Name: SOLUTION

MATH 4377/6308 - Advanced linear algebra I - Summer 2024

Quiz 6

(1) Prove that if $A, B \in M^{n,n}(F)$ are similar, then $\det(A) = \det(B)$.

Let A, B be similar matrices, that is, there exists an invertible matrix Q such that

$$B = Q^{-1}AQ$$

It follows that

$$\det(B) = \det(Q^{-1}AQ) = \det(Q^{-1})\det(A)\det(Q) = (\det(Q))^{-1}\det(A)\det(Q) = \det(A)$$

(2) Mark each statement True or False. If true, cite appropriate facts or theorems. If false, give a counterexample that shows why the statement is not true.

a) Every 2×2 matrix has 2 distinct eigenvalues.

b) The sum of two eigenvalues of a matrix A is also an eigenvalue of A.

c) The sum of two eigenvectors of a matrix A is always an eigenvector of A

d) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.

(a) False. For instance, the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ acting on $\mathbb{R}^2$ has only the eigenvalue $\lambda = 1$.

(b) False. For instance, the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ has two eigenvalues 3 and 2 but the sum 5 is not an eigenvalue of the same matrix.

(c) False. For instance, the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ has eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ but their sum $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not an eigenvector of the same matrix.

(d) False. For instance, the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ has linearly independent eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ corresponding to the same eigenvalue.

(3) The matrix $A = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}$ has eigenvalues $\lambda_1 = -4, \lambda_2 = 1$. Find a matrix Q such that

$$Q^{-1}AQ = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}$$

For $\lambda_1 = -4$, we have $\det(A + 4I)x = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ so that we have the eigenvector $u_1 = \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix}$.

For $\lambda_2 = 1$, we have $\det(A + 4I)x = \begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ so that we have the eigenvector $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

NOTE: This implies that, choosing $Q = \begin{pmatrix} 1 & 1 \\ -\frac{3}{2} & 1 \end{pmatrix}$, then

$$Q^{-1}AQ = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}$$