Solutions to Quiz 10

Spring 2008 MATH 2433

Name:———————————————————— PS Id:————

INSTRUCTIONS You have 20 minutes for this quiz. Do not write on the back of the sheet, work on the back will not be graded. Show all your work. Full credit will not be given if either the work is completely missing or is incorrect. This quiz will be graded on 20 points.

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1. **(7 points)** State whether TRUE or FALSE, briefly giving reasons

   (a) If g has a maxima at x_0 with side condition f, then $\nabla g(x_0) = 0$.**Solution:** FALSE

   (b) If a function f attains a minimum at x with side condition g, then $\nabla f(x) \times \nabla g(x) = 0$.**Solution:** TRUE

   (c) The differential of a function f gives the rate of increment of f.**Solution:** FALSE. Differential gives increment, not rate of increment

   (d) A continuous function f defined on a closed and bounded domain will always attain an absolute maximum and a minimum on the domain.**Solution:**TRUE

   (e) $f(x,y,z) = xyz$ defined on $D = \{(x,y,z) : x^2 + y^2 \leq 1\}$ must attain an absolute maximum.**Solution:** FALSE. Let z become very large

   (f) The gradient of f points in the direction of fastest increase of f.**Solution:** TRUE

   (g) The tangent plane to a surface at a point is unique.**Solution:**TRUE

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2. **(5 points)** Assume that the Celsius temperature T at a point (x,y,z) on the sphere $x^2 + y^2 + z^2 = 1$ is given by $T(x,y,z) = 10xy^2z$. Find points on the sphere at which the temperature is the greatest and points where it is the least.
   Solution:$\nabla T = 10y^2z\mathbf{i} + 20xzy\mathbf{j} + 10xy^2\mathbf{k}$ and $\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. $\nabla T = \lambda \nabla g \implies 1 - y^2z = 2\lambda x, 20xyz = 2\lambda y, 10xy^2 = 2\lambda z$. Note that f can be seen to have gradient equal to 0 only at the point (0, 0,0) and this point does not satisfy g. So, $\lambda \neq 0$. Multiply the second equation by y and divide by λ to get $x^2 = \frac{y^2}{2}$ and multiply the third equation by z and divide by λ to get $z^2 = \frac{y^2}{2}$. Now substitute these values in g to get $4x^2 = 1$ which gives $x = \pm 1/2$ and $z = \pm 1/2$ and $y^2 = 1/2$. Substitute in T to get the max value of $5/4$ at $(1/2, \pm 1/\sqrt{2}, 1/2)$and $(-1/2, \pm 1/\sqrt{2}, -1/2)$ and the min value of $-5/4$ at $(1/2, \pm 1/\sqrt{2}, -1/2)$and $-1/2, \pm 1/\sqrt{2}, 1/2$.

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3. **(4 points)** Calculate

$$\sqrt{3.06^2 + 3.88^2}$$

   Do not use a calculator. Show all work.
   Solution: $x = 3, y = 4, \Delta x = 0.06, \Delta y = -0.12$, $\nabla f = \frac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2}}\mathbf{j}$.

$$df = \frac{x}{\sqrt{x^2+y^2}}\Delta x + \frac{y}{\sqrt{x^2+y^2}}\Delta y = \frac{3}{5}(0.06) - \frac{4}{5}(0.12) = -0.06$$

   $f(3.06, 3.88) = f(3,4) + df = 5 - 0.06 = 4.94$

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4. **(4 points)** Determine the path of steepest descent from the point $(0, 0, A)$ along the surface

$$z = A + x + 2y - x^2 - 3y^2$$

Solution: We can view z as a function of x and y and can find the path of steepest descent.

$$\nabla z = (1 - 2x)\mathbf{i} + (2 - 6y)\mathbf{j}$$

so we set

$$x'(t) = 2x(t) - 1, y'(t) = 6y(t) - 2$$

(recall that the gradient points in the directions of fastest **increase**). These solve to give

$$x = \frac{C_1 e^{2t} + 1}{2}$$

and

$$y = \frac{C_2 e^{6t} + 2}{6}$$

When $t = 0$, $x = 0$ and $y = 0$. Using this initial condition, we get $C_1 = -1$ and $C_2 = -2$. Eliminating t by observing that

$$e^{6t} = (1 - 2x)^3 = 1 - 3y$$

we get the curve to be

$$3y = 1 - (1 - 2x)^3$$

Also note that as t increases, $x = \frac{-e^{2t}+1}{2}$ decreases. The curve points towards the decreasing direction of x.

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5. **(4 points)** Find the distance between the lines $x = \frac{1}{2}y = \frac{1}{3}z$ and $x = y - 2 = z$
Solution: Set $x = \frac{1}{2}y = \frac{1}{3}z = t$ and $x = y - 2 = z = s$. Then points on these lines look like $(t, 2t, 3t)$ and $(s, 2 + s, s)$. The square of the distance between these points is

$$f(s, t) = (t - s)^2 + (2t - 2 - s)^2 + (3t - s)^2 = 14t^2 - 12ts + 3s^2 - 8t + 4s + 4$$

$$\nabla f = (28t - 12s - 8)\mathbf{i} + (-12t + 6s + 4)\mathbf{j}$$

Setting $\nabla f = 0$ and solving we obtain $t = 0$, $s = \frac{-2}{3}$. On computing second partial derivatives, we get $f_{tt} = 28, f_{ts} = -12, f_{ss} = 6, D = (-12)^2 - 6(28) = -24 < 0$. By the second partials test, we have a minimum at $t = 0, s = \frac{-2}{3}$ and the minimum distance is $\sqrt{f(0, -2/3)} = \frac{2}{3}\sqrt{6}$. Also, this minimum is absolute (Why? Think about the geomteric picture. Two lines must have a unique point at which they are closest, which means the distance between them must have a unique minimum. We found a local minimum, so it must be the absolute minimum.)