BASES & COMPARISON RESULTS FOR LINEAR ELLIPTIC EIGENPROBLEMS.

GILES AUCHMUTY

Abstract. This paper describes some results about constructing and comparing, sequences of eigenvalues and eigenvectors of a pair (a,m) of continuous symmetric bilinear forms on a real Hilbert space V. The results are used to describe the properties of the eigenvalues and eigenfunctions for some elliptic eigenproblems on $H^1(\Omega)$ where Ω is a nice bounded region in $\mathbb{R}^N$, $N \geq 2$. These include eigenproblems with Robin type boundary conditions and Steklov eigenproblems. Also a class of eigenproblems where the eigenvalue is in both the equation and the boundary conditions. Different variational principles for the eigenvalues and eigenvectors are introduced and convex analysis is used. Both minimax and maximin characterizations of higher eigenvalues are described. The results are used to prove that the eigenfunctions are orthogonal bases of specific spaces or subspaces and to describe comparison results for the eigenvalues for different pairs of bilinear forms. Also to obtain spectral formulae for weak solutions of parametrized linear systems.

1. Introduction

This paper will describe the construction, and some properties, of specific orthogonal sequences of eigenvectors of pairs (a, m) of continuous symmetric bilinear forms on a real Hilbert space V. This construction is used to prove a variety of orthogonal decomposition results as well as conditions that guarantee that these sequences generate orthonormal bases of (closed subspaces of) V. They enable the proofs of some very general comparison theorems for the eigenvalues of two pairs $(a_1, m_1), (a_2, m_2)$ of bilinear forms on V. Moreover the results hold under rather simple conditions on the forms - different to many criteria used previously. Finally they are used to provide explicit spectral formulae for solutions of systems of equations posed in a weak form.

This analysis was motivated by a desire for such results for second-order divergence form elliptic eigenproblems on bounded regions in $\mathbb{R}^N$ satisfying boundary conditions other than the common Dirichlet condition. Such problems are not well formulated as linear operator problems on densely defined subspaces of $L^2(\Omega)$ - as the usual theory assumes. Rather the boundary conditions and the interior equations are combined in weak formulations using bilinear forms. Classes to be described here include eigenproblems involving Robin boundary conditions in section 7 and Steklov eigenproblems in section 8. Problems

Date: January 14, 2012.

Key words and phrases. Bases of Sobolev spaces, Eigenvalue comparison, Robin eigenproblem, Steklov eigenproblem, Spectral representation of weak solutions.

This research has been supported by DMS 0808115.

MSC2010 classification; Primary 49R05, 35P10, Secondary 35J25, 49J35.

with eigenparameters in both the differential equation and in the boundary condition are treated in section 9. Each of these examples involves eigenproblems posed on (subspaces of) $H^1(\Omega)$ where the bilinear form includes both domain and boundary integrals. One of the original motivations was to answer a question about comparison results for Steklov eigenproblems related to a model in cell biology described in Kazmierczek and Lipniacki [19]. The results also are related to questions about a particular system studied by Bandle, von Below and Reichel [6].

The analysis here uses methods that are appropriate for problems posed on general closed subspaces of $H^1(\Omega)$. Thus weak formulations and bilinear forms are studied directly rather than invoking properties of the associated linear operators between dual spaces. The systematic use of real bilinear forms enables comparison results when there are different boundary conditions as well as different coefficients in the differential operator.

Sections 3 and 4 describe variational principles for generating specific sequences of eigenvectors. These variational principles involve the finding suprema and infima of functionals on closed convex subsets of V with the extremality conditions found using properties of subdifferentials of indicator functionals of closed convex sets.

The proofs here use convex analysis results in place of the usual analysis involving Rayleigh quotients and Lagrange multiplier rules - neither of which are used here. In particular this analysis doesn't depend on the use of two Hilbert spaces (V,H) as has been common. Minmax and maximin versions of these variational principles for the k-th eigenvalue are described in sections 5 and 6.

For the classical treatment of many of these issues see Weinberger [20] chapter 3, sections 7-9 which includes a succinct description of some monotonicity results for the eigenvalues and historical references. Subsequent descriptions of these eigenproblems may be found in Bandle [5], chapter III which includes an introduction to Steklov eigenproblems. A treatment of eigenproblems based on bilinear forms on pairs (V, H) of real Hilbert spaces may be found in Blanchard and Bruning [10], chapter 6. They, and many other PDE texts, use such formulations to prove results for Dirichlet eigenproblems. In particular comparison results for the spectra of self-adjoint compact linear operators based on min-max or max-min principles may be found in Zeidler [21], section 22.11.

In this analysis, the bilinear form m is generally required to be weakly continuous but need not be positive. In particular a version of Hestenes' decomposition theorem is given in section 4. The analysis of indefinite eigenvalue problems has been studied by many authors and many results are summarized by de Figueiredo [13] and Belgacem [8]. Some different varional principles for such problems were described in Auchmuty [2]. A linear elliptic eigenvalue problem that involves both boundary integrals and indefinite forms is analyzed in Bandle, von Below and Reichel [6] for use in representing the solutions of some unusual linear parabolic equations.

2. Bilinear Forms and Eigenproblems.

Throughout this paper V is a real, separable, infinite dimensional, Hilbert space. The inner product and norm on V is denoted $\langle ., . \rangle_V$ and $\| \,.\, \|_V$. The dual space of V will be denoted V^* and is again a separable infinite dimensional Hilbert space with the usual dual norm and inner product. Through section 6 our treatment is quite general but the aim is to provide the results appropriate for the examples studied thereafter that involve eigenproblems for elliptic forms on $H^1(\Omega)$.

Our interest is in finding non-trivial solutions $(\lambda, u) \in \mathbb{R} \times V$ of

$$a(u,v) \;=\; \lambda\, m(u,v) \quad \text{for all} \quad v \in V. \tag{2.1}$$

Here $a : V \times V \to \mathbb{R}$ and $m : V \times V \to \mathbb{R}$ are continuous symmetric bilinear forms on V, This will be called the *(a,m) eigenproblem.*

A bilinear form $b : V \times V \to \mathbb{R}$ is said to be *symmetric* if $b(u,v) = b(v,u)$ for all $(u,v) \in V \times V$. When b is a symmetric continuous bilinear form on V, then vectors $u, v \in V$ are said to be *b-orthogonal* provided $b(u,v) = 0$. A subset $\mathcal{S}$ of V is said to be a *basis* of V if it is a maximal linearly independent set in V with respect to inclusion. $\mathcal{S}$ is a *b-orthogonal basis* of V when $\mathcal{S}$ is a basis that also is b-orthogonal. Note that if e is an eigenvector of (2.1) corresponding to an eigenvalue $\lambda \neq 0$, then

$$m(u,e) \;=\; 0 \iff a(u,e) \;=\; 0.$$

Define

$$\mathcal{A}(u) \;:=\; a(u,u), \text{ and } \quad \mathcal{M}(u) \;:=\; m(u,u). \tag{2.2}$$

$\mathcal{A}, \mathcal{M}$ are the quadratic forms on V associated with a, m respectively.

Our results about the eigenproblem for (a,m) will be proved using variational methods subject to some of the following conditions.

(A1): $a(.,.)$ is a continuous symmetric bilinear form that also is *V-coercive.* That is there are $0 < k_0 \leq k_1 < \infty$ such that

$$k_0 \|u\|_V^2 \;\leq\; \mathcal{A}(u) \;\leq\; k_1 \|u\|_V^2 \quad \text{for all} \quad u \in V. \tag{2.3}$$

(A2): $m(.,.)$ is a weakly continuous symmetric bilinear form on V.
(A3): $\mathcal{M}(u) \;>\; 0$ for some u in V, or,
(A4): $\mathcal{M}(u) \;>\; 0$ for all non-zero u in V.

When (A1) holds then the bilinear form $a(.,.)$ defines the *a-inner product* on V which is equivalent to the V inner product. When b_1, b_2 are continuous symmetric bilinear forms on V, the associated quadratic forms as in (2.2) will be denoted $\mathcal{B}_1, \mathcal{B}_2$ and we say that $\mathcal{B}_1 \leq \mathcal{B}_2$ on V provided

$$\mathcal{B}_1(v) \;\leq\; \mathcal{B}_2(v) \quad \text{for all} \quad v \in V.$$

The functional $\mathcal{B}$ is said to be positive when $\mathcal{B} \geq 0$, it is strictly positive when it satisfies (A4).

