Name:

MATH 4377/6308 - Advanced linear algebra I - Summer 2024

Homework 2

Exercises:

1. Mark each statement True or False. Justify each answer.

   a) A subset H of a vector space V is a subspace of V if the zero vector is in H.
   b) A subspace is also a vector space.
   c) If u is a vector in a vector space V , then $(-1)u$ is the same as the negative of u.
   d) A vector space is also a subspace.
   e) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.
   f) If f is a function in the vector space V of all real-valued functions on $\mathbb{R}$ and if $f(t) = 0$ for some t, then f is the zero vector in V .
   g) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.
   h) Any set containing the zero vector is linearly dependent.
   i) Subsets of linearly dependent sets are linearly dependent.

   (a) False. $\{0, 1\} \subset \mathbb{R}$ *is a subset of* $\mathbb{R}$ *but it is not a subspace of it.*

   (b) True. A subspace is also a vector space in its own right.

   (c) True. If u *is a vector in a vector space* V*, then,* $(-1)u = -u \in V$*.*

   (d) True. A vector space is also a subspace of itself.

   (e) False. $\mathbb{R}^2$ *is not a subset of* $\mathbb{R}^3$*.*

   (f) False. If The function $f(t) = 0$ *for some* t*, then* $g + f \neq g$*, in general, so* f *is not the zero vector in* V*.*

   (g) False. The $S = \{v, 0\}$ *is linear dependent, but* v *is not a linear combination of* 0*. If* S *is a linearly dependent set of two or more vectors, then at least one of the vectors in* S *is a linear combination of other vectors in* S*.*

   (h) True. The zero vector is contained in the span of any vector

   (i) False. $\{(1, 0), (0, 1)\}$ *is a linearly independent subset of the linearly dependent set* $\{(1, 0), (0, 1), (1, 1)\}$*.*

2. (2 points) Determine if the following subsets of $\mathbb{R}^3$ are subspaces:

   a) $\{(a, b, c) \in \mathbb{R}^3 : 2a - 3c = 0\}$
   b) $\{(a, b, c) \in \mathbb{R}^3 : a - 2b + c = 1\}$
   c) $\{(a, b, c) \in \mathbb{R}^3 : 2a = c\} = \{(a, b, c) \in \mathbb{R}^3 : 2a - c = 0\}$
   d) $\{(a, b, c) \in \mathbb{R}^3 : 2a = 5c \text{ and } 4b = a + c\} = \{(a, b, c) \in \mathbb{R}^3 : 2a - 5c = 0 \text{ and } 4b - a - c = 0\}$

   (a) This is a subspace. Let $v = (a, b, c)$ *and* α *be a scalar. For* $v' = \alpha v$*, we have* $2\alpha a - 3\alpha c = \alpha(2a - 3c) = 0$*. Also, for* $w = (a, b, c) + (a', b', c') = (a + a', b + b', c + c')$*, we have*

$$2(a + a') - 3(c + c') = 2a - 3c + 2a' - 3c' = 0.$$

(a) This is not a subspace. Let $v = (a, b, c)$ and α be a scalar. For $v' = \alpha v$, we have $\alpha a - 2\alpha b + \alpha c = alpha(a - 2b + c) = \alpha$. This shows the set is not closed under scalar multiplication.

(c) This is a subspace. Argument is very similar to (a). Let $v = (a, b, c)$ and α be a scalar. For $v' = \alpha v$, we have $2\alpha a - \alpha c = \alpha(2a - c) = 0$. Also, for $w = (a, b, c) + (a', b', c') = (a + a', b + b', c + c')$, we have

$$2(a + a') - (c + c') = 2a - c + 2a' - c' = 0.$$

(d) This is a subspace. Argument is very similar to (a). Let $v = (a, b, c)$ and α be a scalar. For $v' = \alpha v$, we have $2\alpha a - 5\alpha c = \alpha(2a - 5c) = 0$; we also have $4\alpha b - \alpha a - \alpha c = \alpha(4b - a - c) = 0$. Also, for $w = (a, b, c) + (a', b', c') = (a + a', b + b', c + c')$, we have

$$2(a + a') - 5(c + c') = 2a - 5c + 2a' - 5c' = 0$$

and

$$4(b + b') - (a + a') - (c + c') = 4b - a - c + 4b' - a' - c' = 0$$

3. Determine if the following subsets of the vector space of 2×2 matrices with real entries are subspaces:

   a) $\left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$

   b) $\left\{ \begin{bmatrix} a & b^2 \\ b & a^2 \end{bmatrix} : a, b \in \mathbb{R} \right\}$

   (a) This is a subspace. Scalar multiplication preserves the properties of the set and the sum of two triangular matrices is a triangular matrix of the same type.

   (b) This is not a subspace. Scalar multiplication does not preserve the properties of the matrix since a negative scalar will change the sign of the (1,2) and (2,2) entries.

4. A real-valued function f defined on the real line is called an *even function* if $f(t) = f(-t)$ for each real number t. Prove that the set of even functions is a subspace with the usual addition and scalar multiplication for functions. (You may assume as true that the set of real-valued functions f defined on the real line is a vector space with the usual addition and scalar multiplication for functions.)

   Clearly multiplication of an even function by a scalar does not affect the event property. Also, if f and g are even functions, then $(f+g)(t) = f(t)+g)(t) = f(-t)+g)(-t) = (f+g)(-t)$, so the sum is also an even function.

5. Suppose $u_1, \ldots, u_p$ and $v_1, \ldots, v_p$ are vectors in a vector space V, and let

$$H = span(u_1, \ldots, u_p), \quad K = span(v_1, \ldots, v_p)$$

   Prove that $H + K = span(u_1, \ldots, u_p, v_1, \ldots, v_p)$

   Let $x = h + k \in H + K$. Since $h \in H$, then there are scalars $a_1, \ldots, a_p$ such that $h = \sum_{i=1}^{p} a_i u_i$; similarly, since $k \in K$, then there are scalars $b_1, \ldots, b_p$ such that $k = \sum_{i=1}^{p} b_i v_i$. It follows *that* $h+k=\sum_{i=1}^{p}(a_iu_i+b_iv_i)\in span(u_1,\ldots,u_p,v_1,\ldots,v_p)$. *This shows that* $H+K\subset span(u_1,\ldots,u_p,v_1,\ldots,v_p)$.

Conversely, $span(u_1,\ldots,u_p,v_1,\ldots,v_p)$ *consists of all* $x=\sum_{i=1}^{p}a_iu_i+\sum_{i=1}^{p}b_iv_i$ *for some scalars* $a_1,\ldots,a_p,b_1,\ldots,b_p$. *This shows that* $span(u_1,\ldots,u_p,v_1,\ldots,v_p)\subset H+K$.