HW 2

Please, write clearly and justify your arguments using the theory covered in class to get credit for your work.

(1) [3Pts] Let S, T be nonempty subsets of $\mathbb{R}$ and suppose that $S \subset T$. Prove that

$$\inf T \leq \inf S \leq \sup S \leq \sup T$$

Proof. *By definition, for every* $t \in T$, *it is* $\inf T \leq t$.

Since $S \subset T$, *then for every* $s \in S$, *it is* $\inf T \leq s$. *This shows that* $\inf T$ *is a lower bond of* S, *hence* $\inf T \leq \inf S$.

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For every $t \in T$, *by definition it is* $t \leq \sup T$. *Since* $S \subset T$ *then* $s \leq \sup T$ *for all* $S \in S$. *This shows that* $\sup T$ *is an upper bound of* S, *hence* $\sup S \leq \sup T$.

Combining these observations, we conclude that

$$\inf T \leq \inf S \leq \sup S \leq \sup T.$$

(2) [3Pts] Let S be a nonempty and bounded subset of $\mathbb{R}$. Prove that $M = \sup S$ is unique.

Proof. *Since the set is nonempty and bounded, it has an upper bound and a supremum* M. *Suppose that there exists another number* $M_1 = \sup S$ *with* $M_1 \neq M$. *Then either* $M_1 > M$ *or* $M_1 < M$. *If* $M_1 > M$ *then* M_1 *would not be* $\sup S$ *since it could not be the least upper bound of* S. *Similarly, if* $M_1 < M$ *then* M *would not be* $\sup S$ *since it could not be the least upper bound of* S. *Thus it must be* $M = M_1$

(3) [3Pts] Let $S = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$. Prove that $\sup S = 1$ and find the accumulation points of S is any. Justify your answer.

Proof. $1 - \frac{1}{n} \leq 1$, *for all* n, *hence* 1 *is an upper bound of* S. *To show that 1 is the least upper bound, observe that, if* $M = 1 - \epsilon$, *for some* $\epsilon > 0$, *by the Archimedean property there exists some* $n \in \mathbb{N}$ *such that* $1 - \frac{1}{n} > 1 - \epsilon$, *so that* M *cannot be an upper bound. Hence* $\sup S = 1$.

1 is an accumulation point of S since, for any interval of the form $(1-\epsilon, 1+\epsilon)$, with $\epsilon > 0$, the Archimedean property implies that there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ so that $1+\epsilon > 1-\frac{1}{n} > 1-\epsilon$. S has no other accumulation points. For any point $x_n = 1-\frac{1}{n}$, the distance to the closest point is $\frac{1}{n(n+1)}$, so that the deleted neighborhood $N(x_n, r_n)$ with $r_n < \frac{1}{n(n+1)}$ has empty intersection with the set S.

(4) [3Pts] Let $X \in \mathbb{R}$ be nonempty and f, g be bounded functions defined on X. Prove that

$$\sup\{f(x)+g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.$$

Give examples to show that the inequality can be either an equality or a strict inequality.

Proof. *For any $x \in X$, $f(x) \leq \sup\{f(y) : y \in X\}$ and $g(x) \leq \sup\{g(y) : y \in X\}$. Hence for any $x \in X$,*

$$f(x)+g(x) \leq \sup\{f(y) : y \in X\} + g(x) \leq \sup\{g(y) : y \in X\}.$$

This shows that the right hand side is an upper bound of the set $\{f(x)+g(x) : x \in X\}$. Hence

$$\sup\{f(x)+g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}.$$

Example (equality). Set $f(x) = x$, $g(x) = 1$, for $x \in [0,1]$ Then

$$\begin{aligned} 2 &= \sup\{f(x)+g(x) : x \in [0,1]\} \\ &= \sup\{f(x) : x \in [0,1]\} + \sup\{g(x) : x \in [0,1]\} \\ &= 1+1. \end{aligned}$$

Example (inequality). Set $f(x) = x$, $g(x) = -x$, for $x \in [0,1]$ Then

$$\begin{aligned} 0 &= \sup\{f(x)+g(x) : x \in [0,1]\} \\ &= \sup\{f(x) : x \in [0,1]\} + \sup\{g(x) : x \in [0,1]\} \\ &= 1+0. \end{aligned}$$

(5) [3Pts] Let $S \subset \mathbb{R}$ be nonempty. Show that S is bounded if and only if there exists a closed bounded interval I such that $S \subset I$.

Proof. *Since the set is nonempty and bounded, it has upper and lower bounds, a supremum M and an infimun m. It follows that, for any $x \in S$, $x \leq M$ and $x \geq m$. Thus S is contained in the interval $I = [m, M]$.*

Conversely, suppose that $S \subset I = [a, b]$, where $a, b \in \mathbb{R}$. It follows that $a \leq \inf S$ and $\sup S \leq b$. Hence S is bounded.