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SINGULARITIES OF GENERIC LIGHTCONE GAUSS MAPS AND LIGHTCONE PEDAL SURFACES OF SPACELIKE CURVES IN MINKOWSKI 4-SPACE

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Abstract. The main goal of this paper is to study singularities of generic lightcone Gauss maps and lightcone pedal surfaces of spacelike curves in Minkowski 4-space. To do this, we first construct lightcone height functions and extended lightcone height functions, and then show the relations between singularities of generic lightcone Gauss maps(resp. lightcone pedal surfaces) and that of lightcone height functions(resp. extended lightcone height functions). In addition some geometric properties of the spacelike curves are studied from geometrical point of view.

1. Introduction

The study of Minkowski 4-space has produced fruitful results, see for example [5, 6, 7, 8]. Motivated by the study of generic lightcone Gauss maps of spacelike curves in Minkowski 4-space, and completing the study of submanifolds in Minkowski 4-space from the singularity theory point of view, we develop the study of singularities of generic lightcone Gauss maps and lightcone pedal surfaces of spacelike curves in Minkowski 4-space. To do this we need to work out local differential geometry tools for the spacelike curves similar to those of curves in Euclidean space[1, 2, 3, 4]. As it was expected, the situation presents certain peculiarities when it is compared with the Euclidean space and the Minkowski 3-space. For instance, the dimension of lightlike normal vector space is 2, then

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there exist many lightcone Gauss maps and lightcone pedal curves. For some basic notions in Lorentzian geometry, see [9].

The paper is organized as follows:

The rest of this section introduces the basic notions of Minkowski 4-space, after these basic notions, we give the local differential geometry of spacelike curves and the main results of this paper. Section 2 first constructs the lightcone height functions, which are useful tools for the study of singularities of lightcone Gauss maps, and then shows the relations between singularities of lightcone Gauss maps and that of lightcone height functions. Sections 3 and 4 give the proofs of the main results of this paper.

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) | x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be a 4-dimensional vector space. For any two vectors $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$. We call $(\mathbb{R}^4, \langle , \rangle)$ a *Minkowski* 4-*space* and denote it by $\mathbb{R}^4_1$.

We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^4_1 \backslash \{0\}$ is a *spacelike vector*, a *lightlike vector* or a *timelike vector* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$ is positive, zero, negative respectively. For a vector $\boldsymbol{n} \in \mathbb{R}^4_1$ and a real number c, a *hyperplane* with pseudo normal $\boldsymbol{n}$ is defined by $LHP(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^4_1 | \langle \boldsymbol{x}, \boldsymbol{n} \rangle = c\}$. The norm of a vector $\boldsymbol{x} \in \mathbb{R}^4_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. For any two vectors $\boldsymbol{x}$, $\boldsymbol{y}$ in $\mathbb{R}^4_1$, we say that $\boldsymbol{x}$ is *pseudo-perpendicular* to $\boldsymbol{y}$ if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$.

An orientation and a timelike orientation of $\mathbb{R}^4_1$ are fixed(i.e., a 4-volume form dV, and future time-like vector field, have been chosen). Let $\gamma : I \to \mathbb{R}^4_1$ be a smooth regular curve in $\mathbb{R}^4_1$ (i.e.,$\dot{\gamma}(t) \neq 0$ for any $t \in I$), where I is an open interval. We say that a smooth regular curve γ is a *spacelike curve* if $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle > 0$ for any $t \in I$. The *arclength* of a spacelike curve γ, measured from $\gamma(t_0)(t_0 \in I)$, is

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(t)\| \, dt.$$

Then a parameter s is determined such that $\|\gamma'(s)\| = 1$, where $\gamma'(s) = d\gamma/ds(s)$. Consequently we say that a spacelike curve γ is *parameterized by arclength* if $\|\gamma'(s)\| = 1$. Throughout the rest of this paper s is assumed arclength parameter. Let $\boldsymbol{t}(s)$ denote $\gamma'(s)$. We call $\boldsymbol{t}(s)$ a *unit tangent vector* of γ at s. The *signature* of $\boldsymbol{x}$ is defined to be

$$\delta(\boldsymbol{x}) = \operatorname{sign}(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} : \text{spacelike;} \\ 0 & \boldsymbol{x} : \text{lightlike;} \\ -1 & \boldsymbol{x} : \text{timelike .} \end{cases}$$

For any $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \mathbb{R}^4_1$, we define a vector $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 = \begin{vmatrix} -e_1, & e_2, & e_3, & e_4 \\ x_1^1, & x_1^2, & x_1^3, & x_1^4 \\ x_2^1, & x_2^2, & x_2^3, & x_2^4 \\ x_3^1, & x_3^2, & x_3^3, & x_3^4 \end{vmatrix}$$

$$= \left(-\begin{vmatrix} x_1^2, & x_1^3, & x_1^4 \\ x_2^2, & x_2^3, & x_2^4 \\ x_3^2, & x_3^3, & x_3^4 \end{vmatrix}, -\begin{vmatrix} x_1^1, & x_1^3, & x_1^4 \\ x_2^1, & x_2^3, & x_2^4 \\ x_3^1, & x_3^3, & x_3^4 \end{vmatrix}, \begin{vmatrix} x_1^1, & x_1^2, & x_1^4 \\ x_2^1, & x_2^2, & x_2^4 \\ x_3^1, & x_3^2, & x_3^4 \end{vmatrix}, -\begin{vmatrix} x_1^1, & x_1^2, & x_1^3 \\ x_2^1, & x_2^2, & x_2^3 \\ x_3^1, & x_3^2, & x_3^3 \end{vmatrix} \right)$$

where $\boldsymbol{x}_i = (x_i^1, x_i^2, x_i^3, x_i^4)$.

For a unit speed spacelike curve $\gamma : I \to \mathbb{R}^4_1$ with $\|\gamma''(s)\| \neq 0$ and $||\boldsymbol{n}_1'(s) + \delta_1 k_1(s)\boldsymbol{t}(s)|| \neq 0$, then we can construct a pseudo-orthogonal frame $\{\boldsymbol{t}(s), \boldsymbol{n}_1(s), \boldsymbol{n}_2(s), \boldsymbol{n}_3(s)\}$, which satisfies the following Frenet-Serret type formulae, of $\mathbb{R}^4_1$ along γ.