Some standard notations from convex analysis wiil be used. $\overline{\mathbb{R}} := [-\infty, \infty]$ is the extended real numbers and is totally ordered in the standard manner. A number $c \in \overline{\mathbb{R}}$ is positive (respectively negative) if $c \geq (\leq)\ 0$ and strictly positive (negative) if $c > (<)\ 0$.

When $\mathcal{F} : V \to (-\infty, \infty]$ is a given function, then the domain $dom(\mathcal{F})$ of $\mathcal{F}$ is the set of points where $|\mathcal{F}(v)| < \infty$. The functional $\mathcal{F}$ is *G-differentiable* at $v \in V$ provided there is a functional $D\mathcal{F}(v) \in V^*$ such that

$$\lim_{t \to 0+} t^{-1} [\mathcal{F}(v + tw) - \mathcal{F}(v)] = D\mathcal{F}(v)(w) \quad \text{for all} \quad w \in V$$

For a symmetric bilinear form b on V, the G-derivative of the associated quadratic form $\mathcal{B}$ is

$$D\mathcal{B}(v)\,(w) = 2\,b(v, w) \quad \text{for all} \quad w \in V.$$

A vector $G \in V^*$ is a *subgradient* of $\mathcal{F}$ at a point $v \in dom(\mathcal{F})$ provided

$$\mathcal{F}(v + h) \geq \mathcal{F}(v) + G(h) \quad \text{for all} \quad h \in V.$$

The set of all subgradients of $\mathcal{F}$ at v is called the *subdifferential* of $\mathcal{F}$ at v and is denoted $\partial\mathcal{F}(v)$.

When C is a closed convex set in V, then the indicator functional I_C of C is the function with $I_C(u) := 0$ for $u \in C$ and $I_C(u) := \infty$ otherwise. The following standard extremality condition will be used a number of times.

Theorem 2.1. *Let C be a closed convex subset of the real Hilbert space V and $\mathcal{F} : V \to \mathbb{R}$ be a Gateaux differentiable functional. If $\hat{u}$ maximizes $\mathcal{F}$ on C, then it is a solution of $D\,\mathcal{F}(u) \in \partial\, I_C\,(u)$. If $\hat{u}$ minimizes $\mathcal{F}$ on C, then $0 \in D\,\mathcal{F}(\hat{u}) + \partial\, I_C\,(\hat{u})$.*

3. Variational Principles and Comparison for λ_1.

Here a variational principle for finding, and comparing, the smallest strictly positive eigenvalue λ_1 of eigenvalue problems for (a, m) will be described. It involves the maximization of the quadratic form $\mathcal{M}$ on a closed convex subset of V.

Let $C_1 := \{u \in V : \mathcal{A}(u) \leq 1\}$ be the closed unit ball in V with respect to the a-inner product and $S_1 := \{u \in V : \mathcal{A}(u) = 1\}$. Consider the variational principle $(\mathcal{P}_1)$ of maximizing $\mathcal{M}$ on C_1 and finding

$$\beta_1 := \beta(\mathcal{M}, C_1) := \sup_{u \in C_1} \mathcal{M}\,(u). \tag{3.1}$$

The basic existence result for this variational problem is the following.

Theorem 3.1. *Assume V is a real Hilbert space and (A1)-(A3) hold. Then $\beta_1 > 0$ is finite and there are vectors $\pm e_1 \in S_1$ at which this supremum is attained. e_1 is an eigenvector of (a, m) corresponding to the eigenvalue $\lambda_1 := {\beta_1}^{-1}$. Moreover λ_1 is the smallest strictly positive eigenvalue of (a,m) and*

$$\mathcal{A}(u) \geq \lambda_1\, \mathcal{M}\,(u) \quad \textit{for all} \quad u \in V. \tag{3.2}$$

Proof. First note that the assumptions on $\mathcal{A}$ imply that C_1 is a bounded closed convex set in V and thus is weakly compact. Thus $\mathcal{M}$ will attains its supremum on C_1 as it is weakly continuous. β_1 is strictly positive from (A3) and thus will be attained on S_1 by homogeneity of $\mathcal{M}$. Let e_1 be such a maximizer then so also is $-e_1$ as $\mathcal{M}$ and $\mathcal{A}$ are even.

From theorem 2.1, a maximizer e of the G-differentiable function $\mathcal{M}$ on C_1 satisfies

$$2\,m(e,v) \;=\; F(v) \quad \text{for some} \quad F \in \partial I_{C_1}(e)$$

where I_{C_1} is the indicator functional of C_1. A standard calculation yields that $\partial\, I_{C_1}(u) = \{0\}$ when $\mathcal{A}(u) < 1$ and that $\partial\, I_{C_1}(u) = \{\mu\ a(u,.) : \mu \geq 0\}$ when $\mathcal{A}(u) = 1$. Thus since the maximizers are in S_1, there is a $\mu \geq 0$ such that

$$m(e,v) \;=\; \mu\ a(e,v) \quad \text{for all} \quad v \in V.$$

If $\mu = 0$, here then $\mathcal{M}\,(e) = 0$. Such vectors are not maximizers as (A3) holds. Thus $\mu > 0$ and $\pm e_1$ are eigenvector of (2.1) corresponding to a strictly positive eigenvalue. Put $v = e_1$ then $\mathcal{M}\,(e_1) \;=\; \beta_1 = \mu\mathcal{A}(e_1) = \mu$, so e_1 is an eigenvector of (a,m) corresponding to the eigenvalue $\lambda_1 := {\beta_1}^{-1}$. The other statements of the theorem now are consequences of the fact that $\pm e_1$ maximize $\mathcal{M}$ on C_1. □

Note that, when $\mathcal{M}$ is not a strictly positive functional, λ_1 need not be the least eigenvalue of (a,m); there may also be zero or negative eigenvalues.

Suppose that $(a_1,m_1),(a_2,m_2)$ are two pairs of quadratic forms that satisfy (A1)-(A3) that have least strictly positive eigenvalues $\lambda_1^{(1)}$, $\lambda_1^{(2)}$ respectively. The following is the basic comparison result.

Theorem 3.2. *Assume (A1)-(A3) hold for (a_1,m_1) and (a_2,m_2) with $\mathcal{A}_1 \leq \mathcal{A}_2$ and $\mathcal{M}_2 \leq \mathcal{M}_1$ on V, then* $\lambda_1^{(1)} \;\leq\; \lambda_1^{(2)}$.

Proof. Let $\beta_1^{(1)},\beta_1^{(2)}$ be the values of the maximization problems of the form $(\mathcal{P}_1)$ for $(a_1,m_1),(a_2,m_2)$ respectively. Then

$$\beta_1^{(1)} \;:=\; \sup_{\mathcal{A}_1(u)\leq 1} \;\mathcal{M}_1(u) \;\geq\; \sup_{\mathcal{A}_1(u)\leq 1} \;\mathcal{M}_2(u) \;\geq\; \sup_{\mathcal{A}_2(u)\leq 1} \;\mathcal{M}_2(u) \;=\; \beta_1^{(2)}.$$

Here the first inequality here holds as $\mathcal{M}_2 \leq \mathcal{M}_1$, while the second holds as the domain for the last maximization is a subset of the preceding domain. This inequality and the formula for λ_1 from theorem 3.1 imply the conclusion. □

4. Orthonormal Sequences of Eigenvectors.

The preceding maximization problem may be iterated to generate an a-orthonormal sequence of eigenvectors of (a,m).

Suppose that we have found $k-1$ a-orthonormal eigenvectors $E_{k-1} \;:=\; \{e_1,e_2,\ldots,e_{k-1}\}$ of (2.1) corresponding to successive smallest strictly positive eigenvalues $0 <\; \lambda_1 \;\leq\; \lambda_2 \;\leq$

$\ldots \leq \lambda_{k-1}$ of (a,m). Let V_{k-1} be the vector space spanned by these vectors and W_{k-1} be the subspace of all vectors that are a-orthogonal to V_{k-1}. That is $w \in W_{k-1}$ if and only if

$$a(w, v_j) \ = \ 0 \quad \text{for} \quad 1 \leq j \leq k-1.$$

Define $C_k \ := \ \{u \in C_1 : \ a(u, e_j) = 0 \quad \text{for} \quad 1 \ \leq \ j \ \leq \ k-1 \ \}$ so that C_k is the closed unit ball in W_{k-1} with respect to the a-inner product. Consider the problem $(\mathcal{P}_k)$ of maximizing $\mathcal{M}$ on C_k and finding

$$\beta_k \ := \ \beta(\mathcal{M}, C_k) \ := \ \sup_{u \in C_k} \ \mathcal{M}(u). \tag{4.1}$$

To obtain the extremality conditions satisfied by such maximizers we need expressions for the subdifferential of the indicator function $I_k(v) \ := \ I_{C_k}(v)$ of C_k. The following result provides the relevant expressions.

