Name: SOLUTION

MATH 4377/6308 - Advanced linear algebra I - Summer 2024

Quiz 4

Exercises:

(1) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$T(a_1, a_2) = (a_1 + a_2, a_1 - a_2, 2a_2 - a_1).$$

Write $[T]_\beta^{\tilde{\gamma}}$ with $\beta = \{(1,0),(0,1)\}$ and $\tilde{\gamma} = \{(1,2,0),(1,1,0),(1,0,1)\}$.

Let $v_1 = (1,0), v_2 = (0,1)$. *then*

$$T(v_1) = (1,1,-1) = -(1,2,0) + 3(1,1,0) - (1,0,1), \to [T(v_1)]_{\tilde{\gamma}} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$$

$$T(v_2) = (1,-1,2) = 0(1,2,0) - (1,1,0) + 2(1,0,1), \to [T(v_2)]_{\tilde{\gamma}} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

Hence $[T]_\beta^{\tilde{\gamma}} = \begin{pmatrix} -1 & 0 \\ 3 & -1 \\ -1 & 2 \end{pmatrix}$

(2) Let $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ and $U : P_1(\mathbb{R}) \to \mathbb{R}^2$ be the linear transformations defined by

$$T(p(x)) = p'(x) + 2p(x), \quad U(a + bx) = (a + b, a)$$

Let β and γ be the standard ordered bases of $P_1(\mathbb{R})$ and $\mathbb{R}^2$), respectively. Find $[T]_\beta$, $[U]_\beta^\gamma$ and $[U \circ T]_\beta^\gamma$.

The standard ordered bases are $\beta = \{1, x\}$, $\gamma = \{(1,0),(0,1)\}$. *We have*

$$T(1) = 2 = 2(1) + 0(x) \to [T(v_1)]_\beta = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad T(x) = 1 + 2x = 1(1) + 2(x) \to [T(v_2)]_\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Hence $[T]_\beta = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

Similarly, we have

$$U(1) = (1,1) = 1(1,0) + 1(0,1) \to [U(v_1)]_\beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad U(x) = (1,0) = 1(1,0) + 0(0,1) \to [U(v_2)]_\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Hence $[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

It follows that $[U \circ T]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$.