MATH 4377 - MATH 6308

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Outline

Vectors

Definition

In $\mathbb{R}^2$, a vector is an ORDERED pair of real numbers where addition and multiplication by scalars hold:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

$$\alpha(a_1, a_2) = (\alpha a_1, \alpha a_2).$$

In $\mathbb{R}^3$, a vector is an ORDERED triple of real numbers where addition and multiplication by scalars hold:

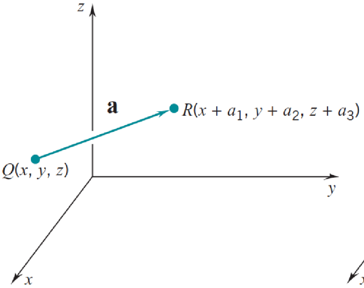
$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1 +, a_2 + b_2, a_3 + b_3)$$

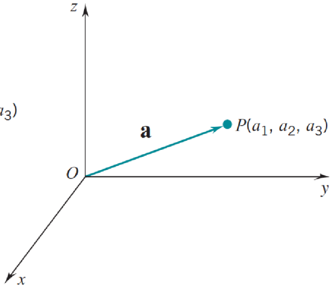
$$\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha s_3).$$

Vectors have many physical applications, e.g., velocity, force, etc

Vectors

We depict vectors with arrows. To depict the vector (a_1, a_2, a_3), you can choose any initial point and use the arrow





Notation: $\overrightarrow{QR}$ and $\overrightarrow{OP}$ or $\mathbf{a} = (a_1, a_2, a_3)$.

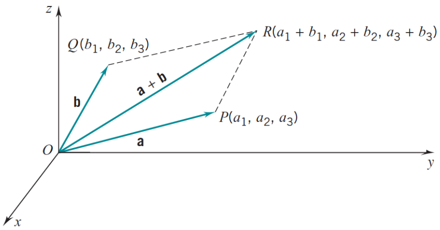
Some properties of vectors

- Commutative:
$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}.$$
- Associative:
$$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c}).$$
- Zero vector:
$$\mathbf{0}=(0,0,0).$$
Note: $\alpha \cdot \mathbf{0}=\mathbf{0}$, α being any real number.

We will see an abstract definition of these obvious properties belwo.

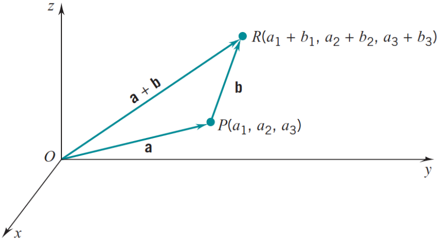
Graphical representation of $\mathbf{a} + \mathbf{b}$

Parallelogram law



or

tail-to-head



Parallel vectors

Definition

Two vectors **a** and **b** are *parallel* if

$$\mathbf{b} = \alpha \mathbf{a}$$

for some real number α.

- if $\alpha > 0$, **a** and **b** have the same direction
- if $\alpha < 0$, **a** and **b** have opposite directions

Lines in 2D and 3D

In 2D, the *line* through the points $A = (x_1, x_2)$ and $B = (y_1, y_2)$ is

$$\{(x_1, x_2) + t(y_1 - x_1, y_2 - x_2) : t \in \mathbb{R}\}.$$

In 3D, the *line* through the points $A = (x_1, x_2, x_3)$ and $B = (y_1, y_2, y_3)$ is

$$\{(x_1, x_2, x_3) + t(y_1 - x_1, y_2 - x_2, y_3 - x_3) : t \in \mathbb{R}\}.$$

Example

Find the line through $(1, 1, 2)$ and $(0, 3, -1)$.

Example

Find the line through $(1,1,2)$ and $(0,3,-1)$.