$$\begin{cases} \boldsymbol{t}(s) & = & \gamma'(s); \\ \boldsymbol{t}'(s) & = & k_1(s)\boldsymbol{n}_1(s); \\ \boldsymbol{n}_1'(s) & = & -\delta_1 k_1(s)\boldsymbol{t}(s) + k_2(s)\boldsymbol{n}_2(s); \\ \boldsymbol{n}_2'(s) & = & \delta_3 k_2(s)\boldsymbol{n}_1(s) + k_3(s)\boldsymbol{n}_3(s); \\ \boldsymbol{n}_3'(s) & = & \delta_1 k_3(s)\boldsymbol{n}_2(s), \end{cases}$$

where $k_1(s) = ||\gamma''(s)||$, $\boldsymbol{n}_1(s) = \frac{\gamma''(s)}{k_1(s)}$, $\boldsymbol{n}_2(s) = \frac{\boldsymbol{n}_1'(s)+\delta_1 k_1(s)\boldsymbol{t}(s)}{||\boldsymbol{n}_1'(s)+\delta_1 k_1(s)\boldsymbol{t}(s)||}$, $\boldsymbol{n}_3(s) = \boldsymbol{t}(s) \wedge \boldsymbol{n}_1(s) \wedge \boldsymbol{n}_2(s)$, and $\delta_i = \delta(\boldsymbol{n}_i(s))$ $(i = 1, 2, 3)$. This is the fundamental formula for the study of generic curves in $\mathbb{R}^3_1$; It is, however, useless at the point $\gamma(s)$ with $||\boldsymbol{n}_1'(s)+\delta_1 k_1(s)\boldsymbol{t}(s)|| = 0$. We now denote $A(s) = \boldsymbol{n}_1'(s)+\delta_1 k_1(s)\boldsymbol{t}(s)(s)$ and $C(s) = \boldsymbol{t}(s) \wedge \boldsymbol{n}_1(s) \wedge A(s)$. If $k_2(s) = 0$, then $A(s)$ is a lightlike vector, so that any pseudo perpendicular vector in the plane normal to $\boldsymbol{t}$ and $\boldsymbol{n}_1$ is parallel to $A(s)$.

On the other hand, there exists a lightlike vector $B(s)$ such that $\langle A(s), B(s)\rangle = 1$, $\langle \boldsymbol{t}(s), B(s)\rangle = 0$, $\langle \boldsymbol{n}_1(s), B(s)\rangle = 0$ and $\{\boldsymbol{t}(s), \boldsymbol{n}_1(s), A(s), B(s)\}$ is a basis of $\mathbb{R}^4_1$.

Let

$$LC_p = \{\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_1 | -(x_1 - p_1)^2 + \sum_{i=2}^{4} (x_i - p_i)^2 = 0\}$$

and

$$S^2_+ = \{\boldsymbol{x} \in LC_0 | \boldsymbol{x} = (1, x_2, x_3, x_4)\},$$

where $p = (p_1, p_2, p_3, p_4) \in \mathbb{R}^4_1$. We call $LC^*_p = LC_p \setminus \{p\}$ a *lightcone* at the vertex p and S^2_+ a *lightlike unite sphere*. If $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ is in LC^*_0, we then have $\widetilde{\boldsymbol{x}} = (1, x_2/x_1, x_3/x_1, x_4/x_1) \in S^2_+$.

If $k_2(s) \neq 0$, for any vector $\boldsymbol{v} = (\lambda_1 \boldsymbol{n}_1 + \mu_1 \boldsymbol{n}_2 + \eta_1 \boldsymbol{n}_3)(s) \in S^2_+ \cap N_s\gamma$, then

$$\begin{cases} \delta_1 \lambda_1^2 + \delta_2 \mu_1^2 + \delta_3 \eta_1^2 = 0; \\ \lambda_1 \boldsymbol{n}_{11}(s) + \mu_1 \boldsymbol{n}_{21}(s) + \eta_1 \boldsymbol{n}_{31}(s) = 1. \end{cases}$$

Thus locally there exist f_1 and g_1 such that $\boldsymbol{v} = (\pm\lambda_1 \boldsymbol{n}_1 + f(\lambda_1)\boldsymbol{n}_2 + g(\lambda_1)\boldsymbol{n}_3)(s)$. where $\lambda_1 \in \mathbb{R}$, $\boldsymbol{n}_a = (n_{a1}, n_{a2}, n_{a3}, n_{a4})(\ a = 1, 2, 3)$.

If $k_2(s) = 0$, for any vector $\boldsymbol{v} = (\lambda_2 \boldsymbol{n}_1 + \mu_2 A + \eta_2 B)(s) \in S^2_+ \cap N_s\gamma$, we have

$$\begin{cases} \lambda_2^2 + 2\mu_2\eta_2 = 0; \\ \lambda_2 \boldsymbol{n}_{11}(s) + \mu_2 A_1(s) + \eta_2 B_1(s) = 1. \end{cases}$$

Thus locally there exist f_2 and g_2 such that $\boldsymbol{v} = (\pm\lambda_2 \boldsymbol{n}_1 + f_2(\lambda_2)A + g_2(\lambda_2)B)(s)$, where $\lambda_2 \in \mathbb{R}$, $\boldsymbol{n}_1 = (n_{11}, n_{12}, n_{13}, n_{14})$, $A = (A_1, A_2, A_3, A_4)$ $B = (B_1, B_2, B_3, B_4)$. When $\boldsymbol{n}_1$ is a spacelike vector, for any vector

$$\boldsymbol{v}_0 = \begin{cases} (\sigma\lambda_0 \boldsymbol{n}_1 + f_1(\lambda_0)\boldsymbol{n}_2 + g_1(\lambda_0)\boldsymbol{n}_3)(s), & k_2(s) \neq 0; \\ (\sigma\lambda_0 \boldsymbol{n}_1 + f_2(\lambda_0)A + g_2(\lambda_0)B)(s), & k_2(s) = 0. \end{cases}$$

in $S^2_+ \cap N_s\gamma$, there exists a rotation map $\rho : S^2_+ \cap N_s\gamma \to S^2_+ \cap N_s\gamma$ such that $\rho(v_0) = \widetilde{A \pm C}(s)$.

We define surface

$$LG^\sigma_\gamma : I \times \mathbb{R} \to S^2_+,$$

by

$$LG^\sigma_\gamma(s, \lambda) = \begin{cases} (\sigma\lambda \boldsymbol{n}_1 + f_1(\lambda)\boldsymbol{n}_2 + g_1(\lambda)\boldsymbol{n}_3)(s), & k_2(s) \neq 0; \\ (\sigma\lambda \boldsymbol{n}_1 + f_2(\lambda)A + g_2(\lambda)B)(s), & k_2(s) = 0. \end{cases}$$

and surface

$$LP^\sigma_\gamma : I \times \mathbb{R} \to LC^*_0,$$

by

$$LP^\sigma_\gamma(s, \lambda) = \langle \gamma(s), \rho(\boldsymbol{v}(s, \lambda)) \rangle \boldsymbol{v}(s, \lambda),$$

where

$$\boldsymbol{v}(s, \lambda) = \begin{cases} (\sigma\lambda \boldsymbol{n}_1 + f_1(\lambda)\boldsymbol{n}_2 + g_1(\lambda)\boldsymbol{n}_3)(s), & k_2(s) \neq 0; \\ (\sigma\lambda \boldsymbol{n}_1 + f_2(\lambda)A + g_2(\lambda)B)(s), & k_2(s) = 0. \end{cases}$$

We call LG^σ_γ the *lightcone Gauss surface* of γ, LP^σ_γ the *lightcone pedal surface* of γ. In fact, for any fixed $\eta_0 \in \mathbb{R}$, $LG^\sigma_\gamma(-, \eta_0)$, denoted by $LG^\sigma_{\gamma,\eta_0}$ is a lightcone Gauss map and $LP^\sigma_\gamma(-, \eta_0)$, denoted by $LP^\sigma_{\gamma,\eta_0}$ is a lightcone pedal curve of γ.

