Name: SOLUTION

MATH 4377/6308 - Advanced linear algebra I - Summer 2024

Homework 6

Exercises:

(1) Mark each statement True or False. Justify each answer. If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case.

a) If B is a matrix obtained from a square matrix A by interchanging any two rows, then $\det(B) = -\det(A)$.

b) If B is a matrix obtained from a square matrix A by multiplying a row of A by a scalar, then $\det(B) = \det(A)$.

c) If B is a matrix obtained from a square matrix A by adding k times row i to row j, then $\det(B) = k\det(A)$.

d) If $A \in M^{n,n}(F)$ has rank n, then $\det(A) = 0$.

e) The determinant of a square matrix may be computed by expanding the matrix along any row or column.

f) If two rows or columns of A are identical, then $\det(A) = 0$.

g) The determinant of a lower triangular $n \times n$ matrix is the product of its diagonal entries.

h) A matrix A is invertible if and only if $\det(A) = 0$.

(a) True. By a result presented in class.
(b) False. By a result presented in class, if B is a matrix obtained from a square matrix A by multiplying a row of A by a scalar k, then $\det(B) = k\det(A)$.
(c) False. By a result presented in class, if B is a matrix obtained from a square matrix A by adding k times row i to row j, then $\det(B) = \det(A)$.
(d) False. By a result presented in class, if $A \in M^{n,n}(F)$ has rank n, then $\det(A) \neq 0$.
(e) True. By a result presented in class, the determinant of a square matrix may be computed by expanding the matrix along any row or column.
(f) True. If two rows or columns of $A \in M^{n,n}(F)$ are identical, then the matrix has rank less than n and $\det(A) = 0$.
(g) True. The determinant of a lower triangular $n \times n$ matrix is the product of its diagonal entries.
(h) False. A matrix A is invertible if and only if it is full rank which is equivalent to $\det(A) \neq 0$.

(2) Prove that if $A, B \in M^{n,n}(F)$ are similar, then $\det(A) = \det(B)$.

Let A, B be similar matrices, that is, there exists an invertible matrix Q such that

$$B = Q^{-1}AQ$$

It follows that

$$\det(B) = \det(Q^{-1}AQ) = \det(Q^{-1})\det(A)\det(Q) = (\det(Q))^{-1}\det(A)\det(Q) = \det(A)$$

(3) Compute the determinant of each of the following matrices

$$(a)\quad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{pmatrix} \quad (b)\quad B = \begin{pmatrix} 1 & 0 & 3 & 4 \\ 2 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{pmatrix} \quad (c)\quad C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -5 \end{pmatrix} \quad (d)\quad D = \begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}$$

$$\det(A) = 5\det\begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 10 \\ 2 & 7 & 11 \end{pmatrix} = 5\det\begin{pmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} = -5\det\begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix} = -(5)(2) = -10$$

$$\det(B) = \det\begin{pmatrix} 1 & 0 & 3 & 4 \\ 2 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & -5 & -3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{pmatrix} = 2\det\begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & -5 & -3 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 3 & 5 \end{pmatrix} = 2\det\begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & -5 & -3 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{7}{2} \end{pmatrix} = 14$$

$$\det(C) = (1)(2)(2)(-5) = -20$$

$$\det(D) = (1)\det\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 5 \\ -2 & 0 & 0 \end{pmatrix} = (1)(-2)\det\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} = (1)(-2)(3)(5) = -30$$

(4) Mark each statement True or False. Justify each answer. If true, cite appropriate facts or theorems. If false, explain why or give a counterexample that shows why the statement is not true in every case.

a) Every linear operator on an n-dimensional vector space has n distinct eigenvalues.

b) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.

c) The sum of two eigenvectors of a linear operator T is always an eigenvector of T.

d) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.

e) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.

f) If λ is an eigenvalue of a linear operator T, then each vector in E_λ is an eigenvector of T.

g) If λ_1 and λ_2 are distinct eigenvalues of a linear operator T, then $E_{\lambda_1} \cap E_{\lambda_1} = \{0\}$.

h) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.

i) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.

(a) False. A linear operator on an n-dimensional vector space does not need to have n distinct eigenvalues. For instance, the matrix I_2 acting on $\mathbb{R}^2$ has only the eigenvalue $\lambda = 1$.

(b) False. For instance, the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ has two eigenvalues 3 and 2 but the sum 5 is not an eigenvalue of the same matrix.

(c) False. For instance, the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ has eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ but their sum $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not an eigenvector of the same matrix.

(d) False. For instance, the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ has only one (distinct) eigenvalue but it is diagonalizable.

(e) False. For instance, the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ *has linearly independent eigenvectors* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *corresponding to the same eigenvalue.*
(f) False. The zero vector 0 in E_λ *is not an eigenvector.*
(g) True. If $v \in E_{\lambda_1} \cap E_{\lambda_2}$ *with* $\lambda_1 \neq |lambda_2$*, then* $T(v) = \lambda_1 v = \lambda_2 v$*, so that* $(\lambda_1 - \lambda_2)v = 0$*, implying that* $v = 0$*.*
(h) False. The test for diagonalization requires that the characteristic polynomial of T *splits.*
(i) True. Let T *be a diagonalization linear operator on a nonzero vector space. The characteristic polynomial of* T *has a degree greater than or equal to one and splits, thus has at least one root. Hence* T *has at least one eigenvalue.*

(5) Find eigenvalues and eigenvectors of $A = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}$.

The characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = (1 + \lambda)(2 + \lambda) - 6 = \lambda^2 + 3\lambda - 4$$

Hence the eigenvalues are $\lambda_1 = -4, \lambda_2 = 1$*.*
For $\lambda_1 = -4$*, we have* $\det(A + 4I)x = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ *so that we have the eigenvector* $u_1 = \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix}$.
For $\lambda_2 = 1$*, we have* $\det(A + 4I)x = \begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ *so that we have the eigenvector* $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
NOTE: This implies that, choosing $Q = \begin{pmatrix} 1 & 1 \\ -\frac{3}{2} & 1 \end{pmatrix}$*, then*

$$Q^{-1}AQ = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}$$

(6) Prove that similar matrices have the same characteristic polynomial and hence the same eigenvalues.
Let A, B *be similar matrices, that is, there exists an invertible matrix* Q *such that*

$$B = Q^{-1}AQ$$

The characteristic polynomial of B *is* $p_B(\lambda) = \det(B - \lambda I)$*. We have that*

$$\begin{aligned} \det(B - \lambda I) &= \det(Q^{-1}AQ - \lambda I) \\ &= \det(Q^{-1}AQ - \lambda Q^{-1}IQ) \\ &= \det(Q^{-1}(A - \lambda I)Q) \\ &= \det(Q^{-1}) \det(A - \lambda I) \det(Q) \\ &= \det(Q^{-1}) \det(Q) \det(A - \lambda I) \\ &= (\det(Q))^{-1} \det(Q) \det(A - \lambda I) \\ &= \det(A - \lambda I) \end{aligned}$$

This proves that A *and* B *have the same characteristic polynomial and, consequently, the same eigenvalues.*

(7) Prove that the eigenvalues of an upper triangular matrix A are the diagonal entries of A.
Let $A \in M^{n,n}$*, with* $A = (a_{ij})$*. If* A *is upper triangular then* $A - \lambda I$ *is also upper triangular. Now the determinant* $\det(A - \lambda I)$ *of a triangular matrix is the product of the diagonal entries so that the characteristic polynomial is*

$$p(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda)$$

The roots of the characteristic polynomial are: $a_{11}, a_{22}, \dots, a_{nn}$.