MATH 4377 - MATH 6308

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Outline

- Section 1.4 - Linear Combinations and System of Linear Equations
- Section 1.5 - Linear Dependance/Independence
- Section 1.6 - Bases and Dimension

Linear Combinations and System of Linear Equations

Section 1.4

Linear combination

Definition

Let V be a vector space over field F and S a nonempty subset of V. We call $\mathbf{v} \in V$ a *linear combination* of vectors in S if there exist vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n \in S$ and scalars $a_1, \ldots, a_n \in F$ such that

$$\mathbf{v} = a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n$$

Exercise: Take $V = \mathbb{R}^3$ and $S = \{(1,0,0),(0,1,0),(0,0,1)\}$. Write $(3,4,1)$ as a linear combination of vectors in S.

Example

Exercise: Write $(3,1,2)$ as a linear combination of $(1,0,1),(0,1,1),(1,2,1)$.

Example

Exercise: Write $(3, 1, 2)$ as a linear combination of $(1, 0, 1), (0, 1, 1), (1, 2, 1)$.

To solve this problem, we need to solve

$$x_1(1, 0, 1) + x_2(0, 1, 1) + x_3(1, 2, 1) = (3, 1, 2)$$

which is gives a system of linear equations:

$$\begin{aligned} x_1 + x_3 &= 3 \\ x_2 + 2x_3 &= 1 \\ x_1 + x_2 + x_3 &= 2 \end{aligned}$$

Solving systems of linear equations

You can simplify the solution of a system of linear equations by performing any of these **elementary row operations**:

- Add a constant multiple of one equation to another.
- Multiply an equation by a nonzero scalar.
- Interchange the order of any two equations.

These there operations DO NOT change the solution of the system!

Solving systems of linear equations

From the system of linear equations

$$\begin{aligned} x_1 + x_3 &= 3 \\ x_2 + 2x_3 &= 1 \\ x_1 + x_2 + x_3 &= 2 \end{aligned}$$

we write the *augmented matrix*

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{array} \right)$$

We will apply elementary row operations until we obtain a simplified matrix which is equivalent to the original one.

Solving systems of linear equations

$r_1 \leftrightarrow r_3$, $r_2 \leftrightarrow r_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$r_2 \rightarrow r_2 - r_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$r_2 \rightarrow -r_2$, $r_3 \rightarrow r_3 - r_2$; then $r_3 \rightarrow \frac{1}{2} r_3$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

The last matrix is a *row-echelon form matrix*.

Solving systems of linear equations

The row-echelon form matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

corresponds to the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_2 &= -1 \\ x_3 &= 1 \end{aligned}$$

which is easily solved: $x_1 = 2, x_2 = -1, x_3 = 1$.

Thus we solve the original linear combination problem as

$$(3,1,2) = 2\,(1,0,1) - (0,1,1) + (1,2,1)$$

Example

Exercise: Write $(3, 1, 2)$ as a linear combination of
$(1, 0, 0), (0, 1, 0), (1, 2, 0)$.

Example

Exercise: Write $(3, 1, 2)$ as a linear combination of $(1, 2, -1)$,$(1, 6, -3)$, $(0, 1, 2)$,$(1, 2, 1)$.

Example

Exercise: Write $(3, 1, 2)$ as a linear combination of $(1, 2, -1)$,$(1, 6, -3)$, $(0, 1, 2)$,$(1, 2, 1)$.

To solve this problem, we need to solve

$$x_1(1, 2, -1) + x_2(1, 6, -3) + x_3(0, 1, 2) + x_4(1, 2, 1) = (3, 1, 2)$$

which is gives a system of linear equations:

$$\begin{aligned} x_1 + x_2 + x_4 &= 3 \\ 2x_1 + 6x_2 + x_3 + 2x_4 &= 1 \\ -x_1 - 3x_2 + 2x_3 + x_4 &= 2 \end{aligned}$$