Lemma 4.1. *Suppose (A1) holds, $w \in C_k$ and $G \in \partial I_k(w)$. If $\mathcal{A}(w) < 1$, then $G(u) = a(v,u)$ for some $v \in V_{k-1}$. If $\mathcal{A}(w) \ = \ 1$, then $G(u) \ = \ a(v,u) + \mu\, a(w,u)$ for some $(v, \mu) \in V_{k-1} \times [0, \infty)$.*

Proof. One has $G \in \partial I_k(w)$ if and only if

$$I_k(w+u) \ \geq \ I_k(w) \ + \ G(u) \quad \text{for all} \quad u \in V.$$

When $\mathcal{A}(w) < 1$, this implies that $G(u) = 0$ for all $u \in W_{k-1}$ which yields the first statement. When $\mathcal{A}(w) = 1$, consider the restriction of the functional I_k to W_{k-1}. The sharp form of the Cauchy-Schwarz inequality shows that $G(u) = \mu\, a(w,u)$ with $\mu \geq 0$. Then the second statement follows. □

The result about this variational principle may be stated as follows.

Theorem 4.2. *Assume (A1)-(A3) hold and E_{k-1} as above are a-orthonormal eigenvectors of (a,m) corresponding to the $k-1$ smallest strictly positive eigenvalues of (a,m). Then either*
(i) $\beta_k \ = \ 0$, there are no other strictly positive eigenvalues of (a,m) and $\mathcal{M}(u) \ \leq \ 0$ for all $u \in W_{k-1}$, or else
(ii) $0 < \beta_k \ \leq \beta_{k-1}$ and there are vectors $\pm e_k \in S_1$ at which this supremum is attained. $\lambda_k \ := \ {\beta_k}^{-1}$ is the smallest eigenvalue of (2.1) greater than or equal to λ_{k-1} and e_k is an eigenvector of (2.1) corresponding to λ_k. Moreover

$$\mathcal{A}(u) \ \geq \ \lambda_k \ \mathcal{M}(u) \quad \textit{for all} \quad u \in W_{k-1}. \tag{4.2}$$

Proof. Note that $\mathcal{M}(0) = 0$, so $\beta_k \geq 0$. If $\beta_k = 0$ then $\mathcal{M}(u) \ \leq \ 0$ for all $u \in C_k$ and (i) holds. Otherwise we have $0 < \beta_k \ \leq \beta_{k-1}$. Just as in the proof of theorem 3.1, the maximizers must lie in S_1 by homogeniety.

From lemma 4.1 and theorem 2.1, the maximizers of $\mathcal{M}$ on C_k satisfy the system

$$2m(e,u) \ = \ a(v,u) \ + \ \mu\, a(e,u) \quad \text{for some} \quad v \in V_{k-1} \text{ and } \quad \mu \geq 0$$

Take $u = v$ here, then since $e \in W_{k-1}$, it follows that $v = 0$, and thus e is a solution of (2.1). Substitute $u = e$, then $2\mathcal{M}(e) = \mu = 2\beta_k$. Then the remaining statements hold just as in the proof of theorem 3.1. □

Suppose that this sequence of variational principles generates k a-orthonormal eigenvectors and then $\beta_{k+1} = 0$. Let V_+ be the space spanned by these eigenvectors, then $\mathcal{M}(v) > 0$ for all non-zero $v \in V_+$. and $dim(V_+) = k$. Note that this dimension is independent of the choice of the form $\mathcal{A}$ satisfying (A1).

When this sequence of successive maximization problems generates an infinite sequence $\mathcal{E} := \{e_k : k \geq 1\}$ of a-orthonormal eigenvectors, define V_+ to be the closed subspace generated by $\mathcal{E}$. Then the following result holds.

Theorem 4.3. *Assume (A1)-(A3) hold and V_+ is infinite dimensional. Then $lim_{k\to\infty} \beta_k = 0$, and $\mathcal{M}(u) \leq 0$ for all u that are a-orthogonal to V_+. When (A4) also holds then $V_+ = V$ and $\mathcal{E}$ is a maximal a-orthonormal subset of V.*

Proof. Let $\mathcal{E}$ be an infinite sequence of a-orthonormal eigenvectors generated as above. Since they are orthonormal with respect to an equivalent inner product on V, the sequence converges weakly to zero. Hence $\mathcal{M}(e_k) = \beta_k$ converges to 0 as $\mathcal{M}$ is weakly continuous.

If $\tilde{u}$ is a-orthogonal to V_+ and $\mathcal{M}(\tilde{u}) > 0$, then $\tilde{v} := \tilde{u}/\sqrt{\mathcal{A}(\tilde{u})}$ has $\mathcal{A}(\tilde{v}) = 1, \mathcal{M}(\tilde{v}) > 0$. Thus there is a K such that $k > K$ implies $\beta_k < \mathcal{M}(\tilde{v})$. This contradicts the definition of e_{K+1}, so we must have $\mathcal{M}(u) \leq 0$ for all u that are a-orthogonal to V_+. When (A4) holds then $\mathcal{M}(u) \leq 0$ implies $u = 0$, so $V_+ = V$ and $\mathcal{E}$ will be an a-orthonormal basis of V. □

When $\mathcal{M}$ obeys (A2) so does $-\mathcal{M}$. If there is a vector $v \in V$ with $\mathcal{M}(v) < 0$, then $-\mathcal{M}$ obeys (A3). In this case let V_- be the closed subspace of V generated, as above, by the eigenvectors of $(a, -m)$ associated with strictly positive eigenvalues. They corresponding to strictly negative eigenvalues of (a, m). Also let $ker(\mathcal{M})$, or V_0, be the maximal closed vector subspace of the zero set of $\mathcal{M}$.

Some useful a-orthogonal decompositions of V hold when these subspaces are non-trivial.

Corollary 4.4. *Suppose (A1)-(A3) hold and $\mathcal{M}(u) \geq 0$ for all $u \in V$. Then the a-orthogonal complement of V_+ is $V_0 = ker(\mathcal{M})$.*

Proof. Let W_+ be the a-orthogonal complement of V_+. From the theorem $\mathcal{M}(w) \leq 0$ for all $w \in W_+$. The positivity of $\mathcal{M}$ then implies that $\mathcal{M}(w) = 0$ for all $w \in W_+$ or $W_+ = ker(\mathcal{M})$. □

The following result may be regarded as a version of results described in an influential paper of Hestenes [18]. See also the monograph of Gregory [16]. It provides an a-orthogonal decomposition of V into the positive, negative and null spaces of the bilinear form m.

Corollary 4.5. *Suppose (A1)-(A3) hold, then the closed subspaces V_+, V_- and V_0 are a-orthogonal and*

$$V \;=\; V_+ \oplus_a V_0 \oplus_a V_- . \tag{4.3}$$

Proof. Define V_+, V_- as above. These subspaces are a-orthogonal as their bases are a-orthogonal. When a vector $w \in V$ is a-orthogonal to V_+, then $\mathcal{M}(w) \;\leq\; 0$. If it also is a-orthogonal to V_- then $\mathcal{M}(w) \;=\; 0$. Any linear combination of vectors that obey these a-orthogonality conditions also obeys them. Thus $w \in V_0 = ker(\mathcal{M})$ and (4.3) holds. □

Quite often these constructions involve situations where the following holds.

(A5): The bilinear form m is an inner product on a Hilbert space H and the imbedding of V into H is compact with dense range.

When (A5) holds then (A2) and (A4) will hold. In this case let

$$\tilde{e_j}(x) \;:=\; \frac{e_j(x)}{\sqrt{\lambda_j}} \quad \text{and} \quad \tilde{\mathcal{E}} \;:=\; \{\tilde{e_j} : j \geq 1\} \tag{4.4}$$

The following result gives a simple criterion for these eigenfunctions to also constitute a basis of H.

Theorem 4.6. *Suppose that (A1) and (A5) hold and $\mathcal{E}$ is constructed as above, then $\tilde{\mathcal{E}}$ defined by* (4.4) *is an m-orthonormal basis of H.*

Proof. Since m is an inner product on H then (A4) will hold for vectors in V. Also (A2) holds as the imbedding is compact and real inner products are symmetric. Thus from theorem 4.3, the set $\mathcal{E}$ is an a-orthonormal basis of V.

