Name: SOLUTION

MATH 4377/6308 - Advanced linear algebra I - Summer 2024

Homework 1

Exercises:

1. Let $A = \{1, 2, 5\}$, $B = \{4, 5\}$, $C = \{4, 6\}$. Explicitly write down the sets:

$$A \cup B,\ A \cap (B \cup C),\ B \cap (A \backslash B), A \times C.$$

SOLUTION:
$A \cup B = \{1, 2, 4, 5\}$, $A \cap (B \cup C) = \{5\}$, $B \cap (A \backslash B) = \emptyset$, $A \times C = \{(1, 4), (1, 6), (2, 4), (2, 6), (5, 4), (5, 6)\}$

2. Let $x, y \in \mathbb{Z}$. Prove if the following relations are equivalence relations or not:

   a) $x \sim y$ if and only if $x - y < 10$.
   b) $x \sim y$ if and only if $x \cdot y \geq 0$.
   c) $x \sim y$ if and only if $x - y$ is even.

SOLUTION:
(a) No. Symmetry fails
(b) No. Transitivity fails. There are $x, y, z \in \mathbb{Z}$ s.t. $x \cdot y \geq 0$ and $y \cdot z \geq 0$, but $x \cdot z < 0$
(c) Yes. (i) $x - x$ is even; (ii) if $x - y = 2m$, then $y - x = 2(-m)$; (iii) if $x - y = 2m$ and if $y - z = 2n$, then $x - z = 2(m + n)$

3. Give an example of a set A and a relation on A which is reflexive and transitive but not symmetric.

SOLUTION:
$x, y \in \mathbb{Z}$, with $x \sim y$ if and only if $x \leq y$
In this case, $x \leq x$ holds, $x \leq y$ and $y \leq z$ implies $x \leq z$. However $x \leq y$ does not imply $y \leq x$

4. Let $f : \{0, 1, 2, 3, 4\} \to \mathbb{N}$, $n \to n^3 + n$.

   a) Find domain, codomain, and range of f.
   b) Is f one-to-one?
   c) Is f onto?

SOLUTION:
(a) domain: $\{0, 1, 2, 3, 4\}$, codomain: $\mathbb{N}$, range: $\{0, 2, 2^2 + 2, 3^3 + 3+, 4^4 + 4\}$
(b) yes. $n^3 + n = m^3 + m$ implies $n = m$
(c) no. There is no n in the domain of f such that $f(n) = 1$

5. Let $f : [0, 2\pi] \to [-1, 1]$ be defined by $f(x) = \sin(x)$.

   a) Is f one-to-one? Is f onto?
   b) Find an interval S, such that $f|_S$ is both one-to-one and onto.

SOLUTION:
(a) f is not 1-1 since $f(0) = f(\pi)$. f is onto.
(b) f one-to-one and onto in the interval $[\pi/2, 3\pi/2]$

6. Let $z = 1 + i2$, $w = 1 - i3$. Write: $\overline{z}$, $z + w$, zw, $\dfrac{1}{w}$ in the form $a + ib$. Finally write $|z|$.

   SOLUTION:
   $\overline{z} = 1 - i2$, $z + w = 2 - i$, $zw = 7 - 3$, $\dfrac{1}{w} = \dfrac{1}{10}(1 + 3i)$, $|z|^2 = 5$, $|z| = \sqrt{5}$

7. Let $x, y \in \mathbb{Z}$. Let $x \sim y$ if and only if $y + 4x$ is an integer multiple of 5. Prove that $\sim$ is an equivalence relation.

   SOLUTION:
   (i) $x + 4x = 5x$ is an integer multiple of 5; (ii) if $y + 4x = 5m$, then $y = 5m - 4x$, hence $x + 4y = x + 20m - 16x = 20m - 15x = 5(4m - 3x)$ which is also a multiple of 5; (iii) if $y + 4x = 5m$ and if $z + 4y = 5n$, then (using these two equations to express z and $4x$) $z + 4x = (5n - 4y) + (5m - y) = 5(n + m) - 5y = 5(n + m - y)$, which is a multiple of 5.