MATH 4377 - MATH 6308

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Lecture : MTuWThF 12:00 PM - 1:40 PM
Office hours : Tu-Th 1:45 PM - 2:45 PM or BY APPOINTMENT

REMARKS

Note about 4377 vs 6308: All students will be treated the same, regardless of their seniority. For simplicity, I will refer to this course usually as MATH 4377 only.

The lecture notes and the homework assignments will be **posted at the class webpage**:

https://www.math.uh.edu/ dlabate/MA4377_Su24.html

I do not use CANVAS for this course

Textbook

LINEAR ALGEBRA, 5-th edition, by Friedberg, Insel, Spence;
ISBN: 9780134860244

The course covers Chapters 1-5.

Classroom participation

- Come to class on time.
- Attendance is NOT mandatory. However, be aware that regular class attendance, participation, and engagement in coursework are critical contributors to student success.
- ASK QUESTIONS. If there is something you do not understand in what I am saying or working on, do not hesitate to ask questions.
- I do not record the class. You can record it for your own use if you want but you are not allowed to share it.

Required background material

Appendix A, B, C, D

Sets

Definition

A *set* is a collection of objects, called elements.

Examples:

- $\{1,2,3\} = \{2,1,3\} = \{2,1,1,1,2,3\}$
- $[0,2]$
- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Later we will see also: $\mathbb{C}, \mathbb{Z}^n, \mathbb{Q}^n, \mathbb{R}^n, \mathbb{C}^n$
- $\left\{ \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$
- $\emptyset$ (the empty set)
- $\{x \in \mathbb{R} : 0 \le x < 1\}$ which can also be written as $[0,1) \subset \mathbb{R}$

Given two sets A and B, the following operations on A and B yield new sets:

- $A \cup B$ (union of A and B)
- $A \cap B$ (intersection of A and B)
- $A \setminus B = \{x \in A : x \notin B\}$ (complement of B in A)
- $A \times B = \{(a, b) : a \in A, b \in B\}$ (product of A and B)

Relation

Definition

Let A be a set. A *relation* on A is a subset S of $A \times A$. For any elements x and y in A, write $x \sim y$ if and only if $(x, y) \in S$.

Examples: Given $A = \{1, 2, 3\}$:

- $S = \{(1,1), (2,2), (3,3)\}$. This relation is "$=$"
- $S = \{(1,2), (1,3), (2,3)\}$. This relation is "$<$"
- $S = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$. This relation is "$\neq$"

Some symbols to remember: $\forall$ ("for all"), $\exists$ ("there exists").

Equivalence relation

Definition

Let A be a set with a relation S. Then S is called an *equivalence relation* if and only if

1. $\forall x \in A : x \sim x$ (reflexive)
2. $\forall x, y \in A : x \sim y \Leftrightarrow y \sim x$ (symmetric)
3. $\forall x, y, z \in A : (x \sim y \text{ and } y \sim z) \Rightarrow x \sim z$ (transitive)

A trivial example is the relation is "$=$" (equality)

Less trivial example of equivalence relation

Problem: Let $A = \mathbb{Z}$. Show that

$$x \sim y \Leftrightarrow \exists k \in \mathbb{Z} : x - y = 5k$$

is an equivalence relation.

Less trivial example of equivalence relation

Problem: Let $A = \mathbb{Z}$. Show that

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is an equivalence relation.

Proof. Need to show that the 3 properties characterizing an equivalence relation are satisfied....

Functions

Definition

Let A, B be sets. A *function* $f : A \to B$ is a rule that associates to EACH element $x \in A$ a UNIQUE element of B, denoted $f(x)$.
The set A is called the *domain*, the set B is called the *codomain*.

We keep considering $f : A \to B$

Definition

For $S \subseteq A, f(S) = \{f(x) : x \in S\}$ is the *image of S under f*.
$f(A)$ is called the *range*.

Definition

Given $f : A \to B$ and $g : A \to B$, $f = g \Leftrightarrow \forall x \in A, \; f(x) = g(x)$

Functions

Definition

$f : A \to B$ is *one-to-one* (or *injective*) if and only if $f(x) = f(y) \Rightarrow x = y$.

Definition

$f : A \to B$ is *onto* (or *surjective*) if and only if $\forall b \in B, \; \exists a \in A : f(a) = b$

Definition

f is *bijective* if and only if f is injective and surjective.

For $S \subset A$, the restriction of f to S is $f|_S : S \to B, x \mapsto f(x)$.

Fields

Definition

Let A be a set. A binary operation is any map $A \times A \to A$.

