A Mathematical Framework for Stochastic Climate Models

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Abstract

There has been a recent burst of activity in the atmosphere-ocean sciences community in utilizing stable linear Langevin stochastic models for the unresolved degrees of freedom in stochastic climate prediction. Here a systematic mathematical strategy for stochastic climate modeling is developed, and some of the new phenomena in the resulting equations for the climate variables alone are explored. The new phenomena include the emergence of both unstable linear Langevin stochastic models for the climate mean variables and the need to incorporate both suitable nonlinear effects and multiplicative noise in stochastic models under appropriate circumstances. All of these phenomena are derived from a systematic self-consistent mathematical framework for eliminating the unresolved stochastic modes that is mathematically rigorous in a suitable asymptotic limit. The theory is illustrated for general quadratically nonlinear equations where the explicit nature of the stochastic climate modeling procedure can be elucidated. The feasibility of the approach is demonstrated for the truncated equations for barotropic flow with topography. Explicit concrete examples with the new phenomena are presented for the stochastically forced three-mode interaction equations. The conjecture of Smith and Waleffe [*Phys. Fluids* **11** (1999), 1608–1622] for stochastically forced three-wave resonant equations in a suitable regime of damping and forcing is solved as a byproduct of the approach. Examples of idealized climate models arising from the highly inhomogeneous equilibrium statistical mechanics for geophysical flows are also utilized to demonstrate self-consistency of the mathematical approach with the predictions of equilibrium statistical mechanics. In particular, for these examples, the reduced stochastic modeling procedure for the climate variables alone is designed to reproduce both the climate mean and the energy spectrum of the climate variables.

Contents

1. Introduction 892
2. Basic Strategy for Stochastic Climate Modeling 894

Communications on Pure and Applied Mathematics, Vol. LIV, 0891–0974 (2001)

3. Stochastic Modeling for the Truncated Barotropic Equations 897
4. Consistent Reduced Stochastic Equations for the Climate Variables Alone 902
5. The Effect of Fast-Wave Averaging in Stochastic Climate Models 913
6. Idealized Climate Models from Equilibrium Statistical Mechanics 927
7. New Phenomena in Low-Order Triad Models 948
8. Concluding Discussion 968
Appendix A. Explicit Properties of the Ornstein-Uhlenbeck Operator 970
Appendix B. The General Derivation 971
Bibliography 973

1 Introduction

An area with great importance for future developments in climate prediction involves simplified stochastic modeling of nonlinear features of the coupled atmosphere/ocean system. The practical reasons for such needs are easy to understand. In the foreseeable future, it will be impossible to resolve the effects of the coupled atmosphere/ocean system through computer models with detailed resolution of the atmosphere on decadal time scales. However, the questions of interest also change. For example, for climate prediction, one is not interested in whether there is a significant deflection of the storm track northward in the Atlantic during a specific week in January of a given year, but rather, whether the mean and variance of the storm track are large during several winter seasons and what the impact of this trend is on the overall poleward transport of heat in both the atmosphere and ocean. The idea of simplified stochastic modeling for unresolved space-time scales in climate modeling is over twenty years old and emerged from fundamental papers by Hasselman [10] and Leith [15]. In the atmosphere/ocean community, there is a recent flourishing of ideas utilizing simple stable linear Langevin stochastic equations to model and predict short-term and decadal climate changes such as El Niño [12, 21], the North Atlantic Oscillation [8, 22], and mid-latitude storm tracks [1, 3, 5, 23, 25] with notable positive results, but this simplified stochastic model has also failed in some circumstances [18].

In this paper, we develop a systematic mathematical strategy for stochastic climate modeling and also explore some of the new phenomena that occur in the resulting stochastic models. The key assumptions in the systematic theory developed below are that the climate variables in a given nonlinear system necessarily evolve on longer time scales than the unresolved variables and that the nonlinear interaction among unresolved variables can be represented stochastically in a suitable simplified fashion (see the detailed discussion in Section 2 of this paper). These two assumptions are implicit in much of the work in stochastic climate modeling mentioned above [1, 3, 5, 8, 10, 12, 15, 18, 21, 22, 23, 25]. In the mathematical approach developed here, once the climate variables are identified via zonal averaging [5], EOF expansions [1, 12, 21, 23], low-pass filtering in time [3, 18, 25], or some other procedure, with the above two assumptions, new closed nonlinear stochastic equations are derived for the climate variables alone on longer time scales. Several new phenomena occur through this systematic approach including the following:

- Systematic nonlinear corrections to the climate dynamics due to the interaction with the unresolved variables.
- The need for multiplicative stochastic noises besides additive noises for the climate variables. Such noises and their structure are deduced in a systematic fashion from the theory.
- Mathematical criteria and examples with unstable linear Langevin equations for the climate variables. Such examples with less stable stochastic models for the climate variables on a longer time scale indicate that interactions with the unresolved variables can diminish predictability in appropriate circumstances.