The study of lightcone Gauss surface of γ is of course a very interesting aspect of the situation, from which one may deduce deep results concerning the curve

γ. In the next step, we shall study that surface. Classically, starting with curve theory, we shall consider the lightcone Gauss map in this paper.

Let $F : \mathbb{R}^4_1 \to \mathbb{R}$ be a submersion and γ a unit speed spacelike curve. We say that γ and $F^{-1}(0)$ have *k-point contact at* s_0 provided the function $g(s) = F\circ\gamma(s)$ satisfies $g(s_0) = g'(s_0) = \cdots = g^{(k-1)}(s_0) = 0$ and $g^{(k)}(s_0) \neq 0$. Dropping the condition $g^{(k)}(s_0) \neq 0$ we say that there is *at least k-point contact.*

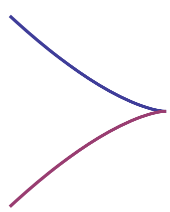
Let $\gamma : S^1 \to \mathbb{R}^4_1$ be a spacelike curve with $k_1(s) \neq 0$. We consider the following properties of γ.

(A_1) The number of points at which γ and hyperplane have 4-point contact is finite.

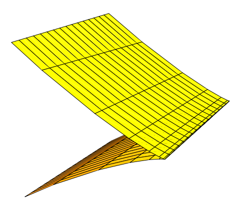
(A_2) There is no point at which γ and hyperplane have at least 5-point contact.

Our main results are the following.

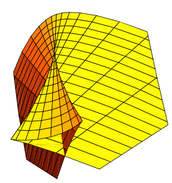
Theorem A *Let $Im(S^1, \mathbb{R}^4_1)$ be a space of spacelike curves equipped with Whitney C^∞-topology. Then the set of spacelike curves that satisfy (A_1) and (A_2) is a residual set in $Im(S^1, \mathbb{R}^4_1)$.*



cusp



cuspidal edge



swallowtail

Theorem B *Under the assumptions of (A_1) and (A_2),*

(a) *The lightcone Gauss map $LG^\sigma_{\gamma,\eta_0}$ of γ has a cusp point at s_0 if and only if $k_2(s_0) = 0$.*

(b) *The lightcone pedal surface LP^σ_γ is locally diffeomorphic to the cuspidal edage if and only if γ and hyperplane $LHP(\rho(\boldsymbol{v}^\sigma_0), c^\sigma_0)$ have 3-point contact.*

(c) *The lightcone pedal surface LP^σ_γ is locally diffeomorphic to the swallowtail if and only if $k_2(s_0) = 0$ (or γ and hyperplane $LHP(\rho(\boldsymbol{v}^\sigma_0), c^\sigma_0)$ have 4-point contact).*

Here

$$\boldsymbol{v}_0^\sigma = \begin{cases} (\sigma\lambda\boldsymbol{n}_1 + f_1(\lambda)\boldsymbol{n}_2 + g_1(\lambda)\boldsymbol{n}_3)(s_0), & k_2(s_0) \neq 0; \\ (\sigma\lambda\boldsymbol{n}_1 + f_2(\lambda)A + g_2(\lambda)B)(s_0), & k_2(s_0) = 0, \end{cases}$$

$c_0^\sigma = \langle \gamma(s_0), \rho(\boldsymbol{v}_0^\sigma)\rangle$ *and* $\sigma = +$ *or* $-$.

2. Lightcone height function on spacelike curve in $\mathbb{R}^4_1$

We first define a function

$$H : I \times S^2_+ \cap N\gamma \to \mathbb{R}$$

by $H(s,\boldsymbol{v}) = \langle \gamma(s), \rho(\boldsymbol{v})\rangle$. We call it the *lightcone height function* of γ. For any fixed $\boldsymbol{v}_0 \in S^2_+ \cap N\gamma$, we let $h_{v_0}(s)$ denote $H(s,\boldsymbol{v}_0)$. Then we have the following proposition.

Proposition 2.1. *Let* $\gamma : I \to \mathbb{R}^4_1$ *be a unit speed spacelike curve with* $k_1(s) \neq 0$. *Then:*

(1) $h'_{v_0}(s_0) = 0$ *if and only if* $\rho(\boldsymbol{v}_0) \in N_{s_0}\gamma$.

(2) $h'_{v_0}(s_0) = h''_{v_0}(s_0) = 0$ *if and only if*

$$\boldsymbol{v}_0 = \begin{cases} (\sigma\lambda_0\boldsymbol{n}_1 + f_1(\lambda_0)\boldsymbol{n}_2 + g_1(\lambda_0)\boldsymbol{n}_3)(s_0), & k_2(s_0) \neq 0; \\ (\sigma\lambda_0\boldsymbol{n}_1 + f_2(\lambda_0)A + g_2(\lambda_0)B)(s_0), & k_2(s_0) = 0. \end{cases}$$

or $\rho(v_0) = \widetilde{B}(s_0)$.

(3) $h'_{v_0}(s_0) = h''_{v_0}(s_0) = h'''_{v_0}(s_0) = 0$ *if and only if*

$$\boldsymbol{v}_0 = \begin{cases} (\sigma\lambda_0\boldsymbol{n}_1 + f_1(\lambda_0)\boldsymbol{n}_2 + g_1(\lambda_0)\boldsymbol{n}_3)(s_0), & k_2(s_0) \neq 0; \\ (\sigma\lambda_0\boldsymbol{n}_1 + f_2(\lambda_0)A + g_2(\lambda_0)B)(s_0), & k_2(s_0) = 0. \end{cases}$$

and $k_2(s_0) = 0$.

(4) $h'_{v_0}(s_0) = h''_{v_0}(s_0) = \cdots = h^{(4)}_{v_0}(s_0) = 0$ *if and only if*

$$\boldsymbol{v}_0 = \begin{cases} (\sigma\lambda_0\boldsymbol{n}_1 + f_1(\lambda_0)\boldsymbol{n}_2 + g_1(\lambda_0)\boldsymbol{n}_3)(s_0), & k_2(s_0) \neq 0; \\ (\sigma\lambda_0\boldsymbol{n}_1 + f_2(\lambda_0)A + g_2(\lambda_0)B)(s_0), & k_2(s_0) = 0. \end{cases}$$

and $k_2(s_0) = \langle A'(s_0), A(s_0)\rangle = 0$.

Proof. (1) $h'_{v_0}(s_0) = 0$ if and only if $\langle \gamma'(s_0), \rho(\boldsymbol{v}_0)\rangle = \langle \boldsymbol{t}(s_0), \rho(\boldsymbol{v}_0)\rangle = 0$, which is equivalent to $\rho(\boldsymbol{v}_0) \in N_{s_0}\gamma$.

