# HW 7

Please, write clearly and justify all your statements using the material covered in class to get credit for your work.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that $f$ is not continuous at $x = 0$.

*Let* $x_n = 2/(n\pi)$. *Clearly* $\lim_n x_n = 0$. *However* $\sin x_n = \sin n\pi$ *and* $\lim_n \sin n\pi = DNE$. *This shows that* $\lim f(x_x) \neq f(0)$, *hence* $f$ *is not continuous at* $x = 0$.

(2) Let

$$f(x) = \begin{cases} \frac{x^2+4x-21}{x-3} & \text{if } x \neq 3 \\ a & \text{if } x = 3. \end{cases}$$

Define $a$ so that $f$ will be continuous at $x = 3$.

*Note that* $\lim_{x\to 3} \frac{x^2+4x-21}{x-3} = \lim_{x\to 3} \frac{(x+7)(x-3)}{x-3} = 10$. *If we set* $f(3) = 10$, *then* $\lim_{x\to 2} f(x) = 10 = f(3)$ *and the function is continuous at* $x = 3$.

(3) Determine a condition (a bound independent on $x$) on $|x-1|$ such that
(a) $|x^2 - 1| < 1/2$.
(b) $|x^2 - 1| < 0.01$.

*Write* $|x^2 - 1| = |x-1||x+1|$. *If* $|x-1| < 1$, *then* $|x| < 2$ *and* $|x^2 - 1| = |x+1||x-1| < (|x|+1)|x-1| < 3|x-1|$.

*Hence, to have* $|x^2 - 1| < \epsilon$, *it is sufficient to require that* $3|x-1| < \epsilon$ *and* $|x-1| < 1$. *Let* $\delta = \min(1, \epsilon/3)$. *Then, if* $|x-1| < \delta$ *it follows that* $|x^2 - 1| < \epsilon$.

*Thus, for part (a), we can choose* $|x-1| < 1/6$; *for part (b), we can choose* $|x-1| < \frac{1}{3} \cdot 0.01$.

(4) Let $f : D \to \mathbb{R}$ and $c$ be an accumulation point of $D$. Suppose that $\lim_{x\to c} f(x) = L$.

(a) Prove that $\lim_{x\to c} |f(c)| = |L|$.

**Proof.** *By the definition, given $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x - c| < \delta$ and $x \in D$, then $|f(x) - L| < \epsilon$. The proof then follows by the inequality*

$$||f(x)| - |L|| = ||f(x) - L + L| - |L|| \leq |f(x) - L|.$$

(b) If $f(x) \geq 0$ for all $x \in D$, prove that $\lim_{x\to c} \sqrt{f(x)} = \sqrt{L}$.

**Proof.** *We consider first the case where $L > 0$. In this case, we have that given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - c| < \delta$ and $x \in D$, then $|f(x)| < \epsilon^2$. This implies that $\sqrt{f(x)} < \epsilon$, hence* $\lim_{x\to c} \sqrt{f(x)} = 0$.

*Let us consider now the case where $L > 0$. Given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - c| < \delta$ and $x \in D$, then $|f(x) - L| < \epsilon/\sqrt{L}$. This implies that*

$$|\sqrt{f(x)} - \sqrt{L}| = \frac{|f(x) - L|}{\sqrt{f(x)} + \sqrt{L}} \leq \frac{|f(x) - L|}{\sqrt{L}} < \epsilon.$$

*Hence* $\lim_{x\to c} \sqrt{f(x)} = \sqrt{L}$.