Example

$$\left(\begin{array}{rrrr|r} 1 & 1 & 0 & 1 & 3 \\ 2 & 6 & 1 & 2 & 1 \\ -1 & -3 & 2 & 1 & 2 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 0 & 1 & 3 \\ 0 & 4 & 1 & 0 & -5 \\ 0 & -2 & 2 & 2 & 5 \end{array}\right) \to$$

$$\left(\begin{array}{rrrr|r} 1 & 1 & 0 & 1 & 3 \\ 0 & -2 & 2 & 2 & 5 \\ 0 & 4 & 1 & 0 & -5 \end{array}\right) \to \left(\begin{array}{rrrr|r} 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -1 & -5/2 \\ 0 & 0 & 5 & 4 & 5 \end{array}\right)$$

Hence we have the solution:

$$5x_3 = 5 - 4x_4, x_2 = -5/2 + x_3 + x_4, x_1 = 3 - x_2 - x_4$$

Choosing any value of $x_4 \in \mathbb{R}$, we find a solution of the linear combination problem.

Span

Definition

Let V be a vector space and S a nonempty subset of V. We call span(S) the set of all vectors in V that can be written as a linear combination of vectors in S.

Exercise: Let $S = \{(1,0,0),(0,1,0),(2,1,0)\}$. What is span($S$)?

Theorem

Theorem

The span of any subset S of a vector space V is a subspace of V.

Proof

Theorem

Theorem

The span of any subset S of a vector space V is a subspace of V.

Proof

Solution: need to show that span S is closed under the operations of addition and scalar multiplication.

Examples

Exercise: Does $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ span $\mathbb{R}^3$?

Examples

Exercise: Does $S = \{(1,2),(2,1)\}$ span $\mathbb{R}^2$?

Examples

Exercise: Does $S = \{(1,2),(2,1)\}$ span $\mathbb{R}^2$?

To solve this problem, we need to verify that, for any $(a,b) \in \mathbb{R}^2$ we can solve

$$x_1(1,2) + x_2(2,1) = (a,b)$$

This gives the system of linear equations:

$$\begin{aligned} x_1 + 2x_2 &= a \\ 2x_1 + x_2 &= b \end{aligned}$$

Examples

Exercise: Does $S = \{(1,2),(2,1)\}$ span $\mathbb{R}^2$?

To solve this problem, we need to verify that, for any $(a,b) \in \mathbb{R}^2$ we can solve

$$x_1(1,2) + x_2(2,1) = (a,b)$$

This gives the system of linear equations:

$$\begin{aligned} x_1 + 2x_2 &= a \\ 2x_1 + x_2 &= b \end{aligned}$$

This is equivalent to the row-reduced system

$$\begin{aligned} x_1 + 2x_2 &= a \\ -3x_2 &= b - 2a \end{aligned}$$

showing that the system has always a solution.

Examples

Exercise: Does $S = \{(1, 2)\}$ span $\mathbb{R}^2$?

Using the argument above, we can see that not every element in $\mathbb{R}^2$ can be written as a linear combination of S.

Examples

Exercise: Which (a, b, c) are in $\text{span}(\{(1, 1, 2), (0, 1, 1), (2, 1, 3)\})$?

Examples

Exercise: Which (a, b, c) are in $\text{span}(\{(1,1,2),(0,1,1),(2,1,3)\})$?

To solve this problem, we can examine the linear system

$$x_1(1,1,2) + x_2(0,1,1) + x_3(2,1,3) = (a, b, c)$$

which is associated with the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 1 & 1 & 1 & b \\ 2 & 1 & 3 & c \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & -1 & b-a \\ 0 & 1 & 1 & c-2a \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & -1 & b-a \\ 0 & 0 & 2 & c-b-a \end{array}\right)$$

Since the linear system can be solved for any $(a, b, c) \in \mathbb{R}^3$, then $\text{span}(\{(1,1,2),(0,1,1),(2,1,3)\}) = \mathbb{R}^3$

Linear Dependance and Linear Independence

Section 1.5

Linear dependence

Goal: given a vector space V, we want to find the SMALLEST set $S \subset V$ such that $\text{span}(S) = V$.

Linear dependence

Definition

A subset S of a vector space V is called *linearly dependent* if there exist a finite number of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n \in S$ and scalars $a_1, \ldots, a_n$, NOT ALL EQUAL TO ZERO, such that

$$a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n = \mathbf{0}.$$

If the vectors in S are not linearly dependent, we say that they are *linearly independent*.