From (4.4) and the definition of eigenvectors, one sees that $m(\tilde{e_j}, \tilde{e_k}) = \delta_{jk}$ so that $\tilde{\mathcal{E}}$ is m-orthonormal. Given $\epsilon > 0, h \in H$, there will be a $v_\epsilon \in V$ satisfying $\|v_\epsilon - h\|_H < \epsilon$ since V is dense in H. Since $\mathcal{E}$ is a basis of V, there is a finite linear combination w_ϵ of elements of $\mathcal{E}$ such that $\|v_\epsilon - w_\epsilon\|_V < \epsilon$. Thus $\|w_\epsilon - h\|_H < C\epsilon$ for a finite constant C since the imbedding of V into H is continuous. Hence $\tilde{\mathcal{E}}$ is a basis of H as any vector in H can be approximated by a finite linear combination of vectors in $\tilde{\mathcal{E}}$. □

5. Maximin Principles for the k-th Eigenvalue.

Here some *maximin principles* for the k-th positive eigenvalue λ_k of (a, m) will be described and used to prove a comparison result for the λ_k. Eigenvalues are counted with multiplicity. Let $\Gamma_k \;:=\; \{v_1 \ldots, v_k\}$ be an a-orthonormal subset of V and V_k be the subspace spanned by Γ_k. Define $S_k \;:=\; \{v \in V_k : \mathcal{A}(v) = 1\}$ and consider the problem of minimizing $\mathcal{M}$ on S_k and finding

$$\alpha_k \;:=\; \alpha(\mathcal{M}, S_k) \;:=\; \inf_{v \in S_k} \; \mathcal{M}(v) \tag{5.1}$$

The following result summarizes some properties of this problem.

Theorem 5.1. *Assume (A1)-(A2) hold and Γ_k is a set of k a-orthonormal vectors in V. Let S_k be generated by Γ_k as above then α_k is finite. When there are at least k strictly positive eigenvalues of (a, m) then*

$$\lambda_k^{-1} \ = \ \sup_{\Gamma_k} \ \inf_{v \in S_k} \ \mathcal{M}(v) \ > \ 0. \tag{5.2}$$

Proof. First note that S_k is a compact subset of the finite dimensional vector space V_k and $\mathcal{M}$ is continuous on V_k. So from Weierstrass' theorem α_k is finite and attained in S_k.

The last statement is proved by induction. It holds for $k = 1$ from the analysis of section 3. Assume it is true for $k-1$ and let $E_{k-1} := \{e_j : 1 \leq j \leq k-1\}$ be a-orthonormal vectors corresponding to the first $k-1$ strictly positive eigenvalues of (a, m).

Since $dim(S_k) = k$, there are at least 2 vectors $\pm w \in S_k$ satisfying $a(w, e_j) = 0$ for all $1 \leq j \leq k-1$. Thus $0 < \alpha_k \leq \lambda_k^{-1}$ from (4.2). Moreover the supremum in (5.2) is λ_k^{-1} when S_k is a set of k a-orthonormal eigenvectors of (a, m) corresponding to the first k strictly positive eigenvalues of (a, m) so (5.2) follows. □

Suppose $m_2 \leq m_1$ on V, a obeys (A1) and $\lambda_j^{(1)}$, $\lambda_j^{(2)}$ are the j-th strictly positive eigenvalue of $(a, m_1), (a, m_2)$ respectively. Then the following comparison theorem for the eigenvalues holds.

Theorem 5.2. *Assume (A1)-(A3) hold for $(a, m_1), (a, m_2)$ and $\mathcal{M}_2 \leq \mathcal{M}_1$ on V. If (a, m_2) has k strictly positive eigenvalues, so does (a, m_1) and moreover $\lambda_j^{(2)} \ \geq \ \lambda_j^{(1)} > 0$ for $1 \leq j \leq k$.*

Proof. For a given Γ_k, we see that $\alpha_k^{(2)} \leq \alpha_k^{(1)}$ from the definition (5.1) as $\mathcal{M}_2 \leq \mathcal{M}_1$. Take the respective suprema in (5.2) to obtain the inequalities. □

6. Minimax Principles for the k-th Eigenvalue.

The preceding section described a comparison theorem when the bilinear form a is fixed. Here a different *minimax* characterization of the k-th strictly positive eigenvalue, will provide a comparison result when the form m is fixed and there are two bilinear forms a_1, a_2 obeying (A1) with $\mathcal{A}_1 \leq \mathcal{A}_2$.

Assume m is a bilinear form satisfying (A2) and (A3) with $dim\,(V_+) \geq k$. Let $\Gamma_k := \{w_1 \ldots, w_k\}$ be an m-orthonormal subset of V, Z_k be the subspace spanned by Γ_k and $S_k := \{z \in Z_k : \mathcal{M}(z) = 1\}$. Consider the optimization problem of maximizing $\mathcal{A}$ on S_k and finding

$$\beta_k \ := \ \beta(\mathcal{A}, S_k) \ := \ \sup_{z \in S_k} \ \mathcal{A}(z) \tag{6.1}$$

The following result summarizes some properties of this problem.

Theorem 6.1. *Assume (A1)-(A3) hold and there are at least k strictly positive eigenvalues of* (a, m)*. Let* Γ_k *be a set of k m-orthonormal vectors in V and* S_k *as above, then* β_k *is finite and*

$$\lambda_k \;=\; \inf_{\Gamma_k} \; \sup_{z \in S_k} \; \mathcal{A}(z). \tag{6.2}$$

Proof. Here S_k is a bounded closed set in the finite dimensional space Z_k so β_k defined by (6.1) is finite as $\mathcal{A}$ is continuous on Z_k.

When $k = 1$, this result follows from theorem 3.1. Assume the result holds for $k - 1$ and let $E_{k-1} \; := \; \{\tilde{e_j} : 1 \leq j \leq k-1\}$ be m-orthonormal vectors corresponding to the first $k - 1$ strictly positive eigenvalues of (a, m).

Since $dim(S_k) = k$, there are at least 2 vectors $\pm w \in S_k$ satisfying $m(w, e_j) = 0$ for all $1 \leq j \leq k-1$. Thus $\beta_k \geq \mathcal{A}(w) \geq \; \lambda_k$ from (4.2). However $\beta_k \; = \; \lambda_k$ when S_k is a set of m-orthogonal eigenvectors corresponding to the m smallest strictly positive eigenvalues of (a, m). Thus (6.2) holds. □

When a_1, a_2 are two bilinear forms that obey (A1), m is a bilinear form with at least k strictly positive eigenvalues let $\lambda_j^{(1)}, \; \lambda_j^{(2)}$ be the j-th strictly positive eigenvalues of $(a, m_1), (a, m_2)$ respectively. Then the following comparison theorem for the eigenvalues holds.

Theorem 6.2. *Assume (A1)-(A3) hold for* $(a_1, m), (a_2, m), \; \mathcal{A}_2 \geq \mathcal{A}_1$ *and m has at least k strictly positive eigenvalues. Then* $\lambda_j^{(2)} \; \geq \; \lambda_j^{(1)}$ *for* $1 \leq j \leq k$*.*

Proof. Use induction so it suffices to prove this for $j = k$. Given Γ_k, we see that $\beta_k^{(2)} \geq \beta_k^{(1)}$ as $\mathcal{A}_2 \geq \mathcal{A}_1$. Take the respective infima in (6.2) then the result follows. □

The following theorem combines results from the last two sections to generalize theorem 3.2 to higher strictly positive eigenvalues λ_j of these problems.

Theorem 6.3. *Assume (A1)-(A3) hold for* (a_1, m_1) *and* (a_2, m_2) *with* $\mathcal{A}_1 \leq \mathcal{A}_2$ *and* $\mathcal{M}_2 \leq \mathcal{M}_1$ *on V. If* (a_2, m_2) *has at least k strictly positive eigenvalues then* $\lambda_j^{(2)} \; \geq \; \lambda_j^{(1)}$ *for* $1 \leq j \leq k$*.*

Proof. Again, using induction, only the case $j = k$ needs to be proved. From theorem 6.3 this inequality holds for the pair of forms $(a_1, m_2), (a_2, m_2)$. Apply theorem 5.2 to the pair $(a_1, m_1), (a_2, m_1)$ and the result follows. □

In particular we have the following corollary.

