Section 6.2 – Double and Half Angle Formulas

We know trig values of many angles on the unit circle. Can we use them to find values for more angles?

For example, we know all trig values of 45^0 ; can we use that information to find trig values of 22.5^0 (which is half of 45^0)?

Or, if we know that $\sin(x)=\frac{1}{4}$, is there a way to find $\sin(2x)$?

NOTE: $\sin(2x)\neq 2\sin(x)$ cot, sec, csc as well.

$\cos(2x)\neq 2\cos(x)$ $\cos(3x) \neq 3\cdot\cos(x)$

$\tan(2x)\neq 2\tan(x)$

To see this:

$$\sin(30^0)=\frac{1}{2} \quad\rightarrow\quad 2\sin(30^0)=2\cdot\frac{1}{2}=1$$

$$\sin(2\cdot 30^0)=\sin(60^0)=\frac{\sqrt{3}}{2}$$

$$\sin(2\cdot 30^0)\neq 2\sin(30^0)$$

Sine Formula:

Now suppose we are interested in finding $\sin(2A)$. We can use the **sum formula** for sine to develop this identity:

sin(A+B) = sinA cosB + sinB·cosA

A A A

$$\begin{aligned}\sin(2A) &= \sin(A+A)\\ &= \sin A\cos A + \sin A\cos A\\ &= 2\sin A\cos A\end{aligned}$$

This can be written as: $\sin(2x) = 2\sin(x)\cos(x)$

formula

Check: double

$$\sin\left(60^0\right) = \sin\left(2\cdot 30^0\right) = 2\sin\left(30^0\right)\cos\left(30^0\right) = \cancel{2}\cdot\frac{1}{\cancel{2}}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

$$\sin\left(90^0\right) = \sin\left(2\cdot 45^0\right) = 2\sin\left(45^0\right)\cos\left(45^0\right) = \cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}}\cdot\frac{\sqrt{2}}{2} = 1$$

Example: If $\sin(x) = \frac{3}{5}$ and $\cos(x) = \frac{4}{5}$, then

$$\sin(2x) = 2\sin(x)\cos(x) = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$$

Cosine Formula $\cos(A+B) = \cos A \cos B - \sin A \sin B$

Similarly, we can develop a formula for $\cos(2A)$:

$$\begin{aligned}\cos(2A) &= \cos(A+A)\\ &= \cos A \cos A - \sin A \sin A\\ &= \cos^2 A - \sin^2 A\end{aligned}$$

This can be written as: $\cos(2x) = \cos^2(x) - \sin^2(x)$

Check: $2\cdot 30^\circ$

$$\cos\left(60^0\right) = \cos^2\left(30^0\right) - \sin^2\left(30^0\right) = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$\cos\left(90^0\right) = \cos^2\left(45^0\right) - \sin^2\left(45^0\right) = \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = 0$$

Example: If $\sin(x) = \frac{3}{5}$ and $\cos(x) = \frac{4}{5}$, then

$$\cos(2x) = \cos^2(x) - \sin^2(x) = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \boxed{\frac{7}{25}}$$

NOTE: We can restate this formula in terms of sine only or in terms of cosine only by using the Pythagorean identities and making a substitution. So we have:

$$\begin{aligned}\cos(2A) &= \boxed{\cos^2(A) - \boxed{\sin^2(A)}} \\ &= \cos^2(A) - \left(1-\cos^2(A)\right) \\ &= \cos^2(A) - 1 + \cos^2(A) \\ &= 2\cos^2(A) - 1\end{aligned}$$

$\cos^2(A) = 1 - \sin^2(A)$

$\sin^2(A) = 1 - \cos^2(A)$

ex: $\cos(A) = \frac{1}{4}$

$\cos(2A) = ?$

OR:

$$\begin{aligned}\cos(2A) &= \boxed{\cos^2(A)} - \sin^2(A) \\ &= \left(1-\sin^2(A)\right) - \sin^2(A) \\ &= 1 - 2\sin^2(A)\end{aligned}$$

ex: $\sin(A) = \frac{1}{5}$

$\cos(2A) = ?$

These can be written as:

$$\cos(2x) = 2\cos^2(x) - 1$$

Or

$$\cos(2x) = 1 - 2\sin^2(x)$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