Remark: Linear dependence is equivalent to say that at least one vector in S can be written as a linear combinations of the others. Linear independence on the other hand implies that no vector in the set can be expressed as a linear combination of the others

Example

Let $S = \{(1,1,1),(2,2,2)\}$.

S linearly dependent since

$$2(1,1,1) = (2,2,2)$$

equivalently

$$2(1,1,1) - (2,2,2) = 0$$

Let $R = \{(2,0,0),(0,1,0)\}$.

R linearly independent since

$$a_1(2,0,0) + a_2(0,1,0) = (2a_1, a_2, 0) = 0$$

implies that $a_1 = a_2 = 0$, showing that R is not linearly dependent.

Remark

If $S = \{\mathbf{u}_1, \mathbf{u}_2\} \subset V$, then S is linearly dependent if and only if there exists a constant $\alpha \neq 0$ such that $\mathbf{u}_1 = \alpha \mathbf{u}_2$.

If S consists of more then two vectors, verifying linear dependence or independence requires more work.

Example

Let $S = \{(1,1,1), (-2,0,-3), (3,1,4)\}$.

S linearly dependent since

$$(3,1,4) = (1,1,1) - (-2,0,-3)$$

Let $R = \{(2,0,0), (0,1,0), (0,0,4)\}$.

R linearly independent since

$$a_1(2,0,0) + a_2(0,1,0) + a_3(0,0,4) = (2a_1, a_2, 4a_3) = 0$$

implies that $a_1 = a_2 = a_3 = 0$, showing that R is not linearly dependent.

Linear independence

Remark

Let S be a subset of a vector space V and let $\mathbf{u}_1, \ldots, \mathbf{u}_n \in S$. These vectors are *linearly independent* if and only if

$$a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n = \mathbf{0} \Rightarrow a_1, \ldots, a_n = 0.$$

Theorem

Theorem

Let V be a vector space. If $S_1 \subseteq S_2$ and S_1 is linearly dependent, then S_2 is linearly dependent.

Proof

It follows form the definition.

Another theorem

Theorem

Let S be a linearly independent subset of V. Let $\mathbf{v} \in V \setminus S$. Then $S \cup \{\mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} \in \text{span}(S)$.

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Let S be a linearly independent subset of V. Let $\mathbf{v} \in V \setminus S$. Then $S \cup \{\mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} \in \text{span}(S)$.

Let $S = \{u_1, \ldots, u_m\}$

Proof for ($\Leftarrow$). If $v \in \text{span}(S)$, then v is a linear combination of elements in $\{u_1, \ldots, u_m\}$, hence $\{v, u_1, \ldots, u_m\}$ is linearly dependent.

Proof for ($\Rightarrow$). If $S \cup \{v\}$ is linearly dependent, then there are constants $c_1, \ldots, c_m, c_{m+1}$ not all zero such that

$$c_1 u_1 + \cdots + c_m u_m + c_{m+1} v = 0$$

In this sum, it must be $c_{m+1} \neq 0$. If not, the rest of the sum would be 0 with $c_1, \ldots, c_m$ not all zero, violating the hypothesis that S is linearly independent. Since $c_{m+1} \neq 0$, we can then write

$$v = -\frac{1}{c_{m+1}}(c_1 u_1 + \cdots + c_m u_m)$$

showing that $v \in \text{span}(S)$.

Linear dependence and homogeneous systems of equations

A homogeneous system of equations like

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 3x_1 + 5x_2 + 9x_3 &= 0 \\ 5x_1 + 9x_2 + 3x_3 &= 0 \end{aligned}$$

can be written as a vector equation

$$x_1(1,3,5) + x_2(2,5,9) + x_3(-3,9,3) = (0,0,0)$$

Fact. The vectors $(1,3,5),(2,5,9),(-3,9,3)$ are linearly independent if and only if the trivial solution $x_1 = x_2 = x_3 = 0$ is the only solution of the homogeneous system.

Linear dependence and homogeneous systems of equations

The last observation implies that we can check the linear dependence or independence of a set of vectors by examining the solution set of the associated homogeneous system.
We examine the augmented matrix of the system

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & -1 & 18 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since the row-reduced system has a row of zeros, then the homogeneous system has non-trivial solutions and, thus, the vectors $(1, 3, 5), (2, 5, 9), (-3, 9, 3)$ are **linearly dependent**.