Corollary 6.4. *Suppose (A1) holds for* a_1, a_2 *with* $\mathcal{A}_1 \leq \mathcal{A}_2$ *and (A2), (A4) hold for* m_1, m_2 *with* $\mathcal{M}_2 \leq \mathcal{M}_1$*. Then* $\lambda_j^{(2)} \; \geq \; \lambda_j^{(1)}$ *for all* $j \geq 1$ *and the associated families of eigenvectors defined by the iteration described in section 4 are maximal linearly independent sets in V.*

Proof. When (A4) also holds all the eigenvalues of (a, m) are strictly positive and $\lambda_j^{(2)} \; \geq \; \lambda_j^{(1)}$ for all $j \geq 1$ from the preceding theorem. The families of eigenfunctions are maximal from theorem 4.3. □

7. Bases & Comparison for Robin Eigenvalue Problems.

In the next few sections, we shall concentrate on examples with $V = H^1(\Omega)$ with Ω being a bounded region in $\mathbb{R}^N, N \geq 2$. A region is a non-empty open connected set. The definitions and terminology of Evans and Gariepy [15], will be followed except that $\sigma, d\sigma$, respectively, represent Hausdorff $(N-1)-$dimensional measure and integration with respect to this measure. All functions will take values in $\overline{\mathbb{R}} := [-\infty, \infty]$ and derivatives should be taken in a weak sense. The requirements on Ω are those for which the trace results of Auchmuty [4] hold.

(B1): *Ω is a bounded region in $\mathbb{R}^N$ and its boundary $\partial\Omega$ is the union of a finite number of disjoint closed Lipschitz surfaces; each surface having finite surface area.*

Let $L^p(\Omega), H^1(\Omega)$ be the usual real Lebesgue and Sobolev spaces of functions on Ω. The norm on $L^p(\Omega)$ is denoted $\|.\|_p$. $H^1(\Omega)$ is a real Hilbert space under the standard H^1- inner product

$$[u, v]_1 \ := \int_\Omega \ [u(x).v(x) \ + \ \nabla u(x) \cdot \nabla v(x)] \ dx. \tag{7.1}$$

Here ∇u is the gradient of the function u and the associated norm is denoted $\|u\|_{1,2}$.

The region Ω is said to satisfy *Rellich - Kondrachov (RK) theorem* provided the imbedding of $H^1(\Omega)$ into $L^p(\Omega)$ is compact for $1 \leq p < p_S$ for $p_S = 2N/(N-2)$ when $N \geq 3$ or $p_S = \infty$ when N = 2.

The region Ω is said to satisfy the *L^2-compact trace theorem* provided the trace map of $H^1(\Omega)$ into $L^2(\partial\Omega, d\sigma)$ is compact. Our standard assumption will be

(B2): Ω is a region such that (B1), the Rellich - Kondrachov theorem and the L^2-compact trace theorem hold.

Our first example of the use of the general results is to prove properties of the sequence of Robin eigenfunctions of second order, divergence form, elliptic systems. For other results on problems of this type, see [1], [8], [11] and [12].

Consider the bilinear forms $a, m : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(u, v) \quad := \quad \int_\Omega \ [(A \ \nabla u) \cdot \nabla v \ + \ c\, u\, v] \ dx \ + \int_{\partial\Omega} \ b\, u\, v \ d\sigma \tag{7.2}$$

$$m(u, v) \quad := \quad \int_\Omega \ m_0\, u\, v \ dx \tag{7.3}$$

The following conditions on the coefficients in these forms will be used.

(B3): $A(x) := (a_{jk}(x))$ is a real symmetric matrix whose components are bounded Lebesgue-measurable functions on Ω and there exist constants $0 < k_2 \leq k_3$ such that

$$k_2\, |\xi|^2 \ \leq \ \langle A(x)\xi, \xi\rangle \ \leq \ k_3\, |\xi|^2 \qquad \text{for all} \quad \xi \in \mathbb{R}^N, \ x \in \Omega \tag{7.4}$$

with $\langle ., . \rangle$ the usual Euclidean inner product and $|.|$ the Euclidean norm on $\mathbb{R}^N$.

When A_1, A_2 are two matrix-valued fields on Ω that satisfy (B3) we say that $A_1 \leq A_2$ on Ω provided

$$\langle A_1(x)\xi, \xi\rangle \ \leq\ \langle A_2(x)\xi, \xi\rangle \qquad \text{for all} \quad x \in \Omega,\ \xi \in \mathbb{R}^N. \tag{7.5}$$

(B4): $c \geq 0$ and $c \in L^p(\Omega)$ for some $p \geq N/2$ when $N \geq 3$, $(p > 1$ if $N = 2)$.

(B5): $b \in L^\infty(\partial\Omega)$ with $b \geq 0$ σ *a.e.* on $\partial\Omega$ and

$$\int_\Omega c\ dx + \int_{\partial\Omega} b\ d\sigma \ =\ b_0 \ >\ 0.$$

The eigenproblem to be studied here is to find non-trivial $(\lambda, u) \in \mathbb{R}\times H^1(\Omega)$ satisfying

$$\int_\Omega \left[(A\ \nabla u)\cdot\nabla v + c\,u\,v\right]\ dx + \int_{\partial\Omega} b\,u\,v\ d\sigma \ =\ \lambda \int_\Omega m_0\,u\,v\ dx \tag{7.6}$$

for all $v \in H^1(\Omega)$. This is the weak form of the eigenvalue problem of finding non-trivial solutions of the system

$$Lu(x) \ :=\ -\operatorname{div}\left(A(x)\,\nabla u(x)\right) + c(x)\,u(x) \ =\ \lambda\, m_0(x)\ u(x) \qquad \text{on } \Omega \tag{7.7}$$

$$(A(x)\,\nabla u(x))\cdot\nu(x) + b(x)\,u(x) \ =\ 0 \qquad \text{on } \partial\Omega. \tag{7.8}$$

When $b \equiv 0$ on $\partial\Omega$, this is the *Neumann eigenproblem* for the formal operator L on Ω. Otherwise this is called the *Robin eigenproblem.* As will be seen our results for these eigenproblems are quite similar.

The results of the previous sections depend primarily on properties of the quadratic forms $\mathcal{A}, \mathcal{M}$ associated with a, m.

$$\mathcal{A}(u) \ :=\ \int_\Omega \left[(A\ \nabla u)\cdot\nabla u + c\,u^2\right]\ dx + \int_{\partial\Omega} b\,u^2\ d\sigma \tag{7.9}$$

$$\mathcal{M}(u) \ :=\ \int_\Omega m_0\,u^2\ dx \tag{7.10}$$

The condition required on the weight function m_0 is the following

(B6): $m_0 \in L^q(\Omega)$ for some $q > N/2$ with m_0 positive on Ω and $\|\, m_0 \,\|_1 > 0$.

Lemma 7.1. *Assume (B2) and (B6) holds, then m defined by* (7.3) *obeys (A2) and,*
(i) $\mathcal{M}$ *is weakly continuous and convex on $H^1(\Omega)$,*
(ii) there are infinitely many m-orthogonal vectors in $H^1(\Omega)$ with $\mathcal{M}(v) > 0$.

Proof. When the R-K theorem holds we have that $u \in H^1(\Omega)$ implies $u \in L^p(\Omega)$ for $1 \leq p < p_S$ with the imbedding compact. From Holder's inequality applied to (7.10), $\mathcal{M}(u)$ is bounded when $q > N/2$. It is weakly continuous as the imbedding is compact. Since m_0 is positive on $\Omega, \mathcal{M}(v) \geq 0$ for all $v \in H^1(\Omega)$ so this form is convex with $\mathcal{M}(v) > 0$ for every $v \in H^1(\Omega)$ that is also continuous and strictly positive on $\overline{\Omega}$. Thus (ii) holds. □

The properties of the sequence of eigenfunctions defined by the iterative construction of section 4 is summarized in the following theorem. In particular simple conditions on the coefficients of these systems yield eigenfunction bases of $H^1(\Omega)$ and $L^2(\Omega)$.

Theorem 7.2. *Assume (B2) holds, the $A(x)$ are real symmetric matrices on Ω obeying (B3) and c, b, m_0 satisfy (B4) - (B6). Then there is an increasing sequence of eigenvalues $\Lambda := \{\lambda_k : k \in \mathbb{N}\}$ and an associated sequence $\mathcal{E}$ of eigenfunctions of* (7.6) *with* $\lim_{k\to\infty} \lambda_k = \infty$. *If $\mathcal{M}$ also satisfies (A4) then $\mathcal{E}$ is an a-orthonormal basis of $H^1(\Omega)$. When m_0 also satisfies*
(B7): $m_0 \in L^\infty(\Omega)$ *and there is an c_0 such that $m_0(x) \geq c_0 > 0$ for all $x \in \Omega$,*
then $\tilde{\mathcal{E}}$ defined by (4.4) *is an m-orthonormal basis of $L^2(\Omega)$*

Proof. The bilinear form m satisfies (A2) and (A4) from the preceding lemma. Our conditions on the coefficients imply that a is a bounded bilinear form on $H^1(\Omega)$. To prove that it is coercive we use that fact that each function in $H^1(\Omega)$ has a unique decomposition of the form $u = \bar{u} + v$ where $\bar{u}$ is the mean value of u on Ω and $\int_\Omega v\,dx = 0$. From this decomposition it follows that $\mathcal{A}$ is coercive on $H^1(\Omega)$ whenever (B3) and (B5) hold. Thus $a(u, v)$ defined by (7.6) satisfies (A1) and $\mathcal{E}$ is a basis of V from theorem 4.3 when (A4) aso holds.

