Math 3333 Name: SOLUTION

Quiz 1

Please, write clearly, justify all your steps and use proper notation to receive credit for your work.

(1) Use induction to prove that, for any $n \in \mathbb{N}$, the number $5^n - 3^n$ is divisible by 2.

Solution: We use an argument by induction.
For $n = 1$, 5-3 =3, hence it is true.
Now assume it is true for $n = k$; that is, there is an $m \in \mathbb{N}$ such that $5^k - 3^k = 2m$.
So, for $n = k + 1$ we get

$$\begin{aligned} 5^{k+1} - 3^{k+1} &= 5 \cdot 5^k - 3 \cdot 3^k \\ &= (3+2)5^k - 3 \cdot 3^k \\ &= 3(5^k - 3^k) + 2 \cdot 5^k \\ &= 3 \cdot 2m + 2 \cdot 5^k \\ &= 2(3m + 5^k) \quad \textbf{QED}. \end{aligned}$$

(2) Let R be the relation on $\mathbb{Z}$ defined as follows:

For $a, b \in \mathbb{Z}$, aRb if and only if a is a multiple of b

(a) Is R reflexive?
(b) If R symmetric?
(c) Is R transitive?
(d) Is R and equivalence relation?
For each question, prove it or disprove it using a counterexample.

Solution:
(a) R is reflexive on $\mathbb{Z}$ since, for each $a \in Z$, a is a multiple of itself, that is, $a = 1a$.
(b) R is not symmetric. Consider the counterexample where $a = 2$ and $b = 1$. In this case $a = 2b$ but there integer m such that $b = ma$.
(c) R is transitive. In fact, assume that aRb and bRc. Then there exist integers p and q such that $a = bp$ and $b = cq$.
Using the second equation to make a substitution in the first equation, we see that $a = c(pq)$. Since $pq \in \mathbb{Z}$, we have shown that a is a multiple of c and hence aRc. Therefore, R is a transitive relation
(d) The relation R is reflexive and transitive on $\mathbb{Z}$ but it is not symmetric, hence it is not an equivalence relation on Z.