A

Tangent Formula

We can also develop a formula for $\tan(2A)$ using sum formula:

$$\begin{aligned}\tan(2A) &= \tan(A+A)\\ &= \frac{\tan A + \tan A}{1 - \tan A \tan A}\\ &= \frac{2\tan A}{1-\tan^2 A}\end{aligned}$$

This can also be expressed as: $\tan(2x) = \dfrac{2\tan(x)}{1-\tan^2(x)}$

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These three formulas are called the **double angle formulas for sine, cosine and tangent**.

Double – Angle Formulas

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

Also:

$$\cos(2x) = 2\cos^2(x) - 1$$

$$\cos(2x) = 1 - 2\sin^2(x)$$

$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$$

$$\cot(2x) = \frac{1}{\tan(2x)}$$

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Question#1:
Which of the following equations is/are true for all values of x ?

I. $\cos(2x) = 2\cos x$

II. $\cos(2x) = 2\cos^2 x - 1$

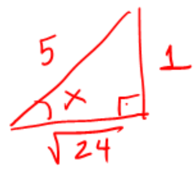
III. $\sin(2x) = \cos^2 x - \sin^2 x$

IV. $\sin(2x) = 2\sin(x)\cos(x)$

Now we'll look at the types of problems we can solve using these identities.

$0 < x < \frac{\pi}{2}$

Example 1: Suppose x is an acute angle such that $\sin(x) = \frac{1}{5}$. Find:



$a^2 + 1^2 = 5^2$

$a^2 = 24$

$\cos x = \frac{\sqrt{24}}{5} = \frac{2\sqrt{6}}{5}$

a) $\sin(2x)$ $= 2 \cdot \underbrace{\sin(x)}_{\checkmark} \cdot \underbrace{\cos(x)}_{?}$

$= 2 \cdot \frac{1}{5} \cdot \frac{2\sqrt{6}}{5} = \boxed{\frac{4\sqrt{6}}{25}}$

b) $\cos(2x)$ $= \cos^2 x - \sin^2 x$

$= \left(\frac{2\sqrt{6}}{5}\right)^2 - \left(\frac{1}{5}\right)^2$

$= \frac{24}{25} - \frac{1}{25} = \boxed{\frac{23}{25}}$

Example 2: Given that $\tan(x)=4$, find $\tan(2x)$.

$$\tan(2x) = \frac{2\cdot\tan(x)}{1-\tan^2(x)} = \frac{2\cdot 4}{1-4^2} = \frac{8}{1-16}$$

$$= \boxed{-\frac{8}{15}}$$

Example: Given that $\cos(x) = \frac{1}{5}$, find $\cos(2x)$.

$$\cos(2x) = 2 \cdot \cos^2(x) - 1$$

$$= 2 \cdot \left(\frac{1}{5}\right)^2 - 1$$

$$= \frac{2}{25} - \frac{1}{1}$$

$$= \boxed{\frac{-23}{25}}$$

Example: Given that $\sin(A)=\frac{1}{6}$, find $\cos(2A)$.

$$\cos(2A) = 1 - 2\sin^2(A)$$

$$= 1 - 2\cdot\left(\frac{1}{6}\right)^2$$

$$= 1 - 2\cdot\frac{1}{36}$$

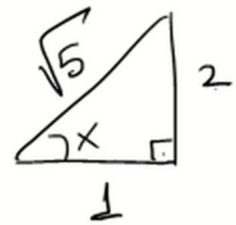
$$= 1 - \frac{1}{18}$$

$$= \boxed{\frac{17}{18}}$$

acute

$\tan x = \frac{2}{1}$

Example: Given that $\underbrace{\tan(x)=2}$ and $0<x<\frac{\pi}{2}$, find $\sin(2x)$.