(2) $h'_{v_0}(s_0) = h''_{v_0}(s_0) = 0$ if and only if $\langle \gamma'(s_0), \rho(\boldsymbol{v}_0)\rangle = \langle \gamma''(s_0), \rho(\boldsymbol{v}_0)\rangle = 0$, which is equivalent to $\rho(\boldsymbol{v}_0) \notin \langle \boldsymbol{t}(s_0), \boldsymbol{n}_1(s_0)\rangle_{\mathbb{R}}$. If $\boldsymbol{n}_1$ is a timelike vector, then $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ are spacelike vector. On the other hand, $\rho(\boldsymbol{v}_0) \in S^2_+ \cap N\gamma$, this is a

contradiction. So $\boldsymbol{n}_1$ is not a timelike vector. Thus $h'_{v_0}(s_0) = h''_{v_0}(s_0) = 0$ if and only if

$$\boldsymbol{v}_0 = \begin{cases} (\sigma\lambda_0 \boldsymbol{n}_1 + f_1(\lambda_0)\boldsymbol{n}_2 + g_1(\lambda_0)\boldsymbol{n}_3)(s_0), & k_2(s_0) \neq 0; \\ (\sigma\lambda_0 \boldsymbol{n}_1 + f_2(\lambda_0)A + g_2(\lambda_0)B)(s_0), & k_2(s_0) = 0. \end{cases}$$

or $\rho(v_0) = \widetilde{B}(s_0)$.

(3) If $k_2 \neq 0$, then $A(s_0)$ is not a lightlike vector. On the other hand, $h'_{v_0}(s_0) = h''_{v_0}(s_0) = h'''_{v_0}(s_0) = 0$ if and only if $\langle(\delta_1 k_1^2 \boldsymbol{t} + k_1{}'\boldsymbol{n}_1 - k_1 k_2 \boldsymbol{n}_2)(s_0), \rho(\boldsymbol{v}_0)\rangle = k_1 k_2 \delta_2(s_0) = 0$, it's a contradiction. So $k_2(s_0) = 0$. Thus $h'_{v_0}(s_0) = h''_{v_0}(s_0) = h'''_{v_0}(s_0) = 0$ if and only if

$$\boldsymbol{v}_0 = \begin{cases} (\sigma\lambda_0 \boldsymbol{n}_1 + f_1(\lambda_0)\boldsymbol{n}_2 + g_1(\lambda_0)\boldsymbol{n}_3)(s_0), & k_2(s_0) \neq 0; \\ (\sigma\lambda_0 \boldsymbol{n}_1 + f_2(\lambda_0)A + g_2(\lambda_0)B)(s_0), & k_2(s_0) = 0. \end{cases}$$

and $k_2(s_0) = 0$.

(4) $h^{(4)}_{v_0}(s_0) = 0$ if and only if $\langle\gamma^{(4)}(s_0), \rho(\boldsymbol{v}_0)\rangle = \langle(3\delta_1 k_1 k_1{}'\boldsymbol{t} + (\delta_1 k_1^3 + k_1{}'' + \delta_3 k_1 k_2^2)\boldsymbol{n}_1 + (-2k_1^{'} k_2 - k_1 k_2^{'})\boldsymbol{n}_2 + k_1 k_2 k_3 \boldsymbol{n}_3)(s_0), \rho(\boldsymbol{v}_0)\rangle = 0$, by (1), (2) and (3), $h'_{v_0}(s_0) = h''_{v_0}(s_0) = \cdots = h^{(4)}_{v_0}(s_0) = 0$ if and only if

$$\boldsymbol{v}_0 = \begin{cases} (\sigma\lambda_0 \boldsymbol{n}_1 + f_1(\lambda_0)\boldsymbol{n}_2 + g_1(\lambda_0)\boldsymbol{n}_3)(s_0), & k_2(s_0) \neq 0; \\ (\sigma\lambda_0 \boldsymbol{n}_1 + f_2(\lambda_0)A + g_2(\lambda_0)B)(s_0), & k_2(s_0) = 0. \end{cases}$$

and $k_2(s_0) = \langle A'(s_0), A(s_0)\rangle = 0$. $\square$

Proposition 2.2. *Let $\gamma : I \to \mathbb{R}^4_1$ be a unit speed spacelike curve. If H is the lightcone height function of γ. Then the following conditions are equivalent:*

(1) *s_0 is a singularity of the lightcone Gauss map $LG^{\pm}_{\gamma,\eta_0}$.*

(2) *$h'''_{v_0}(s_0) = 0$ for $\boldsymbol{v}_0 = LG^{\pm}_{\gamma,\eta_0}(s_0)$.*

(3) *$k_2(s_0) = 0$ and $\boldsymbol{v}_0 = LG^{\pm}_{\gamma,\eta_0}(s_0)$.*

PROOF. Let $B_H = \{\boldsymbol{v} \in S^2_+ \mid h'_v(s) = h''_v(s) = 0\}$. By Proposition 2.1. Then B_H can be written as $B_H = \{\boldsymbol{v} \in S^2_+ \mid \boldsymbol{v}\}$, where

$$\boldsymbol{v} = \begin{cases} (\sigma\lambda_0 \boldsymbol{n}_1 + f_1(\lambda_0)\boldsymbol{n}_2 + g_1(\lambda_0)\boldsymbol{n}_3)(s), & k_2(s) \neq 0; \\ (\sigma\lambda_0 \boldsymbol{n}_1 + f_2(\lambda_0)A + g_2(\lambda_0)B)(s), & k_2(s) = 0. \end{cases}$$

or $\rho(\boldsymbol{v}) = \widetilde{B}(s)$. Thus B_H can be identified with $LG^{\pm}_{\gamma,\eta_0}$ from singularity theory viewpoint, which means (1) is equivalent to (2).

By Proposition 2.1, (2) is equivalent to (3). $\square$

Proposition 2.3. *Let $\gamma : I \to \mathbb{R}^4_1$ be a unit speed spacelike curve. Then:*

(1) *The lightcone Gauss map $LG^{\sigma}_{\gamma,\eta_0}$ is constant if and only if $\gamma \subset LHP(\boldsymbol{v}^{\sigma}, c^{\sigma})$.*

(2) *Both maps, LG^{+}_{γ,η_0} and LG^{-}_{γ,η_0}, are constant if and only if γ is a plane curve.*

Here

$$\boldsymbol{v}^{\sigma}=\begin{cases}(\sigma\lambda_0\boldsymbol{n}_1+f_1(\lambda_0)\boldsymbol{n}_2+g_1(\lambda_0)\boldsymbol{n}_3)(s), & k_2(s)\neq 0;\\ (\sigma\lambda_0\boldsymbol{n}_1+f_2(\lambda_0)A+g_2(\lambda_0)B)(s), & k_2(s)=0.\end{cases}$$

$c^{\sigma}=\langle\gamma(s),\boldsymbol{v}^{\sigma}\rangle$ *is constant, and* $\sigma=+$ *or* $-$.

