Name: SOLUTION

MATH 4377/6308 - Advanced linear algebra I - Summer 2024

Homework 3

Exercises:

(1) Decide if each of the following statements is True or False. Justify each answer. If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case.

Let $v_1, \ldots, v_p$ be vectors in a non-zero finite-dimensional vector space V, and let $S = \{v_1, \ldots, v_p\}$.

a) The set of all linear combinations of $v_1, \ldots, v_p$ is a vector space.

b) If $\{v_1, \ldots, v_{p-1}\}$ spans V, then S spans V.

c) If $\{v_1, \ldots, v_{p-1}\}$ is linearly independent, then so is S.

d) If S is linearly independent, then S is a basis for V.

e) If S is linearly independent, then $dimV = p$.

f) If $V = spanS$, then some subset of S is a basis for V.

g) If $V = spanS$, then $dimV = p$.

h) If $dimV = p$ and $V = spanS$, then S cannot be linearly dependent.

i) A plane in $\mathbb{R}^3$ is a two-dimensional subspace.

(a) True. The set of all linear combinations of $v_1, \ldots, v_p$ is the span of S and is a vector space as proved in class.
(b) True. If $\{v_1, \ldots, v_{p-1}\}$ spans V, then the span S, which is contains the spans of $\{v_1, \ldots, v_{p-1}\}$ will also span V.
(c) False. If $\{v_1, \ldots, v_{p-1}\}$ is l.i., then $\{v_1, \ldots, v_{p-1}\} \cup \{v_p\}$ does not need to be l.i. This is only true if $v_p \notin span\{v_1, \ldots, v_{p-1}\}$.
(d) False. If S is l.i., then it is a basis of V only if it spans S.
(e) False. If S is linearly independent, then, by the Replacement Theorem $dimV \geq p$. One can construct examples where $dimV > p$.
(f) True. If $V = span\{S\}$, then by a result stated in class there is a subset of S that is a basis of V.
(g) False. If $V = span\{S\}$, then by a result stated in class there is a subset of S that is a basis of V, hence $dimV \leq p$. One can construct examples where $dimV < p$.
(h) True. If $dimV = p$ and $V = spanS$, then, by a consequence of the Replacement theorem, S is a basis for V, therefore, S cannot be linearly dependent.
(i) False. A plane in $\mathbb{R}^3$ is a not a subspace unless it is a plane through the origin, since the 0 vector must be contained in the subspace.

(2) The vectors $u_1 = (1, 1, 1, 1)$, $u_2 = (0, 1, 1, 1)$, $u_3 = (0, 0, 1, 1)$, and $u_4 = (0, 0, 0, 1)$ form a basis for $\mathbb{R}^4$. Find a unique representation of an arbitrary vector $(a, b, c, d) \in \mathbb{R}^4$ as a linear combination of u_1, u_2, u_3, and u_4.

We solve the linear system

$$x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4 = (a, b, c, d), \quad a, b, c, d \in \mathbb{R}$$

We write the augmented matrix and next reduce the matrix in row-echelon form:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 1 & 1 & 0 & 0 & b \\ 1 & 1 & 1 & 0 & c \\ 1 & 1 & 1 & 1 & d \end{array}\right) \to \left(\begin{array}{cccc|r} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b-a \\ 0 & 1 & 1 & 0 & c-a \\ 0 & 1 & 1 & 1 & d-a \end{array}\right) \to \left(\begin{array}{cccc|r} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b-a \\ 0 & 0 & 1 & 0 & c-b \\ 0 & 0 & 1 & 1 & d-b \end{array}\right) \to \left(\begin{array}{cccc|r} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b-a \\ 0 & 0 & 1 & 0 & c-b \\ 0 & 0 & 0 & 1 & d-c \end{array}\right)$$

Thus: $x_1 = a, x_2 = b-a, x_3 = c-b, x_4 = d-c$.

(3) Let $L = \{(1,2,1,3),(0,0,1,1)\}$. Let $G = \{(1,2,-2,0),(1,0,0-1),(0,1,1,1),(1,2,2,4)\}$. You can assume without proof that G spans $\mathbb{R}^4$. Find a subset $H \subset G$ such that $H \cup L$ spans $\mathbb{R}^4$. You need to justify that the set you build spans $\mathbb{R}^4$.

Note that

$$(1,2,2,4) = (1,2,1,3) + (0,0,1,1) \quad \textit{and} \quad (1,2,-2,0) = (1,2,1,3) - 3(0,0,1,1)$$

Hence, if G *spans* $\mathbb{R}^4$*, the other two vectors of* G*, that we denote as* $L = \{(1,0,0-1),(0,1,1,1)\}$ *must not belong to the span of* L*. It follows that* $H \cup L$ *spans* $\mathbb{R}^4$*.*

(4) Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be given by

$$T(a_1,a_2,a_3,a_4,a_5) = (a_1 + 2a_2 - a_3, -a_2 + 3a_3, -a_1 - a_2 - 2a_3)$$

(a) Verify that T is linear.
(b) Find bases for the null space and the range of T.

(a) Direct calculation shows that, if α *is a scalar, then*

$$T(\alpha a_1, \alpha a_2, \alpha a_3, \alpha a_4, \alpha a_5) = \alpha(a_1 + 2a_2 - a_3, -a_2 + 3a_3, -a_1 - a_2 - 2a_3),$$

and that $T(v_1 + v_2) = T(v_1) + T(v_2)$ *for any* $v_1, v_2 \in \mathbb{R}^5$.
(b) The null space is determined by the equations

$$a_1 + 2a_2 - a_3 = 0, -a_2 + 3a_3 = 0, -a_1 - a_2 - 2a_3 = 0$$

Note that this equation are in $\mathbb{R}^5$ *even though the variables* x_4 *and* x_5 *do not appear.*

$$\left(\begin{array}{rrrrr|r} 1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 \\ -1 & -1 & -2 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{rrrrr|r} 1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{rrrrr|r} 1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The solution is: $a_2 = 3a_3$, $a_1 = a_3 - 2a_2 = -5a_3$, *where* a_3, a_4, a_5 *can take any value in* $\mathbb{R}$*. Hence the null space has dimension 3 and a basis for the null space is*

$$B = \{(-5,3,1,0,0),(0,0,0,1,0),(0,0,0,0,1)\}$$

The range is determined by the equations

$$a_1 + 2a_2 - a_3 = x_1, -a_2 + 3a_3 = x_2, -a_1 - a_2 - 2a_3 = x_3$$

This gives the augmented matrix

$$\left(\begin{array}{rrr|r} 1 & 2 & -1 & x_1 \\ 0 & -1 & 3 & x_2 \\ -1 & -1 & -2 & x_3 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & -1 & x_1 \\ 0 & 1 & -3 & -x_2 \\ 0 & 1 & -3 & x_1 + x_3 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & -1 & x_1 \\ 0 & 1 & -3 & -x_2 \\ 0 & 0 & 0 & x_1 + x_2 + x_3 \end{array}\right)$$

Hence the range of T *satisfies the condition* $x_1 + x_2 + x_3 = 0$ *A basis for the range is*

$$D = \{(1,1,0),(1,0,-1)\}$$