Name: SOLUTION

MATH 4377/6308 - Advanced linear algebra I - Summer 2024

Quiz 3

(1)[5Pts] The vectors $u_1 = (0, 0, 1)$, $u_2 = (1, 1, 1)$, $u_3 = (0, 1, 1)$ form a basis for $\mathbb{R}^3$. Find a unique representation of an arbitrary vector $(a, b, c) \in \mathbb{R}^3$ as a linear combination of u_1, u_2, u_3.

We solve the linear system

$$x_1 u_1 + x_2 u_2 + x_3 u_3 = (a, b, c), \quad a, b, c \in \mathbb{R}$$

We write the augmented matrix and next reduce the matrix in row-echelon form:

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & a \\ 1 & 1 & 1 & b \\ 0 & 1 & 1 & c \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & 1 & c \\ 0 & 0 & 1 & a \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & b-a \\ 0 & 1 & 0 & c-a \\ 0 & 0 & 1 & a \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & b-c \\ 0 & 1 & 0 & c-a \\ 0 & 0 & 1 & a \end{array}\right)$$

Thus: $x_3 = a, x_2 = c - a, x_3 = b - c$.

(2)[5Pts] Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T(a_1, a_2, a_3) = (a_1 + 2a_2 - a_3, 2a_1 - a_3, 4a_2 + a_3)$$

(a) Find bases for the null space and the range of T.
(b) Find nullity and rank of T.
(a) The null space is determined by the equations

$$a_1 + 2a_2 - a_3 = 0, 2a_1 - a_3 = 0, 4a_2 + a_3 = 0$$

Hence

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 4 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 4 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

The nullspace has only the solution $\{(0, 0, 0)\}$.
By the dimension theorem, the range has dimension 3. WE do not need to do any additional calculation since we can take any basis of $\mathbb{R}^3$, *for instance* $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.
(b) It follows from part (a) that nullity$(T) = 0$ *and rank*$(T) = 3$.

Modified version of problem (2)

(3) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T(a_1, a_2, a_3) = (a_1 + 2a_2 - a_3, 2a_1 - a_3, 4a_2 - a_3)$$

(a) Find bases for the null space and the range of T.
(b) Find nullity and rank of T.
(a) The null space is determined by the equations

$$a_1 + 2a_2 - a_3 = 0, 2a_1 - a_3 = 0, 4a_2 - a_3 = 0$$

Hence

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 4 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 4 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The solution is: $x_3 = 4x_2, x_1 = -6x_2$*, where* x_2 *is a free variable. Hence the null space has dimension 1 and a basis for the null space is*

$$B = \{(-6, 1, -4)\}$$

The range is determined by the equations

$$a_1 + 2a_2 - a_3 = x_1, 2a_1 - a_3 = x_2, 4a_2 - a_3 = x_3$$

This gives the augmented matrix

$$\left(\begin{array}{ccc|l} 1 & 2 & -1 & x_1 \\ 2 & 0 & -3 & x_2 \\ 0 & 4 & -1 & x_3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|l} 1 & 2 & -1 & x_1 \\ 0 & -4 & 1 & x_2 - 2x_1 \\ 0 & 4 & -1 & x_3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|l} 1 & 2 & -1 & x_1 \\ 0 & -4 & -1 & x_2 - 2x_1 \\ 0 & 0 & 0 & x_3 + x_2 - 2x_1 \end{array}\right)$$

Hence the range of T *satisfies the condition* $x_3 + x_2 - 2x_1 = 0$*. A basis for the range is*

$$D = \{(1, 1, 1), (0, 1, -1)\}$$

(b) It follows from part (a) that nullity$(T) = 1$ *and rank*$(T) = 2$*.*