Proof. (1) If $LG^{\sigma}_{\gamma,\eta_0}$ is constant, then

$$d\langle\gamma,LG^{\sigma}_{\gamma,\eta_0}\rangle=\langle d\gamma,LG^{\sigma}_{\gamma,\eta_0}\rangle+\langle\gamma,d(LG^{\sigma}_{\gamma,\eta_0})\rangle=0.$$

Therefore $\langle\gamma,LG^{\sigma}_{\gamma,\eta_0}\rangle=c^{\sigma}$ is constant, which means that $\gamma\subset LHP(\boldsymbol{v}^{\sigma},c^{\sigma})$. Conversely, If $\gamma\subset LHP(\boldsymbol{v}^{\sigma},c^{\sigma})$, then $\langle\gamma,dLG^{\sigma}_{\gamma,\eta_0}\rangle=0$, thus $LG^{\sigma}_{\gamma,\eta_0}$ is constant.

(2) Since $\boldsymbol{v}^{+}$ and $\boldsymbol{v}^{-}$ are linearly independent, $LHP(\boldsymbol{v}^{-},c^{-})$ and $LHP(\boldsymbol{v}^{+},c^{+})$ intersect transversally. By (1), both maps, LG^{+}_{γ,η_0} and LG^{-}_{γ,η_0}, are constant if and only if $\gamma(s)\subset LHP(\boldsymbol{v}^{+},c^{+})\cap LHP(\boldsymbol{v}^{-},c^{-})$ is a plane curve. □

For a unit speed spacelike curve $\gamma:I\to\mathbb{R}^4_1$, we now define *extended lightcone height function* $\widetilde{H}:I\times LC^*_0\cap N\gamma\to\mathbb{R}$ by $\widetilde{H}(s,\boldsymbol{v})=H(s,\widetilde{\boldsymbol{v}})-\boldsymbol{v}_1=\langle\gamma(s),\rho(\widetilde{\boldsymbol{v}})\rangle-\boldsymbol{v}_1$ where H is the lightcone height function of γ. For any $\boldsymbol{v}_0$ in $LC^*_0\cap N\gamma$, let $\widetilde{h}_{v_0}(s)$ denote $\widetilde{H}(s,\boldsymbol{v}_0)$. Then we have the following lemma.

Lemma 2.4. *Let $\gamma:I\to\mathbb{R}^4_1$ be a unit speed spacelike curve with $k_1(s)\neq 0$. Then γ and the hyperplane $LHP(\rho(\boldsymbol{v}^{\pm}_0),c^{\pm}_0)$ have 4-point contact at s_0 if and only if $k_2(s_0)=0$ and $\langle A'(s_0),A(s_0)\rangle\neq 0$, where*

$$\boldsymbol{v}^{\pm}_0=\begin{cases}(\sigma\lambda_0\boldsymbol{n}_1+f_1(\lambda_0)\boldsymbol{n}_2+g_1(\lambda_0)\boldsymbol{n}_3)(s_0), & k_2(s_0)\neq 0;\\ (\sigma\lambda_0\boldsymbol{n}_1+f_2(\lambda_0)A+g_2(\lambda_0)B)(s_0), & k_2(s_0)=0.\end{cases}$$

and $c^{\pm}_0=\langle\gamma(s_0),\rho(\boldsymbol{v}^{\pm}_0)\rangle$.

3. Unfoldings of functions of one-variable

In this section we use some general results of the singularity theory for function germs. Details can be found in [3]. Let $F:(\mathbb{R}\times\mathbb{R}^r,(s_0,x_0))\to\mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f where $f(s)=F(s,x_0)$. We say that f has A_k*-singularity* at s_0 if $f^{(p)}(s_0)=0$ for all $1\leq p\leq k$ and $f^{(k+1)}(s_0)\neq 0$. Let $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s,x_0))(s_0)=\sum_{j=1}^{k-1}\alpha_{ji}s^j$ for $i=1,\dots,r$, where j^{k-1} denotes the $(k-1)$-jet. F is called a (p) *versal unfolding* (resp. *versal*) if the $(k-1)\times r$ (resp. $k\times r$) matrix of coefficients (α_{ji}) (resp. $(\alpha_{0i},\alpha_{ji})$) has rank $k-1(k-1\leq r)$(resp. $k(k\leq r)$), where $\alpha_{0i}=\frac{\partial F}{\partial x_i}(s_0,x_0)$. Before proceeding further, it is convenient to introduce two important sets concerning the unfoldings. The *bifurcation set* of

F is the set $B_F = \{x \in \mathbb{R}^r | \frac{\partial F}{\partial s} = \frac{\partial^2 F}{\partial^2 s} = 0 \ at \ (s,x)\}$. The *discriminant set* of F is the set $D_F = \{x \in \mathbb{R}^r | F = \frac{\partial F}{\partial s} = 0 \ at \ (s,x)\}$. Then we have the following well-known result[4].

Theorem 3.1. *Let $F : (\mathbb{R}\times\mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be a r-parameter unfolding of $f(s)$, which has A_k-singularity at s_0.*
(1) *Suppose that F is a (p) versal unfolding.*
 (a) *If $k = 2$, then B_F is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*
 (b) *If $k = 3$, then B_F is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.*
(2) *Suppose that F is a versal unfolding.*
 (a) *If $k = 1$, then D_F is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*
 (b) *If $k = 2$, then D_F is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.*
 (c) *If $k = 3$, then D_F is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.*
Here, $SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the swallowtail and $C = \{(x_1, x_2) | {x_1}^2 = {x_2}^3\}$ is the ordinary cusp.

Theorem 3.2. *Let $\gamma : I \to \mathbb{R}^4_1$ be a unit speed spacelike curve with $k_1(s) \neq 0$, H the lightcone height function of γ and $\widetilde{H}$ the extended lightcone height function.*
 (1) *If $h(s)$ has A_k-singularity $(k = 2, 3)$ at s_0, then H is the (p) versal unfolding of h.*
 We consider the point $(s, \boldsymbol{v}) \in I \times LC^*_0 \cap N_s\gamma$ such that $\widetilde{H}(s, \boldsymbol{v}) = 0$.
 (2) *If $\widetilde{h}(s)$ has A_k-singularity $(k = 1, 2, 3)$ at s_0, then $\widetilde{H}$ is the versal unfolding of $\widetilde{h}$.*

