4.1 Determinants of Order 2

- Definition
- Linearity
- Inverses
- Orientation of an Ordered Basis
- Area of a Parallelogram

Determinants of Order 2: Definition

Definition

For the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the determinant of A, denoted $\det(A)$ or $|A|$, is the scalar $ad - bc$.

Determinants: Linearity

Theorem (4.1)

$\det : M_{2\times 2}(F) \to F$ *is a linear function of each row of a* 2×2 *matrix when the other row is held fixed. That is, for* $u, v, w \in F^2$ *and* $k \in F$,

$$\det \begin{pmatrix} u + kv \\ w \end{pmatrix} = \det \begin{pmatrix} u \\ w \end{pmatrix} + k \det \begin{pmatrix} v \\ w \end{pmatrix}$$

$$\det \begin{pmatrix} w \\ u + kv \end{pmatrix} = \det \begin{pmatrix} w \\ u \end{pmatrix} + k \det \begin{pmatrix} w \\ v \end{pmatrix}$$

Determinants and Inverses

Theorem (4.2)

The determinant of $A \in M_{2\times 2}(F)$ is nonzero if and only if A is invertible. If A is invertible then

$$A^{-1} = \frac{1}{\det(A)} \det \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Determinant and Orientation of an Ordered Basis

The **orientation** of an ordered basis $\beta = \{u, v\}$ for $\mathbb{R}^2$ is defined by

$$\mathrm{Orient}\begin{pmatrix} u \\ v \end{pmatrix} == \frac{\det\begin{pmatrix} u \\ v \end{pmatrix}}{\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|}$$

Determinant and Left/Right-Handed Coordinate System

Note that $\mathrm{Orient}\begin{pmatrix} u \\ v \end{pmatrix} = \pm 1$, and $\mathrm{Orient}\begin{pmatrix} u \\ v \end{pmatrix} = 1$ if and only if $\{u, v\}$ forms a right-handed coordinate system (u can be rotated in a counterclockwise direction through an angle θ, with $0 < \theta < \pi$, to coincide with v).

Determinant and Area of a Parallelogram

The area of the parallelogram determined by u and v:

$$\mathrm{Area}\begin{pmatrix} u \\ v \end{pmatrix} = \mathrm{Orient}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \det\begin{pmatrix} u \\ v \end{pmatrix} = \left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|$$

4.2 Determinants of Order n

- Definition
- Linearity
- Cofactor Expansions
- Elementary Row Operations
- Triangulation

Determinants of Order n: Definition

For $A \in M_{n\times n}(F)$, for $n \geq 2$, denote the $(n-1)\times(n-1)$ matrix obtained from A by deleting row i and column j by $\tilde{A}_{ij}$.

Example

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad \tilde{A}_{23} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Determinants of Order n: Definition

Recall that $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and we let $\det [a] = a$.

Definition

Let $A = (a_{ij}) \in M_{n\times n}(F)$. If $n = 1$, define $\det(A) = a_{11}$. For $n \geq 2$, define

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \cdot \det(\tilde{A}_{1j})$$
$$= a_{11} \cdot \det(\tilde{A}_{11}) - a_{12} \cdot \det(\tilde{A}_{12}) + \cdots + (-1)^{1+n} a_{1n} \cdot \det(\tilde{A}_{1n}),$$

where $\det(A)$ or $|A|$ is the **determinant** of A.

Determinants: Example

Example

Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

Solution

$$\det A = 1 \det \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} + 0 \det \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_$$

Common notation: $\det \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$.

So

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0 \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix}$$

Determinants and Cofactor Expansion

Cofactor

The $(\mathbf{i},\mathbf{j})$-**cofactor** of A is the number C_{ij} where

$$C_{ij} = (-1)^{i+j} \det \tilde{A}_{ij}.$$

Note that

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \quad + a_{1n}C_{1n},$$

the cofactor expansion along the first row of A.

Example (Cofactor Expansion)

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1C_{11} + 2C_{12} + 0C_{13}$$

(cofactor expansion across row 1)

Determinants: Linearity

Theorem (4.3)

det : $M_{n\times n}(F) \to F$ *is an n-linear function*

$$\det\begin{pmatrix} a_1 \\ \cdot \\ a_{i-1} \\ u+kv \\ a_{i+1} \\ \cdot \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \cdot \\ a_{i-1} \\ u \\ a_{i+1} \\ \cdot \\ a_n \end{pmatrix} + k\det\begin{pmatrix} a_1 \\ \cdot \\ a_{i-1} \\ v \\ a_{i+1} \\ \cdot \\ a_n \end{pmatrix}$$

