(45^\circ) + \sin(45^\circ) \cdot \cos(30^\circ)$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$

$$= \boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$$

$$\sin(A - B) = \sin A \cos B - \sin B \cos A$$

Example 2: Evaluate $\sin\left(15^0\right)$ **using sum and difference formulas.**

Hint: $15^0 = 60^0 - 45^0$

$$\sin(15^\circ) = \sin(60^\circ - 45^\circ)$$

$$= \sin 60^\circ \cdot \cos 45^\circ - \sin 45^\circ \cdot \cos 60^\circ$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$= \boxed{\frac{\sqrt{6} - \sqrt{2}}{4}}$$

easy: $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \ldots$

radians

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

Example 3: Evaluate $\cos\left(\frac{7\pi}{12}\right)$ **using sum and difference formulas.**

Hint: $\frac{7\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3} = \frac{3\pi}{12} + \frac{4\pi}{12} = \frac{7\pi}{12}$

(3) (4)

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$$

$$= \cos\frac{\pi}{4} \cdot \cos\frac{\pi}{3} - \sin\frac{\pi}{4} \cdot \sin\frac{\pi}{3}$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$$

$$= \boxed{\frac{\sqrt{2} - \sqrt{6}}{4}}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$$

Example 4: Evaluate $\tan\left(\frac{\pi}{12}\right)$ **using sum and difference formulas.**

$\tan\frac{\pi}{4} = 1$

Hint: $\frac{\pi}{12} = \underbrace{\frac{\pi}{4}}_{3} - \underbrace{\frac{\pi}{6}}_{2}$ $\qquad \frac{3\pi}{12} - \frac{2\pi}{12} = \frac{\pi}{12}$

$\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{6}\right)}{1 + \tan\left(\frac{\pi}{4}\right) \cdot \tan\left(\frac{\pi}{6}\right)}$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}}$$

$$= \frac{\frac{3-\sqrt{3}}{\cancel{3}}}{\frac{3+\sqrt{3}}{\cancel{3}}} = \frac{(3-\sqrt{3})(3-\sqrt{3})}{(3+\sqrt{3}) \cdot (3-\sqrt{3})}$$

$$= \frac{9 - 6\sqrt{3} + 3}{9 - 3} = \frac{12 - 6\sqrt{3}}{6}$$

$$= \boxed{2 - \sqrt{3}}$$

Remark: In most examples, the "hint" will not be given. You need to brainstorm a bit to see which two "EASY" angles can be used to get the given angle. Many times, there are many ways to do this, try to choose the easiest two angles from the unit circle.

Examples in degrees:

$105^0 = 60^0 + 45^0$ **or** $105^0 = 135^0 - 30^0$

$15^0 = 60^0 - 45^0$ **or** $15^0 = 45^0 - 30^0$

$75^0 = 30^0 + 45^0$ **or** $75^0 = 120^0 - 45^0$

Examples in radians:

$\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$ **or** $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$

$\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$

$\frac{7\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$

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Question#1: Use sum and difference formulas to evaluate: $\cos\left(15^0\right)$

Example 5: Given that $\tan(A) = 2$ and $\tan(B) = 6$, evaluate $\tan\left(A+B\right)$.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

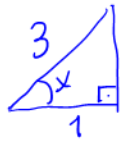
$$= \frac{2+6}{1-2\cdot 6} = \frac{8}{1-12} = \boxed{-\frac{8}{11}}$$

Example 6: Given that $\tan(x) = 5$, evaluate $\tan\left(x - \frac{\pi}{4}\right)$.

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$$

$$\tan\left(x - \frac{\pi}{4}\right) = \frac{\tan x - \tan(\pi/4)}{1 + \tan(x) \cdot \tan\left(\frac{\pi}{4}\right)}$$

$$= \frac{5 - 1}{1 + 5 \cdot 1} = \frac{4}{6} = \boxed{\frac{2}{3}}$$



$\sqrt{8} = 2\sqrt{2}$ $\quad \sin x = \frac{2\sqrt{2}}{3}$

quadrant 1

Example 7: Given that $\boxed{\cos(x) = \frac{1}{3}}$ and $0 < x < \frac{\pi}{2}$, evaluate $\cos\left(x - \frac{\pi}{6}\right)$.

$$\cos\left(x - \frac{\pi}{6}\right) = \cos x \cdot \cos\left(\frac{\pi}{6}\right) + \underbrace{\sin x}_{?\ \text{find!}} \cdot \sin\left(\frac{\pi}{6}\right)$$

$$= \frac{1}{3} \cdot \frac{\sqrt{3}}{2} + \frac{2\sqrt{2}}{3} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}}{6} + \frac{2\sqrt{2}}{6}$$

$$= \boxed{\frac{\sqrt{3} + 2\sqrt{2}}{6}}$$

greek letters: $\begin{cases} \alpha : \text{alpha} \\ \beta : \text{beta} \\ \theta : \text{theta} \end{cases}$

acute

Example 8: Suppose that $\sin\alpha = \dfrac{3}{5}$ and $\cos\beta = \dfrac{5}{13}$ where $0 < \alpha, \beta < \dfrac{\pi}{2}$.

