MATH 4377 - MATH 6308

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Outline

Linear Transformations, null spaces, and ranges

Section 2.1

Linear transformations

Definition

Let V, W be vector spaces over the same field F. We call a function $T : V \to W$ a *linear transformation* from V to W if

1. $\forall \mathbf{x}, \mathbf{y} \in V : T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
2. $\forall c \in F, \forall \mathbf{x} \in V : T(c\mathbf{x}) = cT(\mathbf{x})$

Remark: We can say T is linear, for short.

Example

Show that T is linear:

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(a_1, a_2) = (2a_1 + a_2, a_1)$$

Example

Show that T is linear:

$$T : \mathbb{R}^5 \to \mathbb{R}^7, T(a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, 0, a_4, 0, 0, a_1)$$

Example

Show that T is linear:

$$T : C^{\infty}(\mathbb{R}) \to C^{\infty}(\mathbb{R}),\ T(f) = \frac{df}{dt}$$

Example

Show that T is linear:

$$T : M_{m \times n} \to M_{n \times m}, T(A) = A^T$$

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$$T : M_{m\times n} \to M_{n\times m}, T(A) = A^T$$

Need to show that:

$$T(\alpha A) = \alpha T(A)$$
$$T(A + B) = \alpha T(A) + T(B)$$

Properties of linear transformation

- $T(\mathbf{x} - \mathbf{y}) = T(\mathbf{x}) - T(\mathbf{y})$

- $T(\mathbf{0}) = \mathbf{0}$

- $T(a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n) = a_1 T(\mathbf{v}_1) + \ldots + a_n T(\mathbf{v}_n)$

Null Space

Definition

Let V, W be vector spaces. Let $T : V \to W$ be linear.
The *null space* (or *kernel*) of T is the set

$$N(T) = \{\mathbf{x} \in V \ : \ T(\mathbf{x}) = \mathbf{0}\} \subset V.$$

The range of T is the set

$$R(T) = \{T(\mathbf{x}) \ : \ \mathbf{x} \in V\} \subset W.$$

Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.
Find null space and range.

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Find null space and range.

SOLUTION:

$$N(T) = \{x \in \mathbb{R}^3 : T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3) = 0\}$$

This give the condition $a_3 - 0$ and $a_1 = a_2$.
This implies that $\dim N(T) = 1$.

Example

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.
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SOLUTION:

$$N(T) = \{x \in \mathbb{R}^3 : T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3) = 0\}$$

This give the condition $a_3 - 0$ and $a_1 = a_2$.
This implies that $\dim N(T) = 1$.

$$R(T) = \{T(x) : x \in \mathbb{R}^3\}$$

You can show that $R(T) = \mathbb{R}^2$.

Theorem

Theorem

Let V, W be vector spaces and $T : V \to W$ linear. Then

1. $N(T)$ is a subspace of V
2. $R(T)$ is a subspace of W

Proof. Use definition of subspace.

Another theorem

Theorem

Let V, W be vector spaces and $T : V \to W$ linear. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. Then

$$R(T) = \text{span}\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}.$$

Proof.

Another theorem

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Let V, W be vector spaces and $T : V \to W$ linear. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. Then

$$R(T) = \text{span}\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}.$$

Proof. For any $v \in V$, there are constants $c_1, \ldots, c_n$ such that

$$v = \sum_{i=1}^{n} c_i v_i$$

By linearity,

$$T(v) = T(\sum_{i=1}^{n} c_i v_i) = \sum_{i=1}^{n} c_i T(v_i)$$

Hence,

$$R(T) = \text{span}\{T(v_1), \ldots, T(v_n)\}.$$

Another theorem

Remark.

The theorem below shows that we can represent the span of $R(T)$ using a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V.

However, this does not imply that $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a basis of $R(T)$.