Proof. Let $LC^*_+ = \{\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in LC^*_0 \mid v_1 > 0\}$; $\rho(\boldsymbol{v}) = (v_1, v_2, v_3, v_4) \in LC^*_+$; $\gamma(s) = (x_1(s), x_2(s), x_3(s), x_4(s))$. We define a function $\bar{H} : I \times LC^*_+ \to \mathbb{R}$ by $\bar{H}(s, \boldsymbol{v}) = \langle \gamma(s), \rho(\boldsymbol{v})\rangle$. Since $\bar{H} \mid_{I\times S^2_+ \cap N\gamma} = H$, it is sufficient to verify that $\bar{H}$ is the (p) versal unfolding of $\bar{h}(s) = \bar{H}(s, \boldsymbol{v}_0)$. In fact, since $\rho(\boldsymbol{v}) \in LC^*_+$, $\bar{H}(s, \boldsymbol{v}) = -\sqrt{v_2^2 + v_3^2 + v_4^2}x_1(s) + v_2 x_2(s) + v_3 x_3(s) + v_4 x_4(s)$. $\frac{\partial \bar{H}}{\partial v_2} = -\frac{v_2}{v_1}x_1(s) + x_2(s)$, $\frac{\partial \bar{H}}{\partial v_3} = x_3(s) - (\frac{v_3}{v_1})x_1(s)$, $\frac{\partial \bar{H}}{\partial v_4} = x_4(s) - (\frac{v_4}{v_1})x_1(s)$, the 3-jet at s_0 of $\frac{\partial \bar{H}}{\partial v_2}$, $\frac{\partial \bar{H}}{\partial v_3}$ and $\frac{\partial \bar{H}}{\partial v_4}$ are

$$\begin{aligned}
&(sx_2' + \tfrac{1}{2}s^2x_2'' + \tfrac{1}{6}s^3x_2''') - (\tfrac{v_2}{v_1})(sx_1' + \tfrac{1}{2}s^2x_1'' + \tfrac{1}{6}s^3x_1'''),\\
&(sx_3' + \tfrac{1}{2}s^2x_3'' + \tfrac{1}{6}s^3x_3''') - (\tfrac{v_3}{v_1})(sx_1' + \tfrac{1}{2}s^2x_1'' + \tfrac{1}{6}s^3x_1'''),\\
&(sx_4' + \tfrac{1}{2}s^2x_4'' + \tfrac{1}{6}s^3x_4''') - (\tfrac{v_4}{v_1})(sx_1' + \tfrac{1}{2}s^2x_1'' + \tfrac{1}{6}s^3x_1''').
\end{aligned}$$

$\bar{h}(s)$ has A_2-singularity at s_0 if and only if $\rho(\boldsymbol{v}_0) = \lambda(\boldsymbol{n}_2(s_0) \pm \boldsymbol{n}_3(s_0)), \lambda \neq 0$ and $k_2(s_0) \neq 0$. When $\bar{h}$ has A_2-singularity at s_0, we require matrix $(x_2'(s) - (\frac{v_2}{v_1})x_1'(s), x_3'(s) - (\frac{v_3}{v_1})x_1'(s), x_4'(s) - (\frac{v_4}{v_1})x_1'(s))$ to have rank 1. In fact, Since

$\rho(\boldsymbol{v}) = \lambda(\boldsymbol{n}_2 \pm \boldsymbol{n}_3), k_1k_2 \neq 0$, the determinant of following 3×3 matrix

$$\begin{vmatrix} x_2^{'} - (\frac{v_2}{v_1})x_1^{'} & x_3^{'} - (\frac{v_3}{v_1})x_1^{'} & x_4^{'} - (\frac{v_4}{v_1})x_1^{'} \\ \frac{1}{2}x_2^{''} - (\frac{v_2}{2v_1})x_1^{''} & \frac{1}{2}x_3^{''} - (\frac{v_3}{2v_1})x_1^{''} & \frac{1}{2}x_4^{''} - (\frac{v4}{2v_1})x_1^{''} \\ \frac{1}{6}x_2^{'''}(\frac{v_2}{6v_1})x_1^{'''} & \frac{1}{6}x_3^{'''} - (\frac{v_3}{6v_1})x_1^{'''} & \frac{1}{6}x_4^{'''} - (\frac{v_4}{6v_1})x_1^{'''} \end{vmatrix}$$

$$= \frac{1}{12v_1}\begin{vmatrix} v_1 & v_2 & v_3 & v_4 \\ x_1^{'} & x_2^{'} & x_3^{'} & x_4^{'} \\ x_1^{''} & x_2^{''} & x_3^{''} & x_4^{''} \\ x_1^{'''} & x_2^{'''} & x_3^{'''} & x_4^{'''} \end{vmatrix} = \frac{k_1^2k_2}{12v_1}\langle\rho(\boldsymbol{v}), \boldsymbol{n}_3\rangle$$

is $\pm\frac{\delta_3\lambda k_1^2k_2}{12v_1} \neq 0$, which means that the rank of the above matrix is 3. Whence $\bar{H}$ is the (p) versal unfolding of $\bar{h}(s) = \bar{H}(s, \boldsymbol{v}_0)$. Similarly for others. $\square$

By Proposition 2.1, Theorems 3.1 and 3.2, we have the following theorem.

Theorem 3.3. *If h_v and $\widetilde{h}_w$ have A_k-singularity$(k = 2, 3)$. Then:*

(1) *When $k = 2$, B_H is locally diffeomorphic to $0 \times \mathbb{R}$, $D_{\widetilde{H}}$ is locally diffeomorphic to $C \times \mathbb{R}$.*

(2) *When $k = 3$, B_H is locally diffeomorphic to C. $D_{\widetilde{H}}$ is locally diffeomorphic to SW.*

By Proposition 2.1 and Theorem 3.3, we get the Theorem B.

4. Generic properties of spacelike curves

In this section we consider the notion of Lorentzian Monge-Taylor map of a spacelike curve in Minkowski 4-space. Let $\gamma : I \to \mathbb{R}_1^4$ be a regular spacelike curve in Minkowski 4-space where I is an open connected subset of unit circle S^1. We now choose a smooth family of unit vectors $\boldsymbol{n}_j(t)$, it is pseudo perpendicular to the unit tangent vector $\boldsymbol{t}(t)$ of γ at t, so $\|\boldsymbol{n}_j(t)\| = 1$ and $\langle\boldsymbol{n}_j(t), \boldsymbol{t}(t)\rangle = 0$ for all $t \in I$. Such $\boldsymbol{n}_j(t)$ can be obtained as following: consider the smooth map $\boldsymbol{t} : I \to S_1^3$ which takes t to the unit tangent vector $\boldsymbol{t}(t)$, if V is a vector in S_1^3, we can obtain the vector field $\boldsymbol{n}_j(t)$ by pseudo-orthogonally projecting V onto each of the pseudo normal space and normalizing. Thus $\boldsymbol{n}_j(t) = \frac{V - \langle V, \boldsymbol{t}(t)\rangle\boldsymbol{t}(t)}{\|V - \langle V, \boldsymbol{t}(t)\rangle\boldsymbol{t}(t)\|}$. $\boldsymbol{n}_i(t)$ is obtained similarly:

$$\boldsymbol{n}_i(t) = \frac{W - \langle W, \boldsymbol{n}_j(t)\rangle\boldsymbol{n}_j(t) - \langle W, \boldsymbol{t}(t)\rangle\boldsymbol{t}(t)}{\|W - \langle W, \boldsymbol{n}_j(t)\rangle\boldsymbol{n}_j(t) - \langle W, \boldsymbol{t}(t)\rangle\boldsymbol{t}(t)\|},$$

where $W \in H^3(H^3 = \{p \in \mathbb{R}_1^4 | \langle p, p\rangle = -1\})$. Let $\boldsymbol{n}_k(t) = \boldsymbol{t}(t) \wedge \boldsymbol{n}_j(t) \wedge \boldsymbol{n}_i(t)$. We use the pseudo perpendicular lines spanned by $\boldsymbol{t}(t), \boldsymbol{n}_j(t), \boldsymbol{n}_i(t), \boldsymbol{n}_k(t)$ as axes at $\gamma(t)$ with the unit points on the axes corresponding to the four given vectors.