Linear dependence and homogeneous systems of equations

Fact

Each linear dependence relation among the columns of the matrix A corresponds to a nontrivial solution to $Ax = 0$.
The columns of a matrix A are linearly independent if and only if the equation $Ax = 0$ has only the trivial solution.

Facts about linearly dependent/independent sets

- If a set S in a vector space V contains the 0 vector, then it is linearly dependent (since the linear dependence condition is always satisfied).
- The set of a single element $\{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$ (it follows from the last property).
- If a set S in the vector space $\mathbb{R}^n$ consists of $m > n$ vectors, then S is linearly dependent. It follows from the observation that an homogeneous linear system $Ax = 0$ there the matrix A has more columns than rows has always nontrivial solutions.

Bases and Dimension

Section 1.6

Basis

Definition

Let V be a vector space. A (vector) *basis* B of V is a linearly independent subset of V which satisfies $\text{span}(B) = V$.

Example

Let $S = \{(1,0),(1,1),(2,3)\}$. Is S a basis for $\mathbb{R}^2$?

Example

Let $S = \{(1,0),(1,1),(2,3)\}$. Is S a basis for $\mathbb{R}^2$?

Solution. *No, since A contains 3 vectors in $\mathbb{R}^2$, then the set is linearly dependent.*

Example

Let $S = \{(1,0),(0,1),(0,2)\}$. Is S a basis for $\mathbb{R}^2$?

Example

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Solution. *No, since S contains 3 vectors in $\mathbb{R}^2$, then the set is linearly dependent.*

Example

Let $S = \{(1,0)\}$. Is S a basis for $\mathbb{R}^2$?

Example

Let $S = \{(1,0)\}$. Is S a basis for $\mathbb{R}^2$?

Solution. *No, because the set does not span* S.

Example

Let $S = \{(1,0),(1,1)\}$. Is S a basis for $\mathbb{R}^2$?

Example

Let $S = \{(1,0),(1,1)\}$. Is S a basis for $\mathbb{R}^2$?

Solution. *Yes, because the set is linearly independent and does span S.*

This can be seen by observing the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The columns are A are linearly independent since the matrix is reduced in row-echelon form.

The vectors span $\mathbb{R}^2$ because the matrix $Ax = \begin{pmatrix} a \\ b \end{pmatrix}$ can be solved for any $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$.

Example

Let $S = \{(1,0,0),(0,1,0),(0,0,1)\}$. Is S a basis for $\mathbb{R}^3$?

Example

Let $S = \{(1,0,0),(0,1,0),(0,0,1)\}$. Is S a basis for $\mathbb{R}^3$?

Solution. *Yes, because the set is linearly independent and does span* S*.*

This basis is called the **canonical basis** of $\mathbb{R}^3$.

Similarly we define the **canonical basis** of $\mathbb{R}^n$, for any n.

Theorem for bases

Theorem

Let V be a vector space. Let $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a subset of V. Then

$$B \text{ is a basis of } V \Leftrightarrow \forall \mathbf{v} \in V : \exists!\, a_1, \ldots, a_n \in F, \mathbf{v} = a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n.$$

Proof for $\Rightarrow$ If B is a basis, for every $v \in V$, there are $a_1, \ldots, a_n$ such that $v = a_1 u_1 + \ldots + a_n u_n$ since B spans V. To prove uniqueness, suppose there is another expansion $v = b_1 u_1 + \ldots + b_n u_n$. Then $(a_1 - b_1)u_1 + \ldots + (a_n - b_n)u_n = 0$. By the l.i., it must be $(a_i - b_i) = 0$ for all coefficients. This shows that the expansion must be unique.

Proof for $\Leftarrow$ If for every $v \in V$, there is a unique sequence $a_1, \ldots, a_n$ such that $v = a_1 u_1 + \ldots + a_n u_n$, then B spans V. To show that B is l.i., consider the expansion of the 0 vector, that can be expressed by taking $a_1 = \ldots = a_n = 0$. By the uniqueness, this is the only expansion of the 0 vector. This also implies that B is l.i.

Theorem

Theorem

Let V be a vector space. Let S be a finite subset of V with $\text{span}(S) = V$. Then there exists a subset of S which is a basis for V. In particular, V has a finite basis.