Example: let $A = \mathbb{Q}$ or $A = \mathbb{R}$. We are very familiar with with binary operations $+$ and $\cdot$ and their properties.

Fields

Definition

A **field** F is a set with two binary operations denoted by $+$ and $\cdot$ such that the following property hold

1. commutativity: $a + b = b + a$ and $a \cdot b = b \cdot a$
2. associativity: $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. existence of neutral elements: $\exists 0 \in F : \forall a \in F,\ a + 0 = a$ and $\exists 1 \in F : \forall a \in F,\ 1 \cdot a = a$
4. existence of inverse elements: $\forall a \in F,\ \exists b \in F : a + b = 0$ and $\forall a \in F \backslash \{0\},\ \exists c \in F : a \cdot c = 1$
5. distributive law: $a \cdot (b + c) = a \cdot b + a \cdot c$

Examples of fields

- The set of rational numbers $\mathbb{Q}$ with the usual definitions of addition and multiplication is a field.
- The set of real numbers $\mathbb{R}$ with the usual definitions of addition and multiplication is a field.
- The set of all real numbers of the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$, with the definitions of addition and multiplication as in $\mathbb{R}$ is a field.

Note: the set of integers is NOT a field since it lacks a multiplicative inverse.

Cancellation laws

Theorem (Cancellation laws)

Let F be a field and $a, b, c \in F$.

1. $a + b = c + b \Rightarrow a = c$
2. $a \cdot b = c \cdot b$ and $b \neq 0 \Rightarrow a = c$

Proof for (1)

Proof for (2) is similar to the one for (1).

Corollary to cancellation laws

Corollary

The neutral element of addition is unique.

Proof

Motivation for complex numbers

We all know the solution to:

$$x^2 - 1 = 0$$

in $\mathbb{R}$.
What is the solution to $x^2 + 1 = 0$??

Definition

A *complex number* z is an expression of the form $z = a + ib$, where $a, b \in \mathbb{R}$ are called *real part* and *imaginary part*, respectively.
Sum and product are defined by

$$z + w = (a + ib) + (c + id) = a + c + i(b + d)$$

and

$$zw = (a + ib)(c + id) = (ac - bd) + i(ad + bc).$$

Graphical representation of complex numbers

Representation of $z = a + ib$

Set $\mathbb{C}$

Theorem

The set of complex numbers $\mathbb{C}$ with sum and multiplication as in the above definition is a field.

We don't go over the proof because it's long and not very instructive.

Problem: Find the multiplicative inverse of $z = a + ib$

Theorem

The set of complex numbers $\mathbb{C}$ with sum and multiplication as in the above definition is a field.

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Problem: Find the multiplicative inverse of $z = a + ib$

(solution: $z^{-1} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$ *)*

Complex conjugate

Definition

The *complex conjugate* of $z = a + ib$ is $\bar{z} = a - ib$.

Proposition

1. $\bar{\bar{z}} = z$
2. $\overline{z + w} = \bar{z} + \bar{w}$
3. $\overline{zw} = \bar{z} \cdot \bar{w}$
4. $\overline{\frac{z}{w}} = \frac{\bar{z}}{\bar{w}}$

Modulus

Definition

The *absolute value* (or *modulus*) of $z = a + ib$ is $|z| = \sqrt{a^2 + b^2}$.

Notice that we have

$$z\bar{z} = (a + ib)(a - ib) = a^2 + b^2.$$

Thus,

$$|z| = \sqrt{z\bar{z}}.$$

Properties of the modulus

1. $|zw| = |z||w|$

2. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

3. $|z + w| \leq |z| + |w|$

4. $|z| - |w| \leq |z + w|$

Exercise

Let $z = 1 + i4$ and $w = -4 - i3$.

1. Find $|w|$.
2. Write zw in the form $a + ib$.
3. Write $\dfrac{z}{w}$ in the form $a + ib$.

Fundamental theorem of algebra

Theorem (fundamental theorem of algebra)

Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a complex polynomial, that is $a_i \in \mathbb{C}$. Then $\exists z_0 \in \mathbb{C} : p(z_0) = 0$.

We don't show the proof given because it is rather involved

Corollary to fundamental theorem of algebra

We keep considering the complex polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

Corollary

For p as above, $\exists r_1, \ldots, r_n \in \mathbb{C}$ such that

$$p(z) = a_n (z - r_1) \ldots (z - r_n).$$

Remark

Recall the classical formula to solve quadratic equations $ax^2 + bx + c = 0$, that is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exercise: solve $x^2 - 2x + 5 = 0$