Find each of these:

$\cos\alpha = \frac{4}{5}$

$\sin\beta = \frac{12}{13}$

a. $\sin(\alpha+\beta) = \underbrace{\sin\alpha}_{✓} \cdot \underbrace{\cos\beta}_{✓} + \underbrace{\sin\beta}_{?} \cdot \underbrace{\cos\alpha}_{?}$

formula

$= \frac{3}{5} \cdot \frac{5}{13} + \frac{12}{13} \cdot \frac{4}{5} = \frac{15}{65} + \frac{48}{65} = \boxed{\frac{63}{65}}$

b. $\cos(\alpha-\beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$

$= \frac{4}{5} \cdot \frac{5}{13} + \frac{3}{5} \cdot \frac{12}{13}$

$= \frac{20}{65} + \frac{36}{65} = \boxed{\frac{56}{65}}$

8-15-17

typo corrected: $\tan y = \frac{1}{2}$

quadrant 3: sin: −, cos: −

Example: Suppose $\cos x = -\dfrac{15}{17}$ and $\tan y = +\dfrac{1}{2}$ where $\pi < x, y < \dfrac{3\pi}{2}$.

Find $\cos(x+y)$.

ref. angle for x: α

$\sin x = -\dfrac{8}{17}$

$\sin y = -\dfrac{1}{\sqrt{5}} = -\dfrac{\sqrt{5}}{5}$

$\cos y = -\dfrac{2}{\sqrt{5}} = \dfrac{-2\sqrt{5}}{5}$

$$\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y$$

$$= \left(\frac{-15}{17} \cdot \frac{-2\sqrt{5}}{5}\right) - \left(\frac{-8}{17} \cdot \frac{-\sqrt{5}}{5}\right)$$

$$= \frac{30\sqrt{5}}{85} - \frac{8\sqrt{5}}{85}$$

$$= \frac{30\sqrt{5} - 8\sqrt{5}}{85} = \boxed{\frac{22\sqrt{5}}{85}}$$

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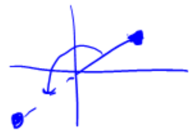
Question#2: Given $\sin(x) = \frac{4}{5}$ **and** $0 < x < \frac{\pi}{2}$ **, use sum and difference formulas to evaluate:** $\sin\left(x + \frac{\pi}{4}\right)$

REMARK:

We have used the following facts in previous sections:

$$\sin(x+\pi) = -\sin x$$



$$\cos(x+\pi) = -\cos x$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

complementary angles

Now, we can prove these facts using sum and difference formulas.

Proofs:

formula

$$\sin(x+\pi) = \underbrace{\sin x}\,\underbrace{\cos\pi}_{-1} + \underbrace{\sin\pi}_{0}\,\underbrace{\cos x} = \underbrace{\sin x\cdot(-1)} + \cancel{0\cdot\cos x} = -\sin x$$

formula

$$\cos(x+\pi) = \cos x\,\underbrace{\cos\pi}_{(-1)} - \sin x\,\underbrace{\sin\pi}_{0} = \cos x\cdot(-1) - \sin x\cdot 0 = \underbrace{-\cos x}$$

$$\sin\left(\frac{\pi}{2} - x\right) = \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1}\underbrace{\cos x} - \underbrace{\sin x}\,\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} = 1\cdot\cos x - \sin x\cdot 0 = \underbrace{\cos x}$$

$\sin A \cdot \cos B + \sin B \cdot \cos(A) = \sin(A+B)$

Example 9: Simplify each.

a) $\sin(2x)\cos(15x)+\sin(15x)\cos(2x)$

$= \sin(2x+15x) = \sin(17x)$

b) $\sin 10^\circ \cos 55^\circ - \sin 55^\circ \cos 10^\circ$ $= \sin(A-B)$

A B

$= \sin(10^\circ - 55^\circ)$

$= \sin(-45^\circ) = -\sin(45^\circ) = \boxed{-\frac{\sqrt{2}}{2}}$

A B

c) $\cos\left(\frac{7\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)+\sin\left(\frac{7\pi}{12}\right)\sin\left(\frac{\pi}{12}\right)$

$= \cos(A-B) = \cos\left(\frac{7\pi}{12} - \frac{\pi}{12}\right) = \cos\left(\frac{\pi}{2}\right) = \boxed{0}$

$\frac{6\pi}{12} = \frac{\pi}{2}$

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Example 10: Simplify each.

