Name: SOLUTION

MATH 4377/6308 - Advanced linear algebra I - Summer 2024

Homework 4

Exercises:

(1) Consider the linear transformation:

$$T: P_2(\mathbb{R}) \to P_3(\mathbb{R}), T(p(x)) = 2p'(x) + \int_0^x p(t)dt$$

Prove that T is one-to-one but not onto.

The map T cannot be onto because the dimension of the co-domain is larger than the dimension of the domain

To show T is 1-1, we will show that $N(T) = \{0\}$. Let $p(x) = a_0 + a_1x + a_2x^2 \in N(T)$, Then

$$2p'(x) + \int_0^x p(t)dt = 2a_1 + 4a_2x + a_0x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} = 0$$

The equation above implies that $a_0 = a_1 = a_2 = 0$ so that $N(T) = \{0\}$.

(2) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T(a_1, a_2) = (a_1 + a_2, a_1 - a_2).$$

(a) Write $[T]_\beta^\gamma$ with $\beta = \{(1,0),(0,1)\}$ and $\gamma = \{(1,0),(0,1)\}$.

(b) Write $[T]_\beta^{\tilde{\gamma}}$ with $\beta = \{(1,0),(0,1)\}$ and $\tilde{\gamma} = \{(1,2),(1,1)\}$.

(a) Let $v_1 = (1,0), v_2 = (0,1)$. then

$$T(v_1) = (1,1) = 1(1,0) + 1(0,1), \to [T(v_1)]_\gamma = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T(v_2) = (1,-1) = 1(1,0) - 1(0,1), \to [T(v_2)]_\gamma = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hence $[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(b) Let $v_1 = (1,0), v_2 = (0,1)$. then

$$T(v_1) = (1,1) = 0(1,2) + 1(1,1), \to [T(v_1)]_{\tilde{\gamma}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T(v_2) = (1,-1) = -2(1,2) + 3(1,1), \to [T(v_2)]_{\tilde{\gamma}} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

Hence $[T]_\beta^{\tilde{\gamma}} = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$

(3) Let $T: P_1(\mathbb{R}) \to P_1(\mathbb{R})$ and $U: P_1(\mathbb{R}) \to \mathbb{R}^2$ be the linear transformations defined by

$$T(p(x)) = p'(x) + 2p(x), \quad U(a + bx) = (a + b, a)$$

Let β and γ be the standard ordered bases of $P_1(\mathbb{R})$ and $\mathbb{R}^2$), respectively. Find $[T]_\beta$, $[U]_\beta^\gamma$ and $[U\circ T]_\beta^\gamma$.

The standard ordered bases are $\beta = \{1, x\}$, $\gamma = \{(1,0),(0,1)\}$. We have

$$T(1) = 2 = 2(1) + 0(x) \to [T(v_1)]_\beta = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad T(x) = 1 + 2x = 1(1) + 2(x) \to [T(v_2)]_\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Hence $[T]_\beta = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

Similarly, we have

$$U(1) = (1,1) = 1(1,0)+1(0,1) \to [U(v_1)]_\beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad U(x) = (1,0) = 1(1,0)+0(0,1) \to [U(v_2)]_\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Hence $[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

It follows that $[U\circ T]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$.

(4) For the following pairs of vector spaces V and W, define an explicit isomorphism or explain why no isomorphism exists between such spaces.
(a) $V = \mathbb{R}^2$, $W = M^{1,1}$
(b) $V = \mathbb{R}^4$, $W = M^{2,2}$
(c) $V = \mathbb{R}^4$, $W = P_4(\mathbb{R})$
(d) $V = \mathbb{R}^4$, $W = P_3(\mathbb{R})$
(e) $V = \mathbb{R}^2$, $W = \mathbb{C}$ (space of complex numbers)

(a) Since $\dim(V) = 2$ *and* $\dim(W) = 1$*, there is no isomorphism between V and W.*
(b) For $(a,b,c,d) \in \mathbb{R}^4$, let $T(a,b,c,d) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *This map is linear and invertible.*
(c) Since $\dim(V) = 4$ *and* $\dim(W) = 5$*, there is no isomorphism between V and W.*
(d) For $(a,b,c,d) \in \mathbb{R}^4$, let $T(a,b,c,d) = a + bx + cx^2 + dx^3$. This map is linear and invertible.
(e) For $(a,b) \in \mathbb{R}^2$, let $T(a,b) = a + ib$. This map is linear and invertible.

(5) Let $\beta' = \{(3,1),(2,4)\}$, $\beta = \{(1,1),(1,-1)\}$. Find the change of coordinates matrix $Q = [I_V]_{\beta'}^\beta$.

Let $\beta' = \{(3,1),(2,4)\} := \{v_1, v_2\}$. Hence

$$I(v_1) = (3,1) = a_1(1,1) + a_2(1,-1) = (a_1 + a_2, a_1 - a_2)$$

Solving the system, one finds $a_1 = 2, a_2 = 1$ Similarly we have

$$I(v_2) = (2,4) = b_1(1,1) + b_2(1,-1) = (b_1 + b_2, b_1 - b_2)$$

Solving the system, one finds $b_1 = 3, b_2 = -1$
Thus, we have that $Q = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$