When m_0 satisfies (B7) then it defines an equivalent inner product on $L^2(\Omega)$ to the usual inner product, so theorem 4.6 yields the last statement. □

Its worth commenting that a common treatment of eigenvalue problems for systems like these has been to transform the problems into eigenvalue problem for self-adjoint compact operators on $L^2(\Omega)$. Then the spectral theorem for such operators is used to prove that they are an orthonormal basis of $L^2(\Omega)$. Note that the analysis here shows that the conditions on the function m for these eigenfunctions to be an orthonormal basis of $H^1(\Omega)$ are weaker than those required to prove they constitute a basis of $L^2(\Omega)$.

More generally when $m_0 = m_+ - m_-$ is the decomposition of m_0 into its positive and negative parts, then a decomposition of $H^1(\Omega)$ into three a-orthogonal subspaces as in Corollary 4.5 may be performed. Some such results are described in Belgacem [8]

Suppose A_1, A_2 are two families of matrix-valued functions on Ω that satisfy (B3) with $A_1 \leq A_2$ a.e. on Ω. Let c_1, c_2 and b_1, b_2 satisfy (B4) and (B5) respectively with $c_1 \leq c_2$ on Ω and $b_1 \leq b_2$ σ *a.e.* on $\partial\Omega$. Define bilinear forms a_1, a_2 on $H^1(\Omega)$ by

$$a_j(u, v) := \int_\Omega [(A_j \nabla u) \cdot \nabla v + c_j\, u\, v\,]\, dx + \int_{\partial\Omega} b_j\, u\, v\, d\sigma \tag{7.11}$$

for $j = 1, 2$. Then $\mathcal{A}_1(u) \leq \mathcal{A}_2(u)$ for each $u \in H^1(\Omega)$ and they both satisfy condition (A1).

Theorem 7.3. *Assume $\mathcal{A}_1(u) \leq \mathcal{A}_2(u)$ as above on $H^1(\Omega)$ and m_2, m_1 obey (B6) with $m_2 \leq m_1$ a.e. on Ω. Then the eigenvalue problems* (7.6) *for* (a_1, m_1) *and* (a_2, m_2) *have infinitely many strictly positive eigenvalues and the corresponding eigenvalues satisfy $\lambda_j^{(2)} \geq \lambda_j^{(1)}$ for $j \geq 1$.*

Proof. Under these assumptions, the existence of such eigenvalue sequences holds from theorem 7.2 and the comparison result follows from theorem 6.3. □

To use this theorem, we generally take one of the problems to have simple forms with $A(x) \equiv a_0 I_N$ on Ω and each of b, c, m_0 being constants. Physically this result provides the direction of change in the eigenvalues as specific data is varied.

8. Bases & Comparison for Steklov Eigenvalue Problems.

Steklov eigenproblems are eigenproblems where the eigenparameter appears only in the boundary conditions. They arise in a variety of applications and have been used for describing a spectral theory of trace spaces. See Auchmuty [3], [4] for introductions to such results.

Here we call an eigenvalue problem of the form (2.1) a Steklov eigenproblem when the bilinear form m only involves boundary integrals. For problems with $V = H^1(\Omega)$ assume $\rho : \partial\Omega \to (0, \infty]$ is nonzero and define

$$m(u,v) \ := \ \int_{\partial\Omega} \rho\, u\, v \ d\sigma, \qquad \mathcal{M}\,(u) \ := \ \int_{\partial\Omega} \rho\, u^2 \ d\sigma. \tag{8.1}$$

In this section we will require that $\rho, \partial\Omega$ satisfy
(B8): $\rho \in L^q(\partial\Omega)$ with $\rho \geq \rho_0 > 0$ σ *a.e.* and $q > N-1$. $\partial\Omega$ satisfies (B2) and the trace map of $H^1(\Omega)$ into $L^p(\partial\Omega, d\sigma)$ is compact for $p < \frac{2(N-1)}{N-2}$ when $N \geq 3$, ($p < \infty$ when N=2).

The boundary trace condition here is known to hold under a variety of conditions on the boundary $\partial\Omega$. See DiBenedetto [14] chapter IX, proposition 18.1 for one proof.

The Steklov eigenproblem for our second order elliptic operator in divergence form is to find non-trivial $(\delta, u) \in \mathbb{R} \times H^1(\Omega)$ that satisfy

$$\int_{\Omega} [(A\ \nabla u)\cdot \nabla v \ + \ c\, u\, v] \ dx \ + \ \int_{\partial\Omega} b\, u\, v \ d\sigma \ = \ \delta \int_{\partial\Omega} \rho\, u\, v \ d\sigma. \tag{8.2}$$

for all $v \in H^1(\Omega)$. The eigenparameter in Steklov eigenproblems will be denoted δ rather than λ so that they may be distinguished. This is the weak form of the eigenvalue problem of finding non-trivial solutions of the system

$$Lu(x) \ := \ -\operatorname{div}(A(x)\,\nabla u(x)) \ + \ c(x)\, u(x) \ = \ 0 \qquad \text{on} \quad \Omega \tag{8.3}$$

$$(A(x)\,\nabla u(x))\cdot \nu(x) \ + \ b(x)\, u(x) \ = \ \delta\ \rho(x)\, u(x) \qquad \text{on} \quad \partial\Omega. \tag{8.4}$$

Here L is just a formal differential operator.

Lemma 8.1. *Assume (B8) holds, then m defined by* (8.1) *obeys (A2) and (A3) and $ker(\mathcal{M}) = H_0^1(\Omega)$. Moreover $\mathcal{M}\,(u) > 0$ on an infinite dimensional subspace of $H^1(\Omega)$. When $\rho \in L^\infty(\partial\Omega, d\sigma)$ then m is an inner product on $L^2(\partial\Omega, d\sigma)$ that is equivalent to the usual inner product on $L^2(\partial\Omega, d\sigma)$.*

Proof. When (B8) holds then the bilinear form m defined by (8.1) is bounded upon using Holder's inequality and the trace condition in (B8). Thus m is weakly continuous as the trace map is compact. Obviously $\mathcal{M}(u) = 0$ and (B8) implies that the trace of u on $\partial\Omega$ is identically zero so $u \in H_0^1(\Omega)$. Also

$$\mathcal{M}(u) \ \geq \ \rho_0 \|u\|^2_{2,\partial\Omega} \quad \text{for all} \quad u \in H^1(\Omega)$$

so it is nonzero on the subspace of functions that are nonzero on $\partial\Omega$ and this is an infinite dimensional subspace. The last sentence follows from another uses of Holder's inequality. This inner product will be called the $\rho-$ inner product on $L^2(\partial\Omega, d\sigma)$. □

When (B8) holds then a function $u \in H^1(\Omega)$ is a-orthogonal to $V_0 := ker(\mathcal{M})$ iff

$$a(u,v) \ = \ 0 \quad \text{for all} \quad v \in H_0^1(\Omega). \tag{8.5}$$

This is the usual criterion for u to be an H^1-solution of the equation $Lu = 0$ on Ω. The set of all such solutions will be denoted $N(L)$ and called the null space of L. Corollary 4.4 applies so we have the orthogonal decomposition

$$H^1(\Omega) \ = \ H_0^1(\Omega) \oplus_a N(L).$$

With this choice of (a, m), the analysis of section 4 may be used to generate an infinite sequence of eigenfunctions $\{e_k\}$ of the system (8.2). Note that each such eigenfunction is an H^1-solution of the homogeneous equation $Lu = 0$. The properties of this sequence of Steklov eigenfunctions may be summarized as follows.