pattern?

a) $\dfrac{\tan 40° + \tan 5°}{1 - \tan 40° \tan 5°} = \tan(A+B) = \tan(40° + 5°) = \tan(45°) = \boxed{1}$

b) $\dfrac{\tan 80° - \tan 15°}{1 + \tan 80° \tan 15°} = \tan(A - B) = \tan(80° - 15°)$

$= \boxed{\tan(65°)}$

formula $\sin(A+B)$ $\sin(A-B)$

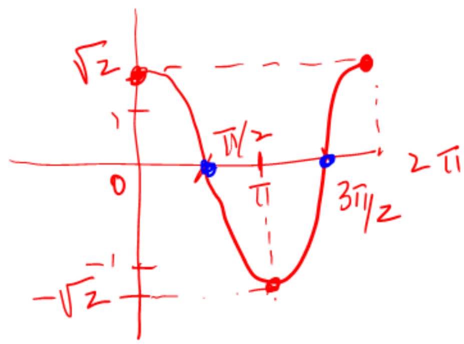
Example 11: Let $f(x) = \boxed{\sin\left(x+\frac{\pi}{4}\right)} - \boxed{\sin\left(x-\frac{\pi}{4}\right)}$.

Simplify the formula for $f(x)$ and find the minimum and maximum values.
Find the $x-$ intercepts of this function over the interval $[0, 2\pi]$.

$$f(x) = \left(\sin x \cdot \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cdot\cos x\right) - \left(\sin x\cdot\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cdot\cos x\right)$$

$$= \cancel{\frac{\sqrt{2}}{2}\cdot\sin x} + \frac{\sqrt{2}}{2}\cdot\cos x - \cancel{\frac{\sqrt{2}}{2}\cdot\sin x} + \frac{\sqrt{2}}{2}\cdot\cos x$$

$$f(x) = \sqrt{2}\cdot\cos x$$



Maximum Value: $\sqrt{2}$

Minimum Value: $-\sqrt{2}$

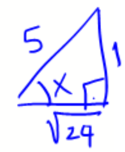
$x-$ intercepts: $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$

$\angle BAD$: angle BAD

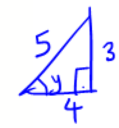
Example 12: In the figure below, $\sin(\angle BAD) = \frac{1}{5}$ and $\sin(\angle ABD) = \frac{3}{5}$.
Find $\sin(\angle BDC)$. *(The image is not drawn to scale.)*

geometry

$\sin(x) = \frac{1}{5}$

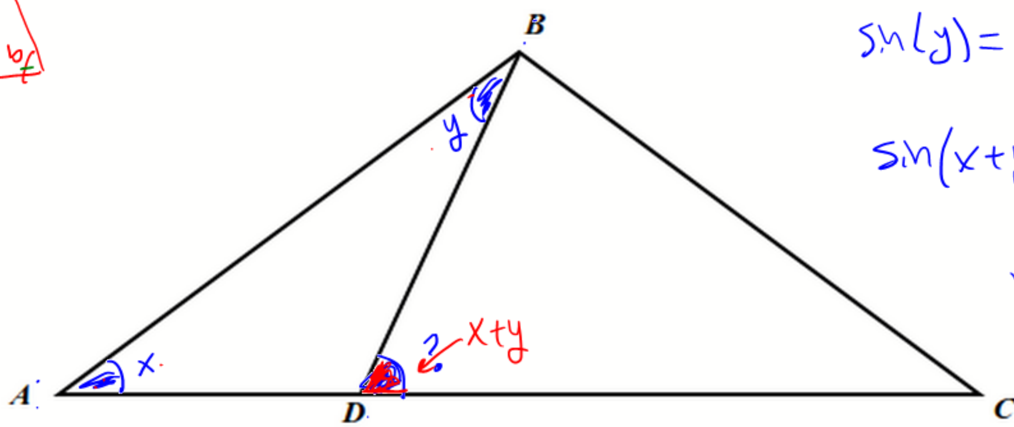


$\sin(y) = \frac{3}{5}$



$\sin(x+y) = ?$

$\sqrt{24} = \sqrt{4 \cdot 6} = 2\sqrt{6}$



$\sin(x+y) = \sin x \cdot \cos y + \sin y \cdot \cos x$

$= \frac{1}{5} \cdot \frac{4}{5} + \frac{3}{5} \cdot \frac{2\sqrt{6}}{5}$

$= \boxed{\frac{4 + 6\sqrt{6}}{25}}$

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Question#3: Given $\sin(x) = \frac{4}{5}$ **and** $\sin(y) = \frac{5}{13}$ **, where** $0 < x, y < \frac{\pi}{2}$ **, use sum and difference formulas to evaluate:** $\sin(x + y)$