Solution:

$$(1,1,2) + t\left((0,3,-1) - (1,1,2)\right), \quad t \in \mathbb{R}$$

$$(1,1,2) + t\left(-1,2,-3\right), \quad t \in \mathbb{R}$$

Planes

The *plane* through the points $A = (x_1, x_2, x_3)$, $B = (y_1, y_2, y_3)$ and $C = (z_1, z_2, z_3)$ (not all three on a line) is

$$\{(x_1, x_2, x_3) + s(y_1 - x_1, y_2 - x_2, y_3 - x_3) + t(z_1 - x_1, z_2 - x_2, z_3 - x_3) : s, t \in \mathbb{R}\}.$$

Example

Find the plane through the points $A = (1, 0, -1)$, $B = (0, 1, 2)$ and $C = (1, 1, 0)$

Example

Find the plane through the points $A = (1, 0, -1)$, $B = (0, 1, 2)$ and $C = (1, 1, 0)$

Solution:

$$\{(1, 0, -1) + s\,((0, 1, 2) - (1, 0, -1)) + t\,((1, 1, 0) - (1, 0, -1)) : s, t \in \mathbb{R}\}$$

$$\{(1, 0, -1) + s\,(-1, 1, 3) + t\,(0, 1, 1) : s, t \in \mathbb{R}\}$$

Vector space

Definition

A *vector space* (or *linear space*) V over a field F (e.g, $\mathbb{R}$ or $\mathbb{C}$) is a set together with two binary operations defined on it, namely, vector addition mapping $V \times V \to V$ and scalar multiplication mapping $F \times V \to V$.

Vector addition and scalar multiplication satisfy the classical vector properties that we spell out in detail below.

Vector space

Definition

A *vector space* (or *linear space*) V over a field F (e.g, $\mathbb{R}$ or $\mathbb{C}$) is a set with a binary operation denoted "$+$" and a map $F \times V \to V$ such that

1. $\forall \mathbf{x}, \mathbf{y} \in V : \mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
2. $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V : (\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
3. $\exists \mathbf{0} \in V : \forall \mathbf{x} \in V,\ \mathbf{x} + \mathbf{0} = \mathbf{x}$
4. $\forall \mathbf{x} \in V,\ \exists \mathbf{y} \in V : \mathbf{x} + \mathbf{y} = \mathbf{0}$
5. $\forall \mathbf{x} \in V : 1\mathbf{x} = \mathbf{x}$, where 1 is the neutral element of multiplication in F
6. $\forall a, b \in F,\ \forall \mathbf{x} \in V : (ab)\mathbf{x} = a(b\mathbf{x})$
7. $\forall a \in F,\ \forall \mathbf{x}, \mathbf{y} \in V : a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$
8. $\forall a, b \in F,\ \forall \mathbf{x} \in V : (a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$

Remark

In the above definition, the elements of F are called **scalars** and the elements of V are called **vectors**.

The map $F \times V \to V$:

$$\forall a \in F,\ \forall \mathbf{x} \in V : a\mathbf{x} \in V$$

is called scalar multiplication.

Example of vector spaces: set of n-tuples

Definition

An object in the form $(a_1, a_2, ..., a_n)$, with $a_1, a_2, ..., a_n$ elements of a field F, is called *n-tuple*.
The elements $a_1, a_2, ..., a_n$ are called *entries* or *components*.

The set of all the n-tuples with entries from F is denoted by F^n. F^n is a vector space over F with componentwise addition and scalar multiplication.

Vectors in F^n may be written as column vectors

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Special cases: $\mathbb{R}^2$, $\mathbb{R}^3$

Generalization: matrices

An $m \times n$ *matrix* with entries from field F is a rectangular array:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The set of all the $m \times n$ matrices with entries in F is denoted by $M_{m\times n}$ and it is a vector space over F with componentwise addition and scalar multiplication.

Other examples of vector spaces

- Let $\mathcal{F}$ be the set of all real-valued functions on a nonempty set S. This is a vector space over $\mathbb{R}$ under the operations of ordinary addition and scalar multiplication of functions:

$$(f+g)(x) = f(x) + g(x)$$

and

$$(af)(x) = af(x)$$

for any $x \in S$

Special cases:
Continuous functions on $[0,1]$, differentiable functions on $[0,1]$.