$$\sin(2x) = 2 \cdot \underbrace{\sin x} \cdot \underbrace{\cos x}$$

$$= 2 \cdot \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}}$$

$$= \boxed{\frac{4}{5}}$$

3rd quadrant sin: −

Example 3: Suppose that $\cos(\theta) = -\frac{2}{3}$ and $\pi < \theta < \frac{3\pi}{2}$. Find

a) $\cos(2\theta) = 2\cos^2(\theta) - 1 = 2\left(\frac{-2}{3}\right)^2 - 1$

$= 2 \cdot \frac{4}{9} - 1$

$= \frac{8}{9} - \frac{1}{1} = \boxed{\frac{-1}{9}}$

b) $\sin(2\theta) = 2 \underbrace{\sin\theta}_{?} \cdot \underbrace{\cos\theta}_{\checkmark}$

$\cos(\theta) = \frac{-2}{3} \Rightarrow \sin^2\theta = 1 - \cos^2\theta = 1 - \frac{4}{9} = \frac{5}{9}$

$\sin\theta = \pm\sqrt{\frac{5}{9}} = -\frac{\sqrt{5}}{3}$

$\sin(2\theta) = 2 \cdot \frac{-\sqrt{5}}{3} \cdot \frac{-2}{3} = \boxed{\frac{4\sqrt{5}}{9}}$

$\sin(2A) = \underbrace{2 \cdot \sin A \cdot \cos A}$

Example 4: Simplify the following expressions:

a) $2\sin(75°)\cos(75°) = \sin(2 \cdot 75°) = \sin(150°) = \boxed{\frac{1}{2}}$

$2\sin(A) \cdot \cos(A)$

b) $\cos^2\left(\frac{\pi}{9}\right) - \sin^2\left(\frac{\pi}{9}\right) = \cos\left(2 \cdot \frac{\pi}{9}\right) = \boxed{\cos\left(\frac{2\pi}{9}\right)}$

$\cos^2(A) - \sin^2(A) = \cos(2A)$

A

c) $\dfrac{2\tan 15°}{1 - \tan^2 15°} = \tan(2 \cdot 15°) = \tan(30°) = \boxed{\frac{\sqrt{3}}{3}}$

Pattern!

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$10 = 5 \cdot 2$

Example 5: Simplify the following expression:

$$\frac{10\sin(x)\cos(x)}{\cos^2(x)-\sin^2(x)} = \frac{5 \cdot \overbrace{\boxed{2 \cdot \sin(x) \cdot \cos(x)}}^{\sin(2x)}}{\underbrace{\boxed{\cos^2(x) - \sin^2(x)}}_{\cos(2x)}}$$

$$= 5 \cdot \frac{\sin(2x)}{\cos(2x)}$$

$$= \boxed{5 \cdot \tan(2x)}$$

Example 6: Simplify the following expression:

$$\frac{1-2\sin^2(x)}{4\sin^2(x)+4\cos^2(x)} = \frac{\cos(2x)}{4(\sin^2 x+\cos^2 x)} = \frac{\cos(2x)}{4\cdot 1}$$

$$= \boxed{\frac{1}{4}\cos(2x)}$$

Remark:

$$\cos^2(x) + \sin^2(x) = 1 \quad \text{(identity)}$$

$$\cos^2(x) - \sin^2(x) = \cos(2x)$$

— o —

formula $\longrightarrow$ $\boxed{1-2\sin^2 x = \cos(2x)}$

$$2\sin^2 x - 1$$

$$= -(1-2\sin^2 x) = -\cos(2x)$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

