Name:

MATH 4377/6308 - Advanced linear algebra I - Summer 2024

Homework 6

Exercises:

(1) Mark each statement True or False. Justify each answer. If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case.

a) If B is a matrix obtained from a square matrix A by interchanging any two rows, then $\det(B) = -\det(A)$.

b) If B is a matrix obtained from a square matrix A by multiplying a row of A by a scalar, then $\det(B) = \det(A)$.

c) If B is a matrix obtained from a square matrix A by adding k times row i to row j, then $\det(B) = k\det(A)$.

d) If $A \in M^{n,n}(F)$ has rank n, then $\det(A) = 0$.

e) The determinant of a square matrix may be computed by expanding the matrix along any row or column.

f) If two rows or columns of A are identical, then $\det(A) = 0$.

g) The determinant of a lower triangular $n \times n$ matrix is the product of its diagonal entries.

h) A matrix A is invertible if and only if $\det(A) = 0$.

(2) Prove that if $A, B \in M^{n,n}(F)$ are similar, then $\det(A) = \det(B)$.

(3) Compute the determinant of each of the following matrices

$$(a)\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{pmatrix} \quad (b)\ \begin{pmatrix} 1 & 0 & 3 & 4 \\ 2 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{pmatrix} \quad (c)\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -5 \end{pmatrix} \quad (d)\ \begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}$$

(4) Mark each statement True or False. Justify each answer. If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case.

a) Every linear operator on an n-dimensional vector space has n distinct eigenvalues.

b) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.

c) The sum of two eigenvectors of a linear operator T is always an eigenvector of T.

d) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.

e) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.

f) If λ is an eigenvalue of a linear operator T, then each vector in E_λ is an eigenvector of T.

g) If λ_1 and λ_2 are distinct eigenvalues of a linear operator T, then $E_{\lambda_1} \cap E_{\lambda_1} = \{0\}$.

h) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.

i) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.

(5) Find eigenvalues and eigenvectors of $A = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}$.

(6) Prove that similar matrices have the same characteristic polynomial and hence the same eigenvalues.

(7) Prove that the eigenvalues of an upper triangular matrix A are the diagonal entries of A.