Theorem 8.2. *Assume the $A(x)$ are real symmetric matrices on Ω obeying (B3), c, b, ρ satisfy (B4), (B5) and (B8). Then there is an increasing sequence of Steklov eigenvalues $\Lambda \ := \ \{\delta_k : k \in \mathbb{N}\}$ with $\lim_{k\to\infty} \ \delta_k = \infty$. The associated sequence $\mathcal{S} := \{e_k : k \in \mathbb{N}\}$ of eigenfunctions of* (8.2)*, is both a-orthonormal in $H^1(\Omega)$ and $\rho-$orthogonal on $\partial\Omega$. If also $\rho \in L^\infty(\partial\Omega, d\sigma)$ then the sequence $\mathcal{S}$ is an $\rho-$orthogonal basis of $L^2(\partial\Omega, d\sigma)$.*

Proof. The assumptions here imply that $\mathcal{A}$ satisfies (A1) just as in the proof of theorem 7.2 and $\mathcal{M}$ satisfies (A2) and (A3) from lemma 8.1. Thus theorem 4.3 yields the existence of infinitely many positive eigenvalues and a-orthonormal eigenfunctions of (8.2). When the value of the k-th problem is β_k, then the corresponding eigenvalue is $\delta_k = \beta_k^{-1}$. These eigenfunctions must converge weakly to zero in $H^1(\Omega)$, so the corresponding eigenvalues δ_k must increase to ∞ just as in the proof of theorem 7.2.

These eigenfunctions span the a-orthogonal complement of $ker(\mathcal{M}) = H_0^1(\Omega)$ from corollary 4.4 and lemma 8.1. This space $N(L)$ is isomorphic to the usual trace space $H^{1/2}(\partial\Omega)$ as it is a quotient space of $H^1(\Omega)$ by $H_0^1(\Omega)$. The eigenfunctions will be a maximal $\rho-$orthonormal sequence of functions in $N(L)$ by the usual argument. When m defines an equivalent inner product on $L^2(\partial\Omega, d\sigma)$ to the usual inner product, then this sequence will then also be a basis of $L^2(\partial\Omega, d\sigma)$ as $H^{1/2}(\partial\Omega)$ is dense in $L^2(\partial\Omega, d\sigma)$. □

When m_1, m_2 are two bilinear forms of the form (8.1), associated with surface densities ρ_1, ρ_2 then $m_2 \ \leq \ m_1$ on $H^1(\Omega)$ iff $\rho_2(x) \ \leq \ \rho_1(x) \ \sigma \ a.e.$ on $\partial\Omega$. In this case the comparison result is the following

Theorem 8.3. *Assume $\mathcal{A}_1(u) \leq \mathcal{A}_2(u)$ as above on $H^1(\Omega)$ and $\rho_2 \leq \rho_1$ satisfy (B8). Then the eigenvalue problems of the form* (8.2) *for* (a_1, m_1) *and* (a_2, m_2) *have infinitely many strictly positive eigenvalues and the corresponding eigenvalues satisfy* $\delta_j^{(2)} \geq \delta_j^{(1)}$ *for* $j \geq 1$.

Proof. Under these assumptions, the existence of infinitely many strictly positive eigenvalues and associated eigenfunctions follows from the preceding theorem and then the comparison result holds from theorem 6.3. □

One motivation for this paper was a question about how the Steklov eigenvalues vary in a model of a problem in cell biology described in Kazmierczak and Lipniacki [19]. For their problem, the Steklov eigenvalues describe the stability of steady state solutions, so comparison results for these Steklov eigenvalues imply results on the stability of steady states as boundary conditions are varied.

9. Bases & Comparison for General Eigenproblems.

Another class of linear elliptic eigenproblems involves systems of the form (2.1) with

$$m(u,v) \ := \ \int_\Omega m_0\, u\, v\ dx \ + \ \int_{\partial\Omega} \rho\, u\, v\ d\sigma. \tag{9.1}$$

That is, we look for nontrivial solutions $(\lambda, u) \in \mathbb{R} \times H^1(\Omega)$ of

$$\int_\Omega [(A\ \nabla u)\cdot \nabla v \ + \ c\, u\, v]\ dx \ + \ \int_{\partial\Omega} b\, u\, v\ d\sigma \ = \ \lambda \left[\int_{\partial\Omega} \rho\, u\, v\ d\sigma \ + \ \int_\Omega m_0\, u\, v\ dx \right] \tag{9.2}$$

for all $v \in H^1(\Omega)$. This is the weak form of the boundary value problem

$$-\operatorname{div}(A(x)\,\nabla u(x)) \ + \ c(x)\, u(x) \ = \ \lambda\ m_0(x)\ u(x) \qquad \text{on } \Omega \tag{9.3}$$

$$(A(x)\,\nabla u(x))\cdot \nu(x) \ + \ b(x)\, u(x) \ = \ \lambda\ \rho(x)\, u(x) \qquad \text{on } \partial\Omega. \tag{9.4}$$

Here the eigenparameter λ enters both the equation and the boundary condition. See Belinsky [9] for a discussion and analysis of some examples of these problems. Bandle, von Below and Reichl [6] study a particular example of this system in connection with the spectral represenation of solutions of a linear parabolic equation. In [7], Bandle describes some monotonicity properties for the first eigenvalue of a problem of this type. Note that these problems have a natural formulation on $H^1(\Omega)$ but not in terms of densely defined closed operators on $L^2(\Omega)$.

Lemma 9.1. *Assume (B7) and (B8) hold, then m defined by* (9.1) *obeys (A2) and (A4).*

Proof. In the last two sections, each of the terms in this definition of m have been shown to satisfy (A2). Hence this m is weakly continuous on $H^1(\Omega)$. Moreover (A4) holds as (B7) and (B8) imply that

$$\mathcal{M}\,(u) \ \geq \ c_0\ \|u\|_2^2 \ + \ \rho_0\ \|u\|_{\partial\Omega,2}^2.$$

□

The properties of the sequence of eigenfunctions defined by the iterative construction of section 4 is summarized in the following theorem.

Theorem 9.2. *Assume (B2) holds, $A(x)$ are real symmetric matrices on Ω obeying (B3), c, b, m_0, ρ satisfy (B4) - (B8). Then there is an infinite sequence $\mathcal{E}$ of eigenfunctions of* (9.2) *with an associated increasing sequence of strictly positive eigenvalues $\Lambda := \{\lambda_k : k \in \mathbb{N}\}$ and $\lim_{k\to\infty} \lambda_k = \infty$. Moreover $\mathcal{E}$ is an a-orthonormal basis of $H^1(\Omega)$.*

Proof. The quadratic form $\mathcal{A}$ satisfies (A1) as in the proof of theorem 7.2. The preceding lemma shows that m satisfies (A2) - (A4). Thus the result follows from theorem 4.3. □

This type of result provides some criteria that may be used to answer question 4 in section 6 of [6]. In their question, the quadratic form associated with m is the difference of two convex functions, so there will be both positive and negative eigenvalues so corollaries 4.4 and 4.5 may be invoked. There also are comparison theorems for these eigenproblems. Suppose m_1, m_2 are bilinear forms of the type (9.1), then $\mathcal{M}_2 \leq \mathcal{M}_1$ on $H^1(\Omega)$ iff $\rho_2 \leq \rho_1$ σ *a.e.* on $\partial\Omega$ and $m_2 \leq m_1$ *a.e.* on Ω.

Theorem 9.3. *Assume $\mathcal{A}_1(u) \leq \mathcal{A}_2(u)$ satisfy (A1) on $H^1(\Omega)$ and $\mathcal{M}_2 \leq \mathcal{M}_1$ are such that (A4) holds. Then the eigenproblems* (9.2) *for (a_1, m_1) and (a_2, m_2) have infinitely many strictly positive eigenvalues and the corresponding eigenvalues satisfy $\lambda_j^{(2)} \geq \lambda_j^{(1)}$ for $j \geq 1$.*

Proof. The existence of the eigenvalues follows from the preceding theorem. The comparison result is a consequence of Corollary 6.4. □

10. Spectral Representations of Solutions of Elliptic Problems.

One of the primary functions of eigenvalue analyses has been to provide spectral representations of the solutions of linear systems. When the system is described by bilinear forms on V as done in the last few sections, one can ask whether there are such representations? Consider the problem of representing the solutions $\hat{u} \in V$ of

$$a(u, v) - \lambda\, m(u, v) = F(v) \quad \text{for all} \quad v \in V. \tag{10.1}$$

Here $\lambda \in \mathbb{R}$, $F \in V^*$ is a continuous linear functional on V and (a, m) will be assumed to satisfy (A1) - (A4). This is a standard weak formulation for elliptic boundary value problems - including many with nonzero boundary data.

Assume that $\mathcal{E} := \{e_k : k \geq 1\}$ is a sequence of a-orthonormal eigenvectors of (a, m) corresponding to the increasing sequence of strictly positive eigenvalues $\Lambda := \{\lambda_k : k \in \mathbb{N}\}$.