Theorem

Let V, W be vector spaces and $T : V \to W$ linear. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. Then

$$R(T) = \text{span}\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}.$$

Example

Find a basis for $R(T)$ when

$$T : \mathbb{R}^3 \to \mathbb{R}^3,\ T(a_1, a_2, a_3) = (a_1 - 2a_2, a_2 + a_3, 2a_1 + a_2 + 5a_3)$$

Example

Find a basis for $R(T)$ when

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \, T(a_1, a_2, a_3) = (a_1 - 2a_2, a_2 + a_3, 2a_1 + a_2 + 5a_3)$$

SOLUTION. Let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$.
By the theorem above,

$$\{T(e_1), T(e_2), T(e_3)\} = \{(1,0,2), (-2,1,1), (0,1,5)\}$$

spans $R(T)$.
Note that $2(1,0,2) + (-2,1,1) = (0,1,5)$, so the 3 vectors are l.d., showing that they do not form a basis of $R(T)$.
However, $\{(1,0,2), (-2,1,1)\}$ are l.i. vectors spanning $R(T)$, hence they form a basis of $R(T)$.

Nullity and Rank

Definition

Let V, W be vector spaces and $T : V \to W$ be linear. If $N(T), R(T)$ are finite dimensional, then let

$$\text{nullity}(T) = \dim N(T), \quad \text{rank}(T) = \dim R(T).$$

Dimension Theorem

Let V, W be vector spaces and $T : V \to W$ be linear. If V is finite-dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim V.$$

Proof. Let $\{v_1, \dots, v_k\}$ be a basis for $N(T) \subset V$, hence, $nullity(T) = k$. By the Replacement theorem, we can find additional l.i. vectors $\{v_{k+1}, \dots, v_n\}$ such that $\{v_1, \dots, v_n\}$ is a basis for V, where $dim(V) = n$. For any $v \in V$, we can write

$$v = \sum_{i=1}^{n} a_i v_i$$

and, by linearity

$$T(v) = T(\sum_{i=1}^{n} a_i v_i) = \sum_{i=1}^{n} a_i T(v_i).$$

Dimension Theorem

Since $T(v_i) = 0$, when $i = 1, \ldots, k$, then
$T(v) \in span\{T(v_{k+1}), \ldots, T(v_n)\}$ and $R(T) = span\{v_{k+1}, \ldots, v_n\}$.

We need to show that the set $\{T(v_{k+1}), \ldots, T(v_n)\}$ is l.i., so that it as basis of $R(T)$.

Suppose $c_{k+1}T(v_{k+1}) + \ldots + c_nT(v_n) = 0$.
This implies that $T(c_{k+1}v_{k+1} + \ldots + c_nv_n) = 0$, so that
$c_{k+1}v_{k+1} + \ldots + c_nv_n \in N(T)$.
Since $\{v_1, \ldots, v_k\}$ is a basis for $N(T)$, we can write

$$c_{k+1}v_{k+1} + \ldots + c_nv_n = a_1v_1 + \ldots + c_kv_k$$

which implies

$$a_1v_1 + \ldots + a_kv_k - c_{k+1}v_{k+1} - \ldots - c_nv_n = 0$$

Since $\{v_1, \ldots, v_n\}$ is a basis, the above equation implies that all coefficients $a_1, \ldots, a_k$ and $c_{k+1}, \ldots, c_n$ are 0, showing that $\{T(v_{k+1}), \ldots, T(v_n)\}$ is l.i. This also implies that $dimR(T) = n - k$.

Theorem

Theorem

Let V, W vector spaces. Let $T : V \to W$ linear. Then T is one-to-one if and only if $N(T) = \{\mathbf{0}\}$.

Proof for $\Rightarrow$ Assume T is one-to-one. Since T is linear $T(0) = 0$.
Suppose we also have that $T(x) = 0$ for some $x \in V$.
Since T is one-to-one, $T(x) = T(0)$ implies $x = 0$. Thus, $N(T) = \{0\}$.