Proof

Exercise

Let $S = \{(1,0),(1,1),(2,3)\}$. We have $\mathbb{R}^2 = \mathrm{span}(S)$ but S is not a basis. Find a subset of S which is a basis for $\mathbb{R}^2$.

Exercise

Let $S = \{(1,0),(0,1),(0,2)\}$. We have $\mathbb{R}^2 = \mathrm{span}(S)$ but S is not a basis. Find a subset of S which is a basis for $\mathbb{R}^2$.

Exercise

Let $S = \{(-1,-1,-1),(5,5,5),(0,2,2),(0,0,3),(0,2,5)\}$. Is S a basis for $\mathbb{R}^3$? If not, can you find a subset of S which is a basis for $\mathbb{R}^3$?

Replacement Theorem

Question: given a vector space V, which is the SMALLEST set $S \subset V$ such that $\text{span}(S) = V$?

Theorem (Replacement Theorem)

Let V be a vector space. Let $V = \text{span}(G)$, where G is a subset of V of cardinality n. Let L be a linearly independent subset of V of cardinality m. Then the following holds.

1. $m \leq n$
2. there exists a subset $H \subseteq G$ of cardinality $n - m$ such that $\text{span}(L \cup H) = V$

In other words

Let's consider two subsets of vector space V:

- $G = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$ (cardinality $n = 5$), such that we have $V = \text{span}(G)$,
- $L = \{\mathbf{v}_1, \mathbf{v}_2\}$ (cardinality $m = 2$) linearly independent.

Replacement theorem tells you that there are 2 vectors in G that can be replaced with the two vectors in L and the new set obtained by this replacement still spans V.

Corollaries to replacement theorem

Corollary 1

Let V be a vector space with a finite basis $B = \{u_1, \ldots, u_n\}$. Then any set containing more than n vectors is linearly dependent.

Corollaries to replacement theorem

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Let V be a vector space with a finite basis $B = \{u_1, \ldots, u_n\}$. Then any set containing more than n vectors is linearly dependent.

Proof Suppose S is a set with $p > n$ vectors. By the Replacement Theorem, S cannot be a a l.i. subset of V.

Corollary 2

Let V be a vector space with a finite basis. Then all bases contain the same number of elements.

Corollaries to replacement theorem

Corollary 2

Let V be a vector space with a finite basis. Then all bases contain the same number of elements.

Proof. Suppose that B_1 and B_1 are two bases of V.
By the definition of basis, both sets are l.i.
By Corollary 1, B_1 cannot contain more elements of B_2, otherwise it would be linearly dependent.
Similarly, by Corollary 1, B_2 cannot contain more elements of B_1, otherwise it would be linearly dependent.
Thus, B_1 and B_1 have the same number of elements.

Dimension of V

Definition

A vector space is called *finite dimensional* if there exists a basis consisting of finitely many vectors.

Definition

The unique cardinality of a basis of a finite dimensional vector space is called the *dimension* of V, denoted $\dim(V)$.

Examples

1. $\dim(\mathbb{R}^n) =$

2. $\dim(M_{n \times m}) =$

Examples

Let P_n be the vector space of the polynomials of degree n. $\dim(P_n) =$

Corollaries to replacement theorem

Corollary 2

Let $S \subset V$. If $V = \text{span}(S)$ and $\#S = \dim(V)$, then S is a basis.

Proof

Corollaries to replacement theorem

Corollary 3

Let $S \subset V$. If S is linearly independent and $\#S = \dim(V)$, then S is a basis.

Proof

Dimension of subspaces

Theorem

Let V be a vector space. Let W be a subspace of V. Assume $\dim V$ is finite. Then $\dim W \leq \dim V$ and equality holds if and only if $V = W$.

Proof

Immediate from Replacement Theorem.

Example

Let $W = \{(a_1, a_2, a_3) \mid a_1 + a_3 = 0 \text{ and } a_1 + a_2 - a_3 = 0\} \subset \mathbb{R}^3$. Find a basis for and the dimension of subspace W.

Example

Let $W = \{(a_1, a_2, a_3, a_4, a_5) \mid a_1 + a_3 + a_5 = 0 \text{ and } a_2 = a_4\} \subset \mathbb{R}^5$. Find a basis for and the dimension of subspace W.