Name: SOLUTION

MATH 4377/6308 - Advanced linear algebra I - Summer 2024

Test 1

You must show your work and justify your steps to receive credit.

Problems:

(1) [4Pts] Let $x, y \in \mathbb{Z}$. Prove that the following relation is a equivalence relations or show that it is not:

$$x \sim y \text{ if and only if } x - y \text{ is a multiple of } 3$$

SOLUTION:

Yes. (i) $x - x = 0$ is a multiple of 3 with multiplicative constant 0;
(ii) if $x - y = 5m$, then $y - x = 5(-m)$, which is also a multiple of 5;
(iii) if $x - y = 5m$ and if $y - z = 5n$, then $x - z = 5(m + n)$

(2) [6Pts] For each one of the statements below, construct an example or explain why such example does not exist.

a) A subspace of $\mathbb{R}^3$ of dimension 1.

b) A non-trivial subset of $\mathbb{R}^3$ that is not a subspace (non-trivial means it should not be the empty set).

c) A linearly independent set of 2 vectors in $\mathbb{R}^3$.

d) A linearly independent set of 4 vectors in $\mathbb{R}^3$.

e) A spanning set of $\mathbb{R}^3$ that is not a basis.

f) An infinite dimensional vector space (that is, a vector spaces with no basis of finite cardinality).

SOLUTION.

(a) The set $\{(a, a, a) : a \in \mathbb{R}\}$
(b) The set $\{(1, 1, 1)\}$.
(c) The set $\{(1, 0, 0), (0, 1, 0)\}$
(d) Not possible since a basis in $\mathbb{R}^3$ has dimension 3 and any set of more than 3 elements is linearly dependent.
(e) The set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (2, 0, 0)\}$.
(f) The space of continuous function with domain in $\mathbb{R}$.

(3) [6Pts] Determine if the following subsets of the vector space of 2×2 matrices with real entries are subspaces:

a) $S = \left\{ \begin{bmatrix} a & b \\ 0 & \frac{1}{a} \end{bmatrix} : a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R} \right\}$

b) $R = \left\{ \begin{bmatrix} a & b - a \\ a - b & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$

SOLUTION.

(a) This is not a subspace. Note that:

$$\begin{bmatrix} a & b \\ 0 & \frac{1}{a} \end{bmatrix} + \begin{bmatrix} a' & b' \\ 0 & \frac{1}{a}' \end{bmatrix} = \begin{bmatrix} a + a' & b + b' \\ 0 & \frac{1}{a} + \frac{1}{a}' \end{bmatrix} \neq \begin{bmatrix} a + a' & (a + a')(b + b') \\ 0 & \frac{1}{a + a'} \end{bmatrix}$$

This shows that the matrix resulting from adding two matrices in S does not belong to S.

(b) This is a subspace.

(i) For any $\alpha \in \mathbb{R}$, $\alpha \begin{bmatrix} a & b-a \\ a-b & b \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha(b-a) \\ \alpha(a-b) & \alpha b \end{bmatrix} = \begin{bmatrix} \tilde{a} & \tilde{b}-\tilde{a} \\ \tilde{a}-\tilde{b} & \tilde{b} \end{bmatrix}$, *for $\tilde{a}, \tilde{b} \in \mathbb{R}$.*

(ii) $\begin{bmatrix} a & b-a \\ a-b & b \end{bmatrix} + \begin{bmatrix} a' & b'-a' \\ a'=b' & b' \end{bmatrix} = \begin{bmatrix} a+a' & (b+b')-(a+a') \\ (a+a'-(b+b')) & b+b') \end{bmatrix} = \begin{bmatrix} \tilde{a} & \tilde{b}-\tilde{a} \\ \tilde{a}-\tilde{b} & \tilde{b} \end{bmatrix}$, *for $\tilde{a}, \tilde{b} \in \mathbb{R}$.*

(4) [6Pts] Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be given by

$$T(a_1, a_2, a_3, a_4) = (a_1 + a_2 + a_3, -a_1 + 2a_2 + a_4, 3a_2 + a_3 + a_4)$$

(a) Find the nullity and the rank of T.
(b) Find bases for the null space and the range of T.

SOLUTION.

(a) The null space is determined by the equations

$$a_1 + a_2 + a_3 = 0, -a_1 + 2a_2 + a_4 = 0, 3a_2 + a_3 + a_4 = 0$$

This gives

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution is: $a_3 = -a_1 - a_2$, $a_4 = a_1 - 2a_2$, with $a_1, a_2 \in \mathbb{R}$. This shows that nullity $= 2$. By the dimension theorem, we also derive that rank$(T) = 2$

(b) From the equations of the nullspace, choosing the free parameters as $(a_1, a_2) = (1, 0)$ and $(a_1, a_2) = (0, a)$ we have that a basis for the null space is

$$B = \{(1, 0, -1, 1), (0, 1, -1, -2)\}$$

The range is determined by the equations

$$a_1 + a_2 + a_3 = x_1, -a_1 + 2a_2 + a_4 = x_2, 3a_2 + a_3 + a_4 = x_3$$

This gives

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & x_1 \\ -1 & 2 & 0 & 1 & x_2 \\ 0 & 3 & 1 & 1 & x_3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & x_1 \\ 0 & 3 & 1 & 1 & x_1 + x_2 \\ 0 & 3 & 1 & 1 & x_3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & x_1 \\ 0 & 3 & 1 & 1 & x_1 + x_2 \\ 0 & 0 & 0 & 0 & x_3 - x_1 - x_2 \end{array}\right)$$

Hence the range of T satisfies the condition $x_3 - x_1 - x_2 = 0$ A basis for the range is

$$D = \{(1, 0, 1), (0, 1, 1)\}.$$

(5) [5Pts] Let $L = \{\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}\}$. Let $G = \{\begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\}$. You can assume without proof that G spans $M^{2\times 2}$. Find a subset $H \subset G$ such that $H \cup L$ spans $M^{2\times 2}$. You need to justify that the set you build spans $M^{2\times 2}$.

Note that the first two matrices of G are already in the span of L since

$$\begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 & 0 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$$

Because G spans $M^{2\times 2}$, and the first two matrices are already in the span of L, then we need to choose the other two matrices of G to form the set H, that is $H = \{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\}$ so that $H \cup L$ spans $M^{2\times 2}$.

(6) [5Pts] Let V be a finite-dimensional vector space and V_0 be a proper subspace of V (where proper means that $V_0 \neq V$). Prove that $\dim V_0 < \dim V$.
SOLUTION.

Suppose that $\dim V_0 = m$ and that $B = \{v_1, \ldots, v_m\}$ be a basis of V_0. Since $V_0 \subset V$, we also have that $S \subset V$.
Since $V_0 \neq V$ and $V_0 \subset V$, it follows that there is a vector $u \in V$ that is not in span$B$. Hence the set $E = \{u, v_1, \ldots, v_m\} \subset V$ is linearly independent. It follows that $\dim V \geq m + 1$.