Example 7: Expand and simplify the following expression:

$$(2\sin x - 2\cos x)^2 = (2\sin x)^2 - 2 \cdot 2\sin x \cdot 2\cos x + (2\cos x)^2$$

$$= 4\sin^2 x - 8\sin x \cdot \cos x + 4\cos^2 x$$

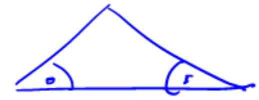
$$= 4(\underbrace{\sin^2 x + \cos^2 x}_{1}) - 8\sin x \cos x$$

$$= 4 - 8\sin x \cdot \cos x$$

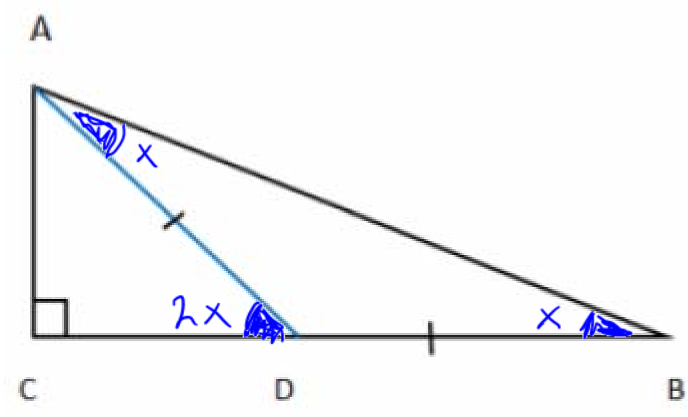
$$= 4 - 4 \cdot \underbrace{2\sin x \cdot \cos x}_{\text{formula}}$$

$$= \boxed{4 - 4 \cdot \sin(2x)}$$

ADB is isosceles



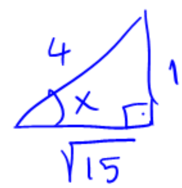
Example 8: Triangle *ABC* with right angle *C* is shown below. Given that $\sin(B)=\frac{1}{4}$ and $|AD|=|BD|$, find $\sin(\angle ADC)$. *(Note: The image is not drawn to scale.)*



$\sin(x) = \frac{1}{4}$

$\angle ADC$ is an exterior angle : $x + x = 2x$

$\sin(x) = \frac{1}{4}$, $\sin(2x) = ?$



$\cos x = \frac{\sqrt{15}}{4}$

$$\sin(2x) = 2 \cdot \underbrace{\sin x}_{\checkmark} \cdot \underbrace{\cos x}_{?}$$

$$= 2 \cdot \frac{1}{\boxed{4}} \cdot \frac{\sqrt{15}}{\boxed{4}}$$

$$= \boxed{\frac{2\sqrt{15}}{16}}$$

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Question#2:

If $\sin(x) = \dfrac{5}{13}$ **and** x **is an acute angle, find the value of** $\sin(2x)$ **.**

$\sin(45°)$ ✓ $\cos(30°)$ ✓

$\sin(22.5°)$? $\cos(15°)$ ✓

Half – Angle Formulas

Using the formula $\cos(2A) = 1 - 2\sin^2(A)$ where $A = \frac{x}{2}$, we get $\cos(x) = 1 - 2\sin^2\left(\frac{x}{2}\right)$. Solving for $\sin\left(\frac{x}{2}\right)$, we can derive the **half-angle formula for sine:**

$2\sin^2\left(\frac{x}{2}\right) = 1 - \cos(x)$

$\sin^2\left(\frac{x}{2}\right) = \frac{1-\cos(x)}{2}$

$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}$

$+$
$-$

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}}$$

quadrant

Similarly, the **half angle formula for cosine** is derived from $\cos(x) = 2\cos^2\left(\frac{x}{2}\right) - 1$:

pick one! quadrant of $\frac{x}{2}$

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos(x)}{2}}$$

Finally, half-angle formula for tangent is:

$$\tan\left(\frac{x}{2}\right) = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

Half angle formulas:

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}}$$

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos(x)}{2}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

Note: In half-angle formulas, the $\pm$ symbol is intended to mean either positive or negative but not both, and the sign before the radical is determined by the quadrant in which the angle $\frac{x}{2}$ terminates.

Now we'll look at the kinds of problems we can solve using half-angle formulas.