Proof for $\Leftarrow$ Assume $N(T) = \{\mathbf{0}\}$. For any $x, y \in V$, suppose $T(x) = T(y)$, which is equivalent to $T(x - y) = 0$. Since $N(T) = \{\mathbf{0}\}$, the last equation implies that $x - y = 0$. This shows that $T(x) = T(y)$ implies $x = y$, hence T is one-to-one.

Theorem

Theorem

Let V, W vector spaces with $\dim V = \dim W$ (both finite!). Let $T : V \to W$ linear. Then the following are equivalent:

1. T is one-to-one
2. T is onto
3. $\text{rank } T = \dim V$

Proof (1) $\Leftrightarrow$ (3). T is one-to-one if and only if $nullity(T) = 0$ Thus, by the Dimension Theorem, using the hypothesis that $\dim V = \dim W$, the statement that T is one-to-one is also equivalent to $\text{rank } T = \dim V$.

Example

We have seen the following linear T:

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(a_1, a_2) = (2a_1 + a_2, a_1)$$

Is T one-to-one? Is T onto?

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Is T one-to-one? Is T onto?

To check T one-to-one, we can verify that $N(T) = \{0\}$

Since T one-to-one and $dimV = dimW$, by previous theorem, T onto.

Remarks

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  By the Dimension Theorem: $\dim V - \dim N(T) = \dim R(T)$
  If $\dim V < \dim W$, then $\dim R(T) > \dim V$, then T cannot be onto by theorem above.

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  By the Dimension Theorem: $\dim V - \dim N(T) = \dim R(T)$
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- If $T : V \to W$ linear and $\dim V > \dim W$, then T cannot be one-to-one.

  By the Dimension Theorem: $\dim V - \dim R(T) = \dim N(T)$
  If $\dim W < \dim V$, then necessarily $\dim N(T) \geq 1$, hence T cannot be one-to-one.

Example

Consider the linear transformation:

$$T : P_2(\mathbb{R}) \to P_3(\mathbb{R}),\, T(p(x)) = 2p'(x) + \int_0^x p(t)dt$$

(1) Is T onto? (2) Is T one-to-one?

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(1) Is T onto? (2) Is T one-to-one?

(1) $\dim(P_2(\mathbb{R})) = 3$ and $\dim(P_3(\mathbb{R})) = 4$.
Thus by above remark T is not onto.

Example

Consider the linear transformation:

$$T : P_2(\mathbb{R}) \to P_3(\mathbb{R}),\, T(p(x)) = 2p'(x) + \int_0^x p(t)dt$$

(1) Is T onto? (2) Is T one-to-one?

(1) $\dim(P_2(\mathbb{R})) = 3$ and $\dim(P_3(\mathbb{R})) = 4$.
Thus by above remark T is not onto.

(2) We compute $N(T) = \{p \in P_2 : T(p) = 0\}$.
Direct calculation shows that $N(T) = \{0\}$), hence T one-to-one.

Theorem

Theorem

Let V, W vector spaces. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V. Let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be a list of arbitrary vectors in W. Then there exists a unique $T : V \to W$ linear such that $T(\mathbf{v}_i) = \mathbf{w}_i$ for all $i = 1, \ldots, n$.

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Proof.

For any $v \in V$, we can write

$$v = \sum_{i=1}^{n} a_i\, v_i$$

and the expansion is unique.

By linearity, with the notation $w_i = T(v_i)$,

$$T(v) = T(\sum_{i=1}^{n} a_i\, v_i) = \sum_{i=1}^{n} a_i\, T(v_i) = \sum_{i=1}^{n} a_i\, W_i.$$

This representation is also unique.

Corollary

Corollary

Let V, W vector spaces. Let $U, T : V \to W$ linear with $U(\mathbf{v}_i) = T(\mathbf{v}_i)$ on a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for V. Then $U = T$.

This follows directly from the theorem.

The Matrix Representation of a Linear Transformation

Section 2.2

Ordered basis

Definition

Let V be a finite dimensional vector space. An *ordered basis* for V is a basis endowed with a specific order.

