Math 3333 Name: **SOLUTION**

HW 1

Please, write clearly and justify your arguments using the theory covered in class to get credit for your work.

(1) [3Pts] Prove that

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1) \quad n \in \mathbb{N}.$$

Solution: For $n = 1$ it is easily verified.
Now assume it works for $n = k$.
For $n = k + 1$, we have

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \\
&= \frac{1}{6}(k+1)\left(k(2k+1) + 6(k+1)\right) \\
&= \frac{1}{6}(k+1)\left(2k^2 + 7k + 6\right) \\
&= \frac{1}{6}(k+1)(k+2)(2k+3) \\
&= \frac{1}{6}(k+1)\left((k+1)+1\right)\left(2(k+1)+1\right) \quad \textbf{QED.}
\end{aligned}$$

(2) [3Pts] Prove that, for any $n \in \mathbb{N}$, the number $9^n - 4^n$ is divisible by 5.

Solution: For $n = 1$, 9-4 =5, hence it is true.
Now assume it is true for $n = k$; that is, there is an $m \in \mathbb{N}$ such that $9^k - 4^k = 5m$.
So, for $n = k + 1$ we get

$$\begin{aligned}
9^{k+1} - 4^{k+1} &= 9 \cdot 9^k - 4 \cdot 4^k \\
&= 9(9^k - 4^k) + 9 \cdot 4^k - 4 \cdot 4^k \\
&= 9 \cdot 5 \cdot m + 5 \cdot 4^k \\
&= 5(9 \cdot m + 4^k) \quad \textbf{QED}.
\end{aligned}$$

(3) [3Pts] Prove that, for any $n \geq 4$ the following inequality holds

$$n^2 \leq 2^n.$$

Solution: It satisfies for $n = 4$ since $4^2 = 2^4 = 16$.
Now assume it is true for $n = k > 4$. That is, $k^2 \leq 2^k$.
Then for $n = k + 1$ we have

$$(k+1)^2 = k^2 + 2k + 1 \leq 2^k + 2k + 1.$$

We observe [1] that, for $k > 4$, $2k + 1 < 2k + k < 3k < k^2$. Hence, going back to the prior inequality and using the inductive step we have

$$(k+1)^2 = k^2 + 2k + 1 \leq k^2 + k^2 \leq 2^k + 2^k = 2^{k+1} \quad \textbf{QED}.$$

(4) [4 Pts]
Let R be the relation on $\mathbb{Z}$ defined as follows:

For $a, b \in \mathbb{Z}$, aRb if and only if a is a multiple of b

(a) Is R reflexive?
(b) If R symmetric?
(c) Is R transitive?
(d) Is R and equivalence relation?
For each question, prove it or disprove it using a counterexample.

Solution:
(a) R is reflexive on $\mathbb{Z}$ since, for each $a \in Z$, a is a multiple of itself, that is, $a = 1a$.
(b) R is not symmetric. Consider the counterexample where $a = 2$ and $b = 1$. In this case $a = 2b$ but there integer m such that $b = ma$.
(c) R is transitive. In fact, assume that aRb and bRc. Then there exist integers p and q such that $a = bp$ and $b = cq$.
Using the second equation to make a substitution in the first equation, we see that $a = c(pq)$. Since $pq \in \mathbb{Z}$, we have shown that a is a multiple of c and hence aRc. Therefore, R is a transitive relation
(d) The relation R is reflexive and transitive on $\mathbb{Z}$ but it is not symmetric, hence it is not an equivalence relation on Z.

[1] Alternative argument: We can show that $2k + 1 < k^2$ for $k > 0$ by observing that the quadratic equation $k^2 - 2k - 1$ has roots at $1 \pm \sqrt{2}$ hence its positive for $k > 1 + \sqrt{2}$.