Other examples of vector spaces

- Let $n \geq 0$ be an integer and

$$P_n = \text{set of all polynomials of degree at most } n \geq 0$$

The elements of P_n have the form

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

where $a_0, \ldots, a_n$ are real numbers. This is a vector space over $\mathbb{R}$ under the operations of ordinary addition and scalar multiplication of functions.

Cancellation law for vector spaces

Theorem (Cancellation law for vector spaces)

Let V be a vector space and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$. If $\mathbf{x} + \mathbf{z} = \mathbf{y} + \mathbf{z}$, then $\mathbf{x} = \mathbf{y}$.

Proof

Corollaries to cancellation law for vector spaces

Corollary 1

The vector $\mathbf{0}$ is unique.

Corollary 2

The additive inverse is unique.

Note: the additive inverse of $\mathbf{x}$ is denoted by $-\mathbf{x}$.

Another theorem for vector spaces

Theorem

Let V be a vector space. Then the following statements are true.

1. $\forall \mathbf{x} \in V : 0\mathbf{x} = \mathbf{0}$
2. $\forall \mathbf{x} \in V, \forall a \in F : (-a)\mathbf{x} = -(a\mathbf{x}) = a(-\mathbf{x})$
3. $\forall a \in F : a\mathbf{0} = \mathbf{0}$

Proof...

Subspaces

Definition

A subset W of a vector space V over the field F is called a *subspace of V* if W is a vector space with $+$ and scalar multiplication from V.

To show that a subset is a subspace, the following theorem is useful:

Theorem

A nonempty subset W of the vector space V is a subspace of V if and only if

1. $\forall \mathbf{x}, \mathbf{y} \in W : \mathbf{x} + \mathbf{y} \in W$
2. $\forall c \in F, \ \forall \mathbf{x} \in W : c \cdot \mathbf{x} \in W$

Example

Is $W = \{(a, b) : a + b = 0\} \subset \mathbb{R}^2$ a subspace?

Example

Is $W = \{(a, b) : a + b = 0\} \subset \mathbb{R}^2$ a subspace?

Answer: Yes.

To prove it, we can use the Theorem above, which requires to verify that

1. $\forall \mathbf{x}, \mathbf{y} \in W : \mathbf{x} + \mathbf{y} \in W$
2. $\forall c \in \mathbb{R}^2, \forall \mathbf{x} \in W : c \cdot \mathbf{x} \in W$

Example

Is $W = \{(a, b, c) : 3a - b + 2c = 0\} \subset \mathbb{R}^3$ a subspace?

Example

Is $W = \{(a, b, c) : 3a - b + 2c = 0\} \subset \mathbb{R}^3$ a subspace?

Answer: Yes.

To prove it, we can use the Theorem above, which requires to verify that

1. $\forall \mathbf{x}, \mathbf{y} \in W : \mathbf{x} + \mathbf{y} \in W$
2. $\forall c \in \mathbb{R}^3, \forall \mathbf{x} \in W : c \cdot \mathbf{x} \in W$

Example

Is $W = \{(a, b, c) : 3a - b + 2c = 1\} \subset \mathbb{R}^3$ a subspace?

Example

Is $W = \{(a, b, c) : 3a - b + 2c = 1\} \subset \mathbb{R}^3$ a subspace?

Answer: No.

To prove it, it is sufficient to show that the sum of two generic elements of W does not belong to W.

Example

The trace of a square matrix $A = (a_{ij})_{i,j=1,\dots,n}$ is $tr(A) = \sum_{i=1}^{n} a_{ii}$. Is $W = \{A = (a_{ij})_{i,j=1,\dots,n} : tr(A) = 0\} \subset M_{n\times n}$ a subspace?