Ex: ordered bases

$$\beta_1 = \{(1,0,0),(0,1,0),(0,0,1)\},$$

$$\beta_2 = \{(0,1,0),(1,0,0),(0,0,1)\}$$

are different!

Coordinate vector

Let $\beta = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ ordered basis for V. We saw earlier:

$$\forall \mathbf{x} \in V, \ \exists! a_1, \ldots, a_n : \mathbf{x} = a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n.$$

Write

$$[\mathbf{x}]_\beta = (a_1, \ldots, a_n)$$

for the *coordinate vector of* $\mathbf{x}$ relative to β.

Ex: Find coordinate vector of $\mathbf{x} = (3, 2, 5)$ relative to $\beta = \{(0, 1, 0), (1, 0, 0), (0, 0, 1)\}$.

Coordinate vector

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$$\forall \mathbf{x} \in V, \ \exists! a_1, \ldots, a_n : \mathbf{x} = a_1\mathbf{u}_1 + \ldots + a_n\mathbf{u}_n.$$

Write

$$[\mathbf{x}]_\beta = (a_1, \ldots, a_n)$$

for the *coordinate vector of* **x** relative to β.

Ex: Find coordinate vector of $\mathbf{x} = (3, 2, 5)$ relative to $\beta = \{(0, 1, 0), (1, 0, 0), (0, 0, 1)\}$.

SOLUTION:

$$(3, 2, 5) = 2(0, 1, 0) + 3(1, 0, 0), 5(0, 0, 1)$$

Hence $[\mathbf{x}]_\beta = (2, 3, 5)$.

Matrix representation of T

Let $T : V \to W$ linear.
Let $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, be a basis for V and $\gamma = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ a basis for W.

Write

$$T(\mathbf{v}_j) = \sum_{i=1}^{m} a_{ij}\mathbf{w}_i \quad \text{for } j = 1, \dots, n$$

We call the matrix (a_{ij}) the *matrix representation of* T with respect to β and γ and denote it by $[T]_\beta^\gamma$.
Notice that the j-th column of the matrix representation is $[T(\mathbf{v}_j)]_\gamma$

Particular case: when $V = W$ and $\beta = \gamma$, we denote the matrix representation by $[T]_\beta$.

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2).$$

Write $[T]_\beta^\gamma$ with $\beta = \{(1, 0), (0, 1)\}$ and $\gamma = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.
(standard bases)

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

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Write $[T]_\beta^\gamma$ with $\beta = \{(1,0),(0,1)\}$ and $\gamma = \{(1,0,0),(0,1,0),\ (0,0,1)\}$.
(standard bases)

SOLUTION:

$$T(v_1) = (1,0,2) = 1(1,0,0) + 0(0,1,0) + 2(0,0,1), \to [T(v_1)]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$T(v_2) = (3,0,-4) = 3(1,0,0) + 0(0,1,0) - 4(0,0,1), \to [T(v_2)]_\gamma = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$$

Hence $[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2).$$

Write $[T]_\beta^{\tilde{\gamma}}$ with $\beta = \{(1,0),(0,1)\}$ and $\tilde{\gamma} = \{(0,1,0),(1,0,0),\ (0,0,1)\}$.

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2).$$

Write $[T]_\beta^{\tilde{\gamma}}$ with $\beta = \{(1,0),(0,1)\}$ and $\tilde{\gamma} = \{(0,1,0),(1,0,0),\ (0,0,1)\}$.

SOLUTION:

$$T(v_1) = (1,0,2) = 0(0,1,0) + 1(1,0,0) + 2(0,0,1), \to [T(v_1)]_\gamma = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$T(v_2) = (3,0,-4) = 0(0,1,0) + 3(1,0,0) - 4(0,0,1), \to [T(v_2)]_\gamma = \begin{pmatrix} 0 \\ 3 \\ -4 \end{pmatrix}$$

Hence $[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 \\ 1 & 3 \\ 2 & -4 \end{pmatrix}$

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2).$$

Write $[T]_{\tilde{\beta}}^{\gamma}$ with $\tilde{\beta} = \{(0,1), (1,0)\}$ and $\gamma = \{(1,0,0), (0,1,0), (0,0,1)\}$.