Note that the curve γ is not necessarily unit speed with $\gamma(t_0) = 0$. We can write $\gamma(I)$ locally as $(g_t(\xi), \xi, f_t(\xi), l_t(\xi)$ with $j^1 f_t(0) = j^1 g_t(0) = j^1 l_t(0) = 0$. If V_k denotes the space of polynomials in ξ of degree ≥ 2 and $\leq k$, then we have a map, the Monge-Taylor map for the spacelike curve γ, $\mu_\gamma : I \to V_k \times V_k \times V_k$ by $\mu_\gamma(t) = (j^k f_t(0), j^k g_t(0), j^k l_t(0))$. $V_k \times V_k \times V_k$ can be identified with $\mathbb{R}_1^{k-1} \times \mathbb{R}_1^{k-1} \times \mathbb{R}_1^{k-1} = \mathbb{R}_1^{3k-3}$ via the coordinate $(a_2, ..., a_k, b_2, ..., b_k, c_2, ..., c_k)$. Of course μ_γ depends rather heavily on our choice of unit normals $\boldsymbol{n}_j(t)$ and $\boldsymbol{n}_i(t)$. Here $a_i = \frac{f_t^{(i)}(0)}{i!}$, $b_i = \frac{g_t^{(i)}(0)}{i!}$, $c_i = \frac{l_t^{(i)}(0)}{i!}(2 \leq i \leq k)$, that is $V_k \times V_k \times V_k = \{(a_2\xi^2 + a_3\xi^3 + \cdots + a_k\xi^k), (b_2\xi^2 + b_3\xi^3 + \cdots + b_k\xi^k), (c_2\xi^2 + c_3\xi^3 + \cdots + c_k\xi^k)\}$. Let P_k denote the set of maps $\psi : \mathbb{R}_1^4 \to \mathbb{R}_1^4$ of the form $\psi(x, y, z, w) = ((\psi_1(x, y, z, w), \psi_2(x, y, z, w), \psi_3(x, y, z, w), \psi_4(x, y, z, w))$, where $\psi_i(x, y, z, w)$ is a polynomial in x, y, z, w of degree $\leq k$. An element $\psi \in P_k$ is determined by the coefficients of the various monomials $x^i y^j z^m w^n$ occurring in $\psi_1, \psi_2, \psi_3, \psi_4$. There are altogether $\frac{(k+4)!}{4!k!}$ monomials $x^i y^j z^m w^n$ of degree $\leq k$, so that P_k can be thought as a Minkowski space $\mathbb{R}_1^{\frac{(k+4)!}{4!k!}}$. It is this space that will provide the required deformations of the curve.

To simplify matters we now assume that the curve γ is compact, that is $I = S^1$. The identity map $1_{\mathbb{R}_1^4} : \mathbb{R}_1^4 \to \mathbb{R}_1^4$ is of course an element of $P_k (k \geq 1)$. By using compactness of γ, it is easy to see that there is an open neighborhood U of $1_{\mathbb{R}_1^4}$ in P_k with the property that if $\psi \in U$, then the linear map $T\psi(\gamma(t)) : \mathbb{R}_1^4 \to \mathbb{R}_1^4; \boldsymbol{v} \longmapsto D\psi(\gamma(t)) \cdot \boldsymbol{v}$ satisfies that it takes a timelike vector(resp. a spacelike vector) to a timelike vector (resp. a spacelike vector), where $D\psi(\gamma(t))$ denotes the derivative of ψ at $\gamma(t)$. If we deform the original curve by the map ψ, then we can also obtain the required two new smooth family of normal vectors $\boldsymbol{n}_{i\psi(t)}, \boldsymbol{n}_{j\psi(t)}$ as follows. Since the map $\psi : \mathbb{R}_1^4 \to \mathbb{R}_1^4$ is a diffeomorphism on some open set containing $\gamma(I)$, vectors $\boldsymbol{n}_j(t), \boldsymbol{n}_i(t)$ will be sent to some new vectors $D\psi(\gamma(t))\boldsymbol{n}_j(t), D\psi(\gamma(t))\boldsymbol{n}_i(t)$, which will be neither zero nor tangent to $\psi \circ \gamma$ at t. Pseudo-orthogonally project $D\psi(\gamma(t))\boldsymbol{n}_j(t), D\psi(\gamma(t))\boldsymbol{n}_i(t)$ onto the pseudo-normal space to $\psi \circ \gamma$ at t and normalize, then we get $\boldsymbol{n}_{j\psi}(t) = \frac{D\psi(\gamma(t))\boldsymbol{n}_j(t) - \langle D\psi(\gamma(t))\boldsymbol{n}_j(t), \boldsymbol{t}_\psi \rangle \boldsymbol{t}_\psi}{\| D\psi(\gamma(t))\boldsymbol{n}_j(t) - \langle D\psi(\psi(t))\boldsymbol{n}_j t), \boldsymbol{t}_\psi \rangle \boldsymbol{t}_\psi \|}$, $\langle \boldsymbol{n}_{j\psi}(t), \boldsymbol{n}_{j\psi}(t) \rangle = 1$,

$$\boldsymbol{n}_{i\psi}(t) = \frac{D\psi(\gamma(t))\boldsymbol{n}_i(t) - \langle D\psi(\gamma(t))\boldsymbol{n}_i(t), \boldsymbol{n}_{j\psi} \rangle \boldsymbol{n}_{j\psi} - \langle D\psi(\gamma(t))\boldsymbol{n}_i(t), \boldsymbol{t}\psi \rangle \boldsymbol{t}\psi}{\| D\psi(\gamma(t))\boldsymbol{n}_i(t) - \langle D\psi(\gamma(t))\boldsymbol{n}_i(t), \boldsymbol{n}_{j\psi} \rangle \boldsymbol{n}_{j\psi} - \langle D\psi(\gamma(t))\boldsymbol{n}_i(t), \boldsymbol{t}\psi \|}$$

and $\langle \boldsymbol{n}_{i\psi}(t), \boldsymbol{n}_{i\psi}(t) \rangle = -1$, where $\boldsymbol{t}_\psi$ denotes the tangent vector of the curve $\psi \circ \gamma$ at t. Assuming as before that $I = S^1$, we choose an open neighborhood U of $1_I \in P_k$ consisting of polynomial maps which map an open set containing $\gamma(S^1)$ diffeomorphic to its image. We have now shown that there is a smooth map

$$\mu : S^1 \times U \to V_k \times V_k \times V_k$$

defined by $\mu(-,\psi) =$ Monge-Taylor map for the curve $\psi \circ \gamma$ using the family of pseudo-normal vectors $\boldsymbol{n}_{j\psi}(t)$ and $\boldsymbol{n}_{i\psi}(t)$. By the same arguments as in the proof of Theorem 9.9 in [4], we have the following theorem.