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}}$$

Example: Use a half-angle formula to find the exact value of $\sin\left(15^0\right)$.

half of $\underline{\underline{30^\circ}}$

$$\sin(15^\circ) = \pm\sqrt{\frac{1-\cos(30^\circ)}{2}}$$

1^{st} quad.

$+$

$$= +\sqrt{\frac{1-\sqrt{3}/2}{2}} \qquad \frac{\frac{2-\sqrt{3}}{2}}{2}$$

$$= \sqrt{\frac{2-\sqrt{3}}{4}} = \boxed{\frac{\sqrt{2-\sqrt{3}}}{2}}$$

Example: Use a half-angle formula to find the exact value of $\cos\left(\frac{\pi}{12}\right)$

half of ......

$\frac{\pi}{12}$ is half of $\frac{\pi}{6}$

$$\frac{\frac{\pi}{6}}{2}$$

$$\frac{1}{2} \cdot \frac{\pi}{6} = \frac{\pi}{12}$$

$$\cos\left(\frac{\pi}{12}\right) = \pm\sqrt{\frac{1+\cos\left(\frac{\pi}{6}\right)}{2}}$$

1st quad.

+

$$= +\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \boxed{\frac{\sqrt{2+\sqrt{3}}}{2}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{\sin(x)}{1+\cos(x)}$$

half of .... ?

Example: Use a half-angle formula to find the exact value of $\tan\left(\frac{\pi}{8}\right)$.

$\frac{\pi}{8}$ is half of $\frac{\pi}{4}$.

$$\frac{\frac{\pi}{4}}{2} = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}$$

$$\tan\left(\frac{\pi}{8}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{1+\cos\left(\frac{\pi}{4}\right)}$$

$$= \frac{\sqrt{2}/2}{1+\frac{\sqrt{2}}{2}} = \frac{\frac{\sqrt{2}}{2}}{\frac{2+\sqrt{2}}{2}} = \frac{\sqrt{2}}{2+\sqrt{2}} \quad (2-\sqrt{2})$$

$$= \frac{\sqrt{2}(2-\sqrt{2})}{4-2} = \frac{2\sqrt{2}-2}{2} = \boxed{\sqrt{2}-1}$$

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Example: Use a half-angle formula to find the exact value of $\cos\left(\frac{7\pi}{12}\right)$.

half of ....

$\frac{7\pi}{12}$ is half of $\frac{7\pi}{6}$

$\frac{1}{2}\cdot\frac{7\pi}{6} = \frac{7\pi}{12}$

$$\cos\left(\underbrace{\frac{7\pi}{12}}\right) = \pm\sqrt{\frac{1+\cos\left(\frac{7\pi}{6}\right)}{2}}$$

unit circle!

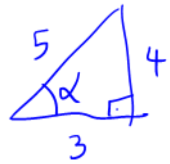
$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

quadrant 2

cos: $-$

$$= -\sqrt{\frac{1+\left(-\frac{\sqrt{3}}{2}\right)}{2}}$$

$$= -\sqrt{\frac{2-\sqrt{3}}{4}} = \boxed{-\frac{\sqrt{2-\sqrt{3}}}{2}}$$



$\cos(x) = -\frac{3}{5}$

$\pi < x < \frac{3\pi}{2}$

half

$\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$

90° 135°

2nd quad.

cosine: −

big

Example: Given $\sin(x) = -\frac{4}{5}$ and $\pi < x < \frac{3\pi}{2}$, find the value of

a) $\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos(x)}{2}} = -\sqrt{\frac{1+\frac{-3}{5}}{2}}$

2nd quad

−

$= -\sqrt{\frac{\frac{2}{5}}{2}} = -\sqrt{\frac{1}{5}} = -\frac{1}{\sqrt{5}}$

$= \boxed{\frac{-\sqrt{5}}{5}}$

b) $\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}$

2nd quad.

sine: +

$= +\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}} = \sqrt{\frac{\frac{8}{5}}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$