By induction on n. If $n = 1$ or $r = 1$, trivial ?. For $n \geq 2$, $r > 1$,

$$\det(A) \overset{?}{=} \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\cdot\det(\tilde{A}_{1j}) \overset{?}{=} \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\cdot\det(\tilde{B}_{1j}+k\tilde{C}_{1j})$$

$$= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\cdot\det(\tilde{B}_{1j}) + k\sum_{j=1}^{n}(-1)^{1+j}a_{1j}\cdot\det(\tilde{B}_{1j})$$

$$\overset{?}{=} \det(B) + \det(C)$$

Determinant of Matrices with a Row of Zeros

Corollary

If $A \in M_{n\times n}(F)$ has a row consisting entirely of zeros, then $\det(A) = 0$.

$$\det(A) = \det\begin{pmatrix} a_1 \\ \cdot \\ a_{i-1} \\ 0 \\ a_{i+1} \\ \cdot \\ a_n \end{pmatrix} \stackrel{?}{=} \det\begin{pmatrix} a_1 \\ \cdot \\ a_{i-1} \\ 0 \\ a_{i+1} \\ \cdot \\ a_n \end{pmatrix} + k\det\begin{pmatrix} a_1 \\ \cdot \\ a_{i-1} \\ 0 \\ a_{i+1} \\ \cdot \\ a_n \end{pmatrix}$$

$$= \det(A) + k\det(A), \quad \forall k \in F,$$

$$\stackrel{?}{\Longrightarrow} \quad \det(A) = 0.$$

Determinant and Cofactor Expansions

Lemma

Let $B \in M_{n\times n}(F)$ with $n \geq 2$. If row i of B equals e_k for some k, $1 \leq k \leq n$, then $\det(B) = (-1)^{i+k}\det(\tilde{B}_{ik})$.

By induction on n. If $n = 1, 2$ or $i = 1$, trivial ?. For $n \geq 3$, $i > 1$,

$$\det(B) \stackrel{?}{=} \sum_{j=1}^{n}(-1)^{1+j} b_{1j} \cdot \det(\tilde{B}_{1j})$$

$$\stackrel{?}{=} \sum_{j<k}(-1)^{1+j} b_{1j} \cdot \det(\tilde{B}_{1j}) + \sum_{j>k}(-1)^{1+j} b_{1j} \cdot \det(\tilde{B}_{1j})$$

$$\stackrel{?}{=} \sum_{j<k}(-1)^{1+j} b_{1j} \cdot \left[(-1)^{(i-1)+(k-1)}\det(C_{1j})\right]$$

$$+ \sum_{j>k}(-1)^{1+j} b_{1j} \cdot \left[(-1)^{(i-1)+(k)}\det(C_{1j})\right]$$

$$\stackrel{?}{=} (-1)^{i+k}\left[\sum_{j<k}(-1)^{1+j} b_{1j} \cdot \det(C_{1j}) + \sum_{j>k}(-1)^{1+(j-1)} b_{1j} \cdot \det(C_{1j})\right]$$

$$\stackrel{?}{=} (-1)^{i+k}\det(\tilde{B}_{ik})$$

Determinant and Cofactor Expansions (cont.)

Theorem (4.4)

The determinant of a square matrix $A = (a_{ij})$ can be evaluated by cofactor expansion along any row i, $1 \leq i \leq n$:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot \det(\tilde{A}_{ij}),$$

For $i = 1$, trivial. For $i > 1$, let row i of A be $a_i = \sum_{j=1}^{n} a_{ij} e_j$, let B_j be the matrix obtained from A by replacing row i of A by e_j.

$$\det(A) \stackrel{?}{=} \sum_{j=1}^{n} a_{ij} \det(B_j) \stackrel{?}{=} \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot \det(\tilde{A}_{ij}).$$

Cofactor Expansion: Theorem

Cofactor Expansion

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$
(expansion across row i)

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$
(expansion down column j)

Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Cofactor Expansion: Example

Example

Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ using cofactor expansion down column 3.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1.$$

Cofactor Expansion: Example

Example

Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}$

Solution:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 3 & 5 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 1 \cdot 2\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 14$$

Triangular Matrices

Method of cofactor expansion is not practical for large matrices

Triangular Matrices

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

(upper triangular) (lower triangular)

Theorem

If A is a triangular matrix, then $\det A$ is the product of the main diagonal entries of A.

Triangular Matrices: Example

Example

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{vmatrix} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = -24$$

Determinant: Properties

Corollary

If $A \in M_{n\times n}(F)$ has two identical rows, then $\det(A) = 0$.

By induction on n. If $n = 2$, trivial. For $n \geq 2$, choose i other than r and s.

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot \det(\tilde{A}_{ij}) = 0,$$

since the induction hypothesis implies $\det(\tilde{A}_{ij}) \overset{?}{=} 0$ for $\forall j$.