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2).$$

Write $[T]_{\tilde{\beta}}^{\gamma}$ with $\tilde{\beta} = \{(0,1),(1,0)\}$ and $\gamma = \{(1,0,0),(0,1,0),\ (0,0,1)\}$.

SOLUTION:

$$T(u_1) = (3,0,-4) = 3(1,0,0) + 0(0,1,0) - 4(0,0,1), \to [T(v_2)]_\gamma = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$$

$$T(u_2) = (1,0,2) = 1(1,0,0) + 0(0,1,0) + 2(0,0,1), \to [T(v_1)]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Hence $[T]_{\beta}^{\gamma} = \begin{pmatrix} 3 & 1 \\ 0 & 0 \\ -4 & 2 \end{pmatrix}$

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2).$$

Write $[T]_\beta^\delta$ with $\beta = \{(1,0),(0,1)\}$ and $\delta = \{(1,1,0),(0,1,1),\ (2,2,3)\}$.

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2).$$

Write $[T]_\beta^\delta$ with $\beta = \{(1,0),(0,1)\}$ and $\delta = \{(1,1,0),(0,1,1),\ (2,2,3)\}$.

SOLUTION: $T(v_1) = (1,0,2) = a_1(1,1,0) + a_2(0,1,1) + a_3(2,2,3) = (a_1 + 2a_3, a_1 + a_2 + 2a_3, a_2 + 3a_3)$

By solving the linear system we obtain $\to [T(v_1)]_\gamma = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$T(v_2) = (3,0,-4) = b_1(1,1,0) + b_2(0,1,1) + b_3(2,2,3) = (b_1 + 2b_3, b_1 + b_2 + 2b_3, b_2 + 3b_3)$

By solving the linear system we obtain $\to [T(v_2)]_\gamma = \begin{pmatrix} 11/3 \\ -3 \\ -1/3 \end{pmatrix}$

Hence $[T]_\beta^\gamma = \begin{pmatrix} -1 & 11/3 \\ -1 & -3 \\ 1 & -1/3 \end{pmatrix}$

Sum and scalar multiplication for linear transfirmations

Definition

Let $U : V \to W$ and $T : V \to W$ be linear. Then

$$(U + T)(\mathbf{x}) = U(\mathbf{x}) + T(\mathbf{x})$$

and

$$(cT)(\mathbf{x}) = cT(\mathbf{x}).$$

Theorem

Theorem

Let V, W be given vector spaces. The set of all linear transformations $V \to W$ is a vector space with $+$ and $\cdot$ defined as above.
Write $\mathcal{L}(V, W)$ for this vector space.

Proof
Check the properties in the definition of vector space

Theorem

Theorem

Let $U, T : V \to W$ linear. Then

1. $[U+T]_\beta^\gamma = [U]_\beta^\gamma + [T]_\beta^\gamma$
2. $[aT]_\beta^\gamma = a[T]_\beta^\gamma$

Theorem

Theorem

Let $U, T : V \to W$ linear. Then

1. $[U + T]_\beta^\gamma = [U]_\beta^\gamma + [T]_\beta^\gamma$
2. $[aT]_\beta^\gamma = a[T]_\beta^\gamma$

Proof for (1):

$$(U + T)(v_j) = U(v_j) + T(v_j) = \sum_{i=1}^{m} a_{ij} w_i + \sum_{i=1}^{m} b_{ij} w_i = \sum_{i=1}^{m} c_{ij} w_i,$$

where $c_{ij} = a_{ij} + b_{ij}$, *showing that* $[U + T]_\beta^\gamma = (c_{ij})$.
Hence $[U + T]_\beta^\gamma = [U]_\beta^\gamma + [T]_\beta^\gamma$.

Proof of (2) is similar.