$= \boxed{\frac{2\sqrt{5}}{5}}$

c) $\tan\left(\frac{x}{2}\right) = \frac{1-\cos x}{\sin x} = \frac{1-\left(-\frac{3}{5}\right)}{-\frac{4}{5}}$

$= \frac{\frac{8}{5}}{-\frac{4}{5}} = \frac{8}{5}\cdot\frac{-5}{4} = \boxed{-2}$

What if

$\tan\left(\frac{x}{2}\right) = \frac{\sin x}{1+\cos x} = \frac{-4/5}{1+\frac{-3}{5}} = \frac{-4/5}{2/5} = \boxed{-2}$

If x is in quadrant 3 : $\pi < x < \frac{3\pi}{2}$

Note: If $A < x < B$, then to determine the quadrant for $\frac{x}{2}$, divide the inequality by 2: $\frac{A}{2} < \frac{x}{2} < \frac{B}{2}$.

For example;

If $\underbrace{\pi < x < \frac{3\pi}{2}}$, then $\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$. So, terminal side for $\frac{x}{2}$ is in Quadrant 2.

(sine: + . cosine: -)

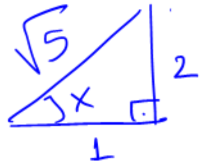
If $\frac{3\pi}{2} < \theta < 2\pi$, then $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$. So, terminal side for $\frac{\theta}{2}$ is in Quadrant ~~3~~ 2.

(sine: + , cosine: -) 135° 180°

If x is acute, then $\frac{x}{2}$ is also acute (in Quadrant 1).

sin: +

cos: +



$\sin x = \frac{2}{\sqrt{5}}$ $\cos x = \frac{1}{\sqrt{5}}$

Example: Given $\tan(x) = \frac{2}{1}$ and $0 < x < \pi$, find the value of

$\tan(2x) + \tan\left(\frac{x}{2}\right)$.

tan: + ⇒ x is acute

double angle → $\tan(2x) = \dfrac{2 \cdot \tan x}{1 - \tan^2 x} = \dfrac{2 \cdot 2}{1 - 2^2} = \dfrac{4}{-3} = -\dfrac{4}{3}$

half angle → $\tan\left(\frac{x}{2}\right) = \dfrac{1 - \cos x}{\sin x} = \dfrac{1 - \frac{1}{\sqrt{5}}}{\frac{2}{\sqrt{5}}} = \dfrac{\frac{\sqrt{5} - 1}{\sqrt{5}}}{\frac{2}{\sqrt{5}}}$

$\tan\left(\frac{x}{2}\right) = \dfrac{\sqrt{5} - 1}{2}$

$$\tan(2x) + \tan\left(\frac{x}{2}\right) = -\frac{4}{3} + \frac{\sqrt{5} - 1}{2}$$

$$= \frac{-8 + 3(\sqrt{5} - 1)}{6}$$

$$= \boxed{\frac{-11 + 3\sqrt{5}}{6}}$$

very small!

$\frac{10x}{2}$

Example: Given $0 < x < \frac{\pi}{20}$, simplify the following expression:

$$12\cos(5x)\underbrace{\sqrt{\frac{1-\cos(10x)}{2}}}$$

$$\sqrt{\frac{1-\cos(A)}{2}} = \sin\left(\frac{A}{2}\right)$$

$A = 10x \qquad \frac{A}{2} = 5x$

$$= 12 \cdot \cos(5x) \cdot \sin(5x)$$

$$= 6 \cdot \underbrace{2 \cdot \cos(5x) \cdot \sin(5x)}_{\text{formula}}$$

$$= 6 \cdot \sin(2 \cdot 5x) = \boxed{6 \cdot \sin(10x)}$$

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Question#3:

If $\cos(x) = \frac{1}{3}$ **and** x **is an acute angle, find the value of** $\sin\left(\frac{x}{2}\right)$ **.**

(Hint: don't forget to rationalize your answer.)