Determinant and Elementary Row Operations

Theorem (4.5)

If $A \in M_{n\times n}(F)$ and B is obtained from A by interchanging any two rows of A, then $\det(B) = -\det(A)$.

$$0 = \det\begin{pmatrix} a_1 \\ \cdot \\ a_r + a_s \\ \cdot \\ a_r + a_s \\ \cdot \\ a_n \end{pmatrix} \overset{?}{=} \det\begin{pmatrix} a_1 \\ \cdot \\ a_r \\ \cdot \\ a_r + a_s \\ \cdot \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \cdot \\ a_s \\ \cdot \\ a_r + a_s \\ \cdot \\ a_n \end{pmatrix}$$

$$\overset{?}{=} \det\begin{pmatrix} a_1 \\ \cdot \\ a_r \\ \cdot \\ a_r \\ \cdot \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \cdot \\ a_r \\ \cdot \\ a_s \\ \cdot \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \cdot \\ a_s \\ \cdot \\ a_r \\ \cdot \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \cdot \\ a_s \\ \cdot \\ a_s \\ \cdot \\ a_n \end{pmatrix} = \begin{matrix} 0+ \\ \det(A) + \det(B) \\ +0 \end{matrix}$$

Determinant and Elementary Row Operations (cont.)

Theorem (4.6)

If $A \in M_{n\times n}(F)$ and B is obtained from A by adding a multiple of one row of A to another row of A, then $\det(B) = \det(A)$.

Determinant and Elementary Row Operations (cont.)

Theorem (4.7)

If $A \in M_{n\times n}(F)$ has rank less than n, then $\det(A) = 0$.

Evaluating Determinants by Elementary Row Operations

Effect of elementary row operations on the determinant of $A \in M_{n\times n}(F)$:

(a) If B is obtained by interchanging any two rows of A, then $\det(B) = -\det(A)$

(b) If B is obtained by multiplying a row of A by nonzero scalar k, then $\det(B) = k\det(A)$

(c) If B is obtained by adding a multiple of one row of A to another row of A, then $\det(B) = \det(A)$

Theorem still holds if the word *row* is replaced

with ________________.

Evaluating Determinants by Elementary Row Operations

Evaluate the determinant using row operations:

- Transform the matrix into an upper triangular form (row operations of types 1 and 3)
- The determinant of an upper triangular matrix is the product of its diagonal entries

Properties of Determinants: Example

Example

Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 10 \\ 2 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 2 & 7 & 11 \end{vmatrix}$$

$$= 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = \text{\_\_\_\_\_\_\_\_\_\_\_\_} = \text{\_\_\_\_}.$$

Properties of Determinants: Example

Theorem (c) indicates that $\begin{vmatrix} * & * & * \\ -2k & 5k & 4k \\ * & * & * \end{vmatrix} = k \begin{vmatrix} * & * & * \\ -2 & 5 & 4 \\ * & * & * \end{vmatrix}$.

Example

Compute $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

$$= 2(-4) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} = 2(-4) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix} = -40$$

Properties of Determinants: Example

Example

Compute $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix}$ by row reduction and cofac. expansion.

Solution $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & 1 \\ 4 & 7 & 3 \\ 1 & 2 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}$

$$= 2\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 0 & -6 \end{vmatrix} = -2\,(1)\,(-1)\,(-6) = -12.$$

Properties of Determinants: Triangulation

Suppose A has been reduced to

$$U = \begin{bmatrix} \blacksquare & * & * & \cdots & * \\ 0 & \blacksquare & * & \cdots & * \\ 0 & 0 & \blacksquare & \cdots & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

by row replacements and row interchanges, then

$$\det A = \begin{cases} (-1)^r \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

4.3 Properties of Determinants

- Determinants of Products of Matrices
- Determinant of Inverse of Matrix
- Determinant of Transpose of Matrix
- Cramer's Rule and Solution of Linear System

Properties of Determinants: Product

Theorem (4.7)

For $A, B \in M_{n\times n}(F)$, $\det(AB) = \det(A)\cdot\det(B)$.

Properties of Determinants: Inverse

Corollary

$A \in M_{nn}(F)$ is invertible if and only if $\det(A) \neq 0$. If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

Properties of Determinants: Transpose

Theorem (4.8)

For $A \in M_{n \times n}(F)$, $\det(A^t) = \det(A)$.

Properties of Determinants: Cramer's Rule

Theorem (4.9 - Cramer's Rule)

Let $Ax = b$ be a system of n linear equations in n unknowns. If $\det(A) \neq 0$, it has a unique solution $x = (x_1, \cdots, x_n)^t$ with $x_k = \frac{\det(M_k)}{\det(A)}$, where M_k is A with column k replaced by b.