Theorem 4.1. *Let Q be a manifold in $V_k \times V_k \times V_k = \mathbb{R}^{3k-3}_1$. For some open set $U_1 \subset U$ containing identity map, the map $\mu : S^1 \times U_1 \to V_k \times V_k \times V_k$ defined by $\mu(t,\psi) = \mu_{\psi\circ\gamma}(t)$ is transverse to Q(In fact, μ is a submersion, so that Q does not enter the argument at all).*

A straightforward computation shows the following lemma. The computations are rather long and tedious, so we omit the details.

Lemma 4.2. *Let γ be a unit speed spacelike curve defined by*

$$\gamma(t) = (g_t(\xi), \xi, f_t(\xi), l_t(\xi)) = (b_2\xi^2 + b_3\xi^3 + \cdots, \xi, a_2\xi^2 + a_3\xi^3 + \cdots, c_2\xi^2 + c_3\xi^3 + \cdots)$$

with $\xi(t_0) = 0$ and $k_1(t_0) \neq 0$. We let $N_{ij} = i!j!(a_ia_j - b_ib_j + c_ic_j)$. Then:

(1) *$f_1(a_2, a_3, b_2, b_3, c_2, c_3) = 0$ at t_0 if and only if $k_2(t_0) = 0$, where $f_1 = N_{32}^2 + N_{22}^3 - N_{33}N_{22}$.*

(2) *$f_2(a_2, a_3, a_4, b_2, b_3, b_4, c_2, c_3, c_4) = 0$ at t_0 if and only if $\langle A', A\rangle(t_0) = 0$, where $f_2 = N_{22}^3N_{34} - 2N_{32}N_{22}^2N_{33} + N_{22}N_{32}[3N_{32}^2 - N_{22}(N_{42} + N_{33}) - N_{22}^3] - N_{32}N_{22}^3N_{31} - N_{22}^2N_{32}N_{42} + 2N_{32}^3N_{22} - N_{32}N_{22}[3N_{32}^2 - N_{22}(N_{42} + N_{33}) - N_{22}^3] + N_{32}^2N_{22}^2N_{21} - N_{22}^4N_{41} + 2N_{32}N_{22}^3N_{31} - N_{22}^2N_{21}[3N_{32}^2 - N_{22}(N_{42} + N_{33}) - N_{22}^3] + N_{32}N_{22}^4N_{11}$.*

Here, ξ is the coordinate along the $\boldsymbol{t}$-direction, $f_t(\xi)$ is the coordinate along the $\boldsymbol{n}_j$-direction, $g_t(\xi)$ is the coordinate along the $\boldsymbol{n}_i$-direction, $l_t(\xi)$ is the coordinate along the $\boldsymbol{n}_k$-direction.

Lemma 4.3. *We consider smooth maps $\rho_i : V_4 \times V_4 \times V_4 = \mathbb{R}^9 \to \mathbb{R}(i = 1, 2)$ given by*

$$\begin{cases} \rho_1 = & N_{32}^2 + N_{22}^3 - N_{33}N_{22}; \\ \rho_2 = & N_{22}^3N_{34} - 2N_{32}N_{22}^2N_{33} + N_{22}N_{32}[3N_{32}^2 - N_{22}(N_{42} + N_{33}) - N_{22}^3] \\ & -N_{32}N_{22}^3N_{31} - N_{22}^2N_{32}N_{42} + 2N_{32}^3N_{22} \\ & -N_{32}N_{22}[3N_{32}^2 - N_{22}(N_{42} + N_{33}) - N_{22}^3] + N_{32}^2N_{22}^2N_{21} \\ & -N_{22}^4N_{41} + 2N_{32}N_{22}^3N_{31} - N_{22}^2N_{21}[3N_{32}^2 - N_{22}(N_{42} + N_{33}) - N_{22}^3] \\ & +N_{32}N_{22}^4N_{11}. \end{cases}$$

Then the set $Q_1 = \{(a_2, a_3, a_4, b_2, b_3, b_4, c_2, c_3, c_4) \in \mathbb{R}^9 \mid \rho_1 = \rho_2 = 0\}$ is a codimension two submanifold in $\mathbb{R}^9$.

An *ordinary(resp. degenerate) vertex* is a point $p=\gamma(t_0)$ of a spacelike curve γ, for which there exists a hyperplane having 4-(resp. at least 5-) point contact with the curve for $t=t_0$. We say γ has a vertex at t_0, or at p.

We now give the proof of Theorem A.

PROOF. Suppose we are given any regular spacelike curve $\gamma : S^1 \to \mathbb{R}^4_1$. Applying Theorem 4.1 to the map μ with Q the submanifold of degenerate vertexes proves that a dense set of curves have only ordinary vertexes. Further taking $k=4$ and Q to be the submanifold of $f_1(t)=0$. Then the vertexes of $\psi\circ\gamma$ correspond to points t with $\mu(t,\psi)\in Q$. However, again by Theorem 4.1, we know that for a dense set of $\psi\in U_1$ the map $\mu(-,\psi)=\nu$ is transverse to Q. Consequently the set $\nu^{-1}(Q)$ of the ordinary vertexes is finite for a dense set of $\psi\in U_1$, whence the result.

We now have to prove that these properties are open. We first show that this is so for the property of only having ordinary vertexes. Let Q denote the set $f_1=f_2=0$ in $V_4\times V_4\times V_4$(degenerate vertexes). Thus Q is closed. Let $\widetilde{\gamma} : S^1\times U \to \mathbb{R}^4_1$ be a family of curves with $\widetilde{\gamma}_0=\widetilde{\gamma}(-,0)$ having only ordinary vertexes. Let $\mu : S^1\times U \to V_k\times V_k\times V_k$ be the corresponding family of Monge-Taylor map. Then the compactness of S^1, together with the fact that $\mu_0=\mu(-,0)$ is transverse to Q(misses Q in fact), implies by [4, Proposition 8.23] that $\mu(S^1\times\{u\})$ misses Q for u in some open neighborhood U' of 0. Hence nearby curves $\widetilde{\gamma}_u$ in the family also possess no degenerate vertex.

It remains to show that, in fact, the property of having finitely many ordinary vertexes and no degenerate vertex is open. First note that, if γ has an ordinary vertex at $t\in S^1$, then the image of the map $\mu : S^1\to V_4\times V_4\times V_4$ meets the submanifold of $f_1=0$ at $\mu(t)$ and is transverse to this submanifold.

Let $\widetilde{\gamma} : S^1\times U \to \mathbb{R}^4_1$ be a family of curves with $\widetilde{\gamma}_0$ having finitely many ordinary vertexes, and no degenerate vertex, so that $\mu(-,0)$ is transverse to the submanifold of $f_1=0$. Since transversality is an open condition when the source is compact and the relevant submanifold are closed[4, Proposition 8.23], it follows that $\nu=\mu(-,u) : S^1\to V_4\times V_4\times V_4$ will also transverse to that submanifold for all u in some neighborhood U_1 of $0\in U$. Consequently, if Q is that submanifold, the set $\nu^{-1}(Q)$ of ordinary vertexes of $\widetilde{\gamma}_u$ is finite and there is no degenerate vertex. This proves the result. $\square$

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