Name: SOLUTION

MATH 4377/6308 - Advanced linear algebra I - Summer 2024

Homework 5

Exercises:

(1) Mark each statement True or False. Justify each answer. (If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case).

a) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then A can be obtained by performing an elementary row operation on B.

b) The rank of a matrix is equal to the number of its nonzero columns.

c) The product of two matrices always has rank equal to the lesser of the ranks of the two matrices.

d) Elementary row operations preserve rank.

e) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.

f) An $n \times n$ matrix having rank n is invertible.

g) Any homogeneous system of linear equations has at least one solution.

h) If the homogeneous system corresponding to a given system of linear equations has a solution, then the given system has a solution.

i) The solution set of any system of m linear equations in n unknowns is a subspace of F_n.

j) If A is an $n \times n$ matrix with rank n, then the reduced row echelon form of A is I_n.

(a) True. Since $B = EA$ with E an elementary matrix, it follows that $A = E^{-1}B$ where the inverse V is also an elementary matrix.

(b) False. For example, the rank of $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ is 1, which is not equal to the number of its nonzero columns.

(c) False. For example, for $A = (10)$ and $B = (01)^t$, both having rank 1, the product $AB = (0)$ has rank 0. What we have is that $rank(AB) \leq \min(rank(A); rank(B))$.

(d) True. From result stated in the book and notes, elementary row operations preserve rank.

(e) True. $rank(A)$ is equal to the dimension of the row space of A and this is equal to the dimension of the column space of A.

(f) True. The liner transformation $L_A : F_n \to F_n$ is invertible if and only if $rank(L_A) = \dim(Fn)$, i.e., $rank(A) = n$. Since A is invertible if and only if L_A is invertible, then A is invertible if and only if $rank(A) = n$.

(g) True. Any homogeneous system of linear equations has zero as a solution.

(h) False. For example, the system that $0x = 1$ has no solution while the corresponding homogeneous system $0x = 0$ has a solution.

(i) False. For example, the solution set of the system $x = 1$ is not a subspace of $F_1 = \mathbb{R}$.

(j) True.

(2) Apply the method of Gaussian elimination to find the solution set of the following system of equations or show that it has no solution. Write the elementary matrices that you apply to transform the matrix

of coefficients.

$$\begin{aligned} 3x+3y+z &= 15 \\ 2x-y+z &= 3 \\ x+4y &= 9 \end{aligned}$$

Row reduce:

$$\left(\begin{array}{ccc|c} 3 & 3 & 1 & 15 \\ 2 & -1 & 1 & 3 \\ 1 & 4 & 0 & 9 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & 0 & 9 \\ 3 & 3 & 1 & 15 \\ 2 & -1 & 1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & 0 & 9 \\ 0 & -9 & 1 & -12 \\ 0 & -9 & 1 & -15 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 4 & 0 & 9 \\ 0 & -9 & 1 & -12 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

Hence: No solution

(3) Determine the values of the parameter k such that the following system of equations has unique solution, no solution or infinitely many solutions.

$$\begin{aligned} ky+kz &= k \\ kx+2y &= 1 \\ -3x+y &= -k \end{aligned}$$

Row reduce: under the assumption $k \neq 0$, we have:

$$\left(\begin{array}{ccc|c} 0 & k & k & k \\ k & 2 & 0 & 1 \\ -3 & 1 & 0 & -k \end{array}\right) \to \left(\begin{array}{ccc|c} -3 & 1 & 0 & -k \\ 0 & k & k & k \\ k & 2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1/3 & 0 & k/3 \\ 0 & 1 & 1 & 1 \\ 0 & 2+k/3 & 0 & 1-k^2/3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & -1/3 & 0 & k/3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2-k/3 & -1-k/3-k^2/3 \end{array}\right)$$

This shows that the system has no solution if $k=-6$.

If $k=0$, the system reduces to $\left(\begin{array}{ccc|c} 0 & 2 & 0 & 1 \\ -3 & 1 & 0 & 0 \end{array}\right)$ which has infinitely many solutions: $y=1/2$, $x=1/6$, $z=$ any number.

If $k \neq 0, \neq -6$, the system had unique solution.

(4) Determine the values of the parameter k such that the following system of equations has unique solution, no solution or infinitely many solutions.

$$\begin{aligned} 2x+y-5z &= k \\ -x+3y+kz &= 0 \\ 3x+2y-9z &= 0 \\ -x+y-2z &= 0 \end{aligned}$$

Row reduce:

$$\left(\begin{array}{ccc|c} 2 & 1 & -5 & k \\ -1 & 3 & k & 0 \\ 3 & 2 & -9 & 0 \\ -1 & 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 1 & -2 & 0 \\ 2 & 1 & -5 & k \\ -1 & 3 & k & 0 \\ 3 & 2 & -9 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 1 & -2 & 0 \\ 0 & 3 & -9 & k \\ 0 & 2 & k+2 & 0 \\ 0 & 5 & -15 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 1 & -2 & 0 \\ 0 & 1 & (k+2)/2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & k/3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} -1 & 1 & -2 & 0 \\ 0 & 1 & (k+2)/2 & 0 \\ 0 & 0 & 4+k/2 & 0 \\ 0 & 0 & 4+k/2 & -k/3 \end{array}\right)$$

If $k \neq 0$ the system has no solution.

If $k = 0$, the matrix of coefficient has rank 3 and, thus, the system has a unique solution $(0, 0, 0)$. There is no value of k for which the system has non-trivial solution.

(5) Consider the following matrices and find the inverse, if it exists. If it does not exist, explain why.

(a) $\begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 2 & -2 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ 1 & 3 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

(a) NOT INVERTIBLE since the first row is linearly dependent with the thord row, then the determinant is 0.
(b) NOT INVERTIBLE since the third row is the sum of the first and second row, hence it is linearly dependent with the the other two rows and the determinant is 0.
(c) NOT INVERTIBLE since the matrix is not square
(d) INVERTIBLE since determinant is non-zero. We compute the inverse:

$$\left(\begin{array}{ccc|ccc} 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1/2 \end{array}\right)$$

This shows that the inverse matrix is $\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$

(6) For each of the following matrices, determine the values of the parameter k such that the matrix is invertible.

(a) $\begin{pmatrix} 2 & k & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & k & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & k & -1 \\ 1 & 1 & -1 \\ 3 & 3 & -3 \end{pmatrix}$

(a) Row reduce:

$$\begin{pmatrix} 2 & k & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & k/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 2-k/2 & 1/2 \end{pmatrix} \to \begin{pmatrix} 1 & k/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & k-4 & -1 \end{pmatrix}$$

Hence the matrix is invertible if and only if $k - 4 \neq 1$, that is, iff $k \neq 5$.
(b) ALWAYS INVERTIBLE since the columns are linearly independent, as can be also seen by observing that the matrix is reducible to row echelon form withouit any row of zeros.
(c) NEVER INVERTIBLE since the second row is linearly dependent with the third row, then the determinant is 0, independently of k.