This will be an a-orthonormal basis of V from theorem 4.3. When $\lambda \notin \Lambda$, substitute $v = e_k$ in (10.1) for each $k \in \mathbb{N}$ to find that the solution $\hat{u}$ has the *spectral representation*

$$\hat{u} \;=\; \sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j - \lambda} F(e_j)\, e_j. \tag{10.2}$$

This is a formal calculation, so one might ask about its validity? For each $K \geq 1$, let $P_K \hat{u}$ be the K-th partial sum, or the *K-th spectral approximation*, of $\hat{u}$. It depends on the choice of the eigenfunctions e_j.

The following result justifies this spectral expression and provides an estimate of the solution in terms of finite approximations. As noted before when (A1) holds then the bilinear form a provides an equivalent inner product on V so we shall write $\|u\|_a^2 \;=\; a(u,u)$ for the associated norm and $\|F\|_{a*}$ for the dual norm on V^*. There are two different cases depending on whether $\lambda \notin \Lambda$ or $\lambda \in \Lambda$.

Theorem 10.1. *(Resolvent representation) Assume (A1) - (A4) hold and $\mathcal{E}$ is a maximal a-orthonormal set of eigenvectors of (a,m) in V. Let Λ be the corresponding set of eigenvalues of (a,m) and suppose $\lambda \notin \Lambda$. For each $F \in V^*$, there is a unique solution $\hat{u}$ of* (10.1) *that is given by* (10.2)*. When $\lambda < \lambda_1$ then*

$$\| \hat{u} \|_a \;\leq\; \frac{\lambda_1}{\lambda_1 - \lambda}\; \|F\|_{a*}. \tag{10.3}$$

When $\lambda_K < \lambda < \lambda_{K+1}$, let $P_K \hat{u}$ be the K-th partial sum of (10.2)*, then*

$$\| \hat{u} \|_a \;\leq\; \frac{\lambda_{K+1}}{\lambda_{K+1} - \lambda}\; \|F\|_{a*} \;+\; \frac{\lambda\,(\lambda_{K+1} - \lambda_1)}{\lambda_1\,(\lambda_{K+1} - \lambda)}\; \| P_K \hat{u} \|_a. \tag{10.4}$$

Proof. The fact that the solution is unique holds as $\lambda \notin \Lambda$. To obtain these bounds, assume $\hat{u} = \sum_{j=1}^{\infty} c_j\, e_j$ and put $u = v = \hat{u}$ in (10.1). Then Paseval's equality and the orthogonality properties yield

$$\| u \|_a^2 \;=\; \sum_{j=1}^{\infty} {c_j}^2 \quad \text{and} \quad \sum_{j=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_j}\right) {c_j}^2 \;=\; F(\hat{u}). \tag{10.5}$$

When $\lambda <\; \lambda_1$ each coefficient in the second sum is positive and greater than the term with $j = 1$. Hence (10.3) holds.

Otherwise, let λ_K be the largest eigenvalue less than λ, then (10.5) becomes

$$\left(1 - \frac{\lambda}{\lambda_{K+1}}\right)\; \| u \|_a^2 \;\leq\; |F(\hat{u})| \;+\; \frac{\lambda\,(\lambda_{K+1} - \lambda_1)}{\lambda_1\, \lambda_{K+1}}\; \| P_K \hat{u} \|_a^2.$$

Rearranging this leads to (10.4). □

This shows that, given a bound on the dual norm of F, then the solution $\hat{u}$ of (10.1) with $\lambda \in (\lambda_K, \lambda_{K+1})$ may be bounded in terms of data from the $(K+1)$-st spectral approximation.

Note also that we had no need to determine any linear operators here nor have we used any pivot spaces such as $L^2(\Omega)$. In particular for parametrized inhomogeneous Robin boundary value systems, this formulation provides results without having to use any property of the dual space of $H^1(\Omega)$ other than a norm of the functional F.

When $\lambda \in \Lambda$, let J_λ be the set of integers j corresponding to this eigenvalue (i.e. such that $\lambda_j = \lambda$). Let E_λ be the finite dimensional eigenspace spanned by these associated eigenfunctions of (a, m). Then the above analysis may be modified to yield.

Theorem 10.2. *(Resonance representation) Assume (A1) - (A4) hold and $\mathcal{E}$ is a maximal a-orthonormal set of eigenvectors of (a, m) in V. Let Λ be the corresponding set of eigenvalues of (a, m) and suppose that $\lambda \in \Lambda$. There are solutions of* (10.1) *if and only if $F(e) = 0$ for all $e \in E_\lambda$. In this case when $\hat{u} \in V$ is a solution of* (10.1)*, there is a vector $v \in E_\lambda$ such that*

$$\hat{u} \;=\; \sum_{j \notin J_\lambda} \frac{\lambda_j}{\lambda_j - \lambda} F(e_j)\, e_j + v. \tag{10.6}$$

Proof. Obviously if $F(e) \neq 0$ for some $e \in E_\lambda$, then (10.1) cannot hold for any $u \in V$ when $v = e$. Hence this is a necessary condition for the existence of a solution. When $F(e) = 0$ for all $e \in E_\lambda$, then the sum in (10.6) may be verified to be a solution of the system, so this condition is also sufficient. The other parts of this proof are standard. □

References

[1] G.A. Afrouzi and K.J. Brown, "On principal eigenvalues for boundary value problems with indefinite weight and Robin boundary conditions", Proc. Amer. Math. Soc. **127** (1999), 125-130.

[2] G. Auchmuty, "Variational Principles for Self-adjoint Elliptic Eigenproblems", in Nonsmooth/Nonconvex Mechanics, ed Gao, Ogden and Stavroulakis, Kluwer, Dordrecht (2001) 15-42.

[3] G. Auchmuty, "Steklov Eigenproblems and the Representation of Solutions of Elliptic Boundary Value Problems", Numerical Functional Analysis and Optimization, **25** (2004) 321-348.

[4] G. Auchmuty, "Spectral Characterizations of the Trace Spaces $H^s(\partial\Omega)$ ", SIAM J Math Analysis **38** (2006), 894-905.

[5] C. Bandle *Isoperimetric Inequalities and Applications*, Pitman, London 1980.

[6] C. Bandle, J. von Below and W. Reichl, "Parabolic problems with dynamical boundary conditions: eigenvalue expansions and blow-up", Rend. Lincei Mat Appl. **17** (2006), 35-67.

[7] C. Bandle, "A Rayleigh-Faber-Krahn Inequality and some Monotonicity Properties for Eigenvalue Problems with Mixed Boundary Conditions", Internat. Ser. Numer. Math., 157, Birkhuser, Basel, (2009), 3-12.

[8] F. Belgacem, *Elliptic Boundary Value Problems with indefinite weights*, Research Notes in Math., Vol 368, Pitman (1997).

[9] B. Belinskiy, "Eigenvalue problems for Elliptic type partial differential operators with spectral parameter contained linearly in boundary conditions", in *Partial Differential and Integral Equations* ed Begehr et al, Kluwer (1999).

[10] P. Blanchard and E. Brüning, *Variational Methods in Mathematical Physics*, Springer Verlag, Berlin (1992).

[11] D. Daners, "Robin boundary value problems on arbitrary domains", Trans Amer. Math. Soc. **352** (2000), 4207-4236.

[12] D. Daners, "Inverse Positivity for General Robin Problems on Lipschitz Domains", Arch. Math. (Basel) 92 (2009), no. 1, 5769.

[13] D.G. de Figueiredo, "Positive Solutions of Semilinear Elliptic Problems", Lecture Notes in Mathematics Vol 957, Springer (1982), 35-87.

[14] E. DiBenedetto, *Real Analysis*, Birkhauser, Boston, (2001).

[15] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton (1992).

[16] J. Gregory, *Quadratic Form Theory and Differential Equations*, Academic Press, N.Y., (1980).

[17] P. Grisvard, *Elliptic problems in non-smooth domains* Pitman, Boston, (1985).

[18] M. Hestenes, "Applications of the Theory of Quadratic Forms in Hilbert Space to the Calculus of Variations", Pagific J. of Mathematics **1**, (1951), 525-581.

[19] B. Kazmierczak and T. Lipniacki, "Regulation of Kinase Activity by Diffusion and Feedback", J. Theoretical Biology, **259** (2009) 291-296.

[20] Hans F. Weinberger, *Variational Methods for Eigenvalue Approximation*, SIAM Philadelphia, (1974).

[21] E. Zeidler, *Nonlinear Functional Analysis and its Applications, Volume IIA:* , Springer Verlag, New York (1985).

Department of Mathematics, University of Houston, Houston, Tx 77204-3008 USA

E-mail address: `auchmuty@uh.edu`