Example

The trace of a square matrix $A = (a_{ij})_{i,j=1,\dots,n}$ is $tr(A) = \sum_{i=1}^{n} a_{ii}$. Is $W = \{A = (a_{ij})_{i,j=1,\dots,n} : tr(A) = 0\} \subset M_{n\times n}$ a subspace?

Answer: Yes.

To prove it, we can use the Theorem above, which requires to verify that

1. $\forall \mathbf{x}, \mathbf{y} \in W : \mathbf{x} + \mathbf{y} \in W$
2. $\forall c \in \mathbb{R}, \forall \mathbf{x} \in W : c \cdot \mathbf{x} \in W$

Example

Is the set of the upper triangular matrices U a subspace of $M_{n \times n}$?

Example

Is the set of the upper triangular matrices U a subspace of $M_{n\times n}$?

Answer: Yes.

To prove it, we can use the Theorem above, which requires to verify that

1. $\forall \mathbf{x}, \mathbf{y} \in W : \mathbf{x} + \mathbf{y} \in W$
2. $\forall c \in \mathbb{R}^3, \forall \mathbf{x} \in W : c \cdot \mathbf{x} \in W$

Sum and direct sum

Definition

Let V be a vector space and let S, T be nonempty subsets of V. Then let $S + T = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in S, \mathbf{y} \in T\}$. We call $S + T$ the *sum* of S and T.

Definition

Let V be a vector space and let W, U be subspaces of V. If $V = W + U$ and $W \cap U = \{\mathbf{0}\}$, we call V the *direct sum* of W and U and write it as $V = W \oplus U$.

Proposition

Let V be a vector space and let W, U be subspaces of V. Then the sum $W + U$ is a subspace of V (containing both W and U).

Example

Consider the sum
$\{(a, b, 0, c) : a, b, c \in \mathbb{R}\} + \{(d, 0, e, f) : d, e, f \in \mathbb{R}\} = \mathbb{R}^4$.
Is this a direct sum?

Example

Consider the sum
$\{(a, b, 0, c) : a, b, c \in \mathbb{R}\} + \{(d, 0, e, f) : d, e, f \in \mathbb{R}\} = \mathbb{R}^4$.
Is this a direct sum?

Answer: No.

To show it, we will examine the intersection of the two sets.
Let $W = \{(a, b, 0, c) : a, b, c \in \mathbb{R}\}$ *and* $V = \{(d, 0, e, f) : d, e, f \in \mathbb{R}\}$
We have that

$$W \cap V = \{(a, 0, 0, c) : a, b, c \in \mathbb{R}\} \neq \emptyset$$

Example

Consider the sum $\{(a, 0, 0, b) : a, b \in \mathbb{R}\} + \{(0, c, d, 0) : c, d \in \mathbb{R}\} = \mathbb{R}^4$.
Is this a direct sum?

Example

Consider the sum $\{(a, 0, 0, b) : a, b \in \mathbb{R}\} + \{(0, c, d, 0) : c, d \in \mathbb{R}\} = \mathbb{R}^4$.
Is this a direct sum?

Answer: Yes.

Let $W = \{(a, b, 0, c) : a, b, c \in \mathbb{R}\}$ *and* $V = \{(d, 0, e, f) : d, e, f \in \mathbb{R}\}$
We have that

$$W \cap V = \{(0, 0, 0, 0)\} = \emptyset$$

We also have that

$$W + V = \{(a, b, c, d) : a, b, c, d \in \mathbb{R}\} = \mathbb{R}^4$$

Example

Set of upper triangular square matrices $+$ Set of lower triangular square matrices $=$ Set of square matrices.
Is this a direct sum?

Example

Set of upper triangular square matrices + Set of lower triangular square matrices = Set of square matrices.
Is this a direct sum?

Answer: No.

To show it, we will examine the intersection of the two sets.
One can verify that this intersection contains the set of diagonal matrices, hence it is non empty.