The theory allows for strong coupling between the climate variables and the unresolved variables. Furthermore, the predicted stochastic evolution equations for the climate variables are given quantitatively so the theory is effectively computable but much simpler than turbulence closure. The key mathematical idea in the systematic theory developed here is to borrow techniques from singular perturbation theory for Markov processes originally developed in the 1970's for limits of linear Boltzmann transport theory by Kurtz [13], Ellis and Pinsky [6], and Papanicolaou [19] who combined the methods in [6, 13] with those developed by Khasminsky [11] to allow for fast averaging. Although the applications to stochastic climate modeling developed here are completely different with several new phenomena and require several new concrete ideas, this connection to mathematical theory for stochastic processes guarantees that the results presented here are mathematically rigorous in a suitable asymptotic limit.

We summarize the contents of the remainder of this paper briefly. In Section 2 we present the basic strategy for stochastic climate modeling, which utilizes the two assumptions listed earlier for the important example of quadratically nonlinear equations. In Section 3 we introduce the equations for barotropic flow on a beta plane with topography and mean flow. This idealized climate model due to Leith [4, 15] provides a simple illustrative example in that section and throughout the remainder of the paper. In Section 4, we summarize the main results of this paper involving consistent reduced stochastic equations for the climate variables alone for the general quadratically nonlinear systems introduced in Section 2. We emphasize the different general phenomena that occur with wave–mean flow interaction or climate scattering interaction alone, which introduce systematic additive and multiplicative noises, respectively, as well as their combination through interaction. The new features that occur naturally in stochastic modeling with fast-wave averaging are developed in Section 5.

The implications of the theory for the truncated barotropic flow equations introduced in Section 3 are developed in Section 6. One key result presented there is systematic self-consistency of the theory developed here with equilibrium statistical mechanics. Also numerical simulations are presented in Section 6 for truncated barotropic flow with topography that demonstrate the validity of the key assumptions in Section 2; another different but related example has already been presented elsewhere by the authors [16]. In Section 7, the range of new phenomena are illustrated in simple explicit examples involving stochastically forced three-mode interaction equations and related examples, which also provide a pedagogical introduction to the general theory. In particular, in Sections 7.1 and 7.2, we solve the interesting conjecture of Smith and Waleffe [24] for stochastically forced three-wave resonant equations in a suitable regime of damping and forcing. The systematic strategy of effective calculation for the general theory is presented in two appendices in order to streamline the presentation.

2 Basic Strategy for Stochastic Climate Modeling

We illustrate the ideas for stochastic climate modeling on an abstract basic model involving quadratically nonlinear dynamics, which is very appropriate for modeling many aspects of atmospheric dynamics. In the abstract model, the unknown variable $\vec{z}$, generally complex, evolves in time in response to an external forcing term $\vec{F}(t)$, a linear operator $L\vec{z}$, and a quadratic or bilinear operator $B(\vec{z}, \vec{z})$, and satisfies

$$\frac{d\vec{z}}{dt} = \vec{f}(t) + L\vec{z} + B(\vec{z}, \vec{z})\,. \tag{2.1}$$

An important example of quadratically nonlinear equations of the type as in (2.1) that will be used as an illustration throughout this paper is given by the equations for barotropic flow on a beta plane with topography and mean flow:

$$\begin{gathered} \frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q + U\frac{\partial q}{\partial x} + \beta\frac{\partial \psi}{\partial x} = 0\,, \quad q = \Delta\psi + h\,, \\ \frac{dU}{dt} = \fint h\frac{\partial \psi}{\partial x}\,. \end{gathered} \tag{2.2}$$

Here $q(x, y, t)$ denotes the small-scale potential vorticity, $U(t)$ is the mean flow, $\psi(x, y, t)$ is the small-scale stream function, and $h(x, y)$ denotes the underlying topography, whereas β approximates the variation of the Coriolis parameter. The bar across the integral sign indicates normalization by the area of the domain of integration and $\nabla^{\perp} = (-\partial_y, \partial_x)$. The equations in (2.2) are discussed in detail in Sections 3, 5.1, and 6. A good general reference for the equations in (2.2) and their geophysical properties is Pedlosky's book [20]. The effects of an interactive mean flow U with topography are discussed in [4] and [17]. Hamiltonian chaos in exact solutions of (2.2) is discussed by Grote, Raggazzo, and one of the authors in [9].

In stochastic climate modeling, the variable $\vec{z}$ is decomposed into an orthogonal decomposition through the variables $\vec{x}$, $\vec{y}$ by $\vec{z} = (\vec{x}, \vec{y})$. The variable $\vec{x}$ denotes the climate state of the system; the climate state necessarily evolves slowly in time compared to the $\vec{y}$-variables, which evolve more rapidly in time and are not resolved in detail in the stochastic climate model. In practice, the climate variables $\vec{x}$ are determined by a variety of procedures, including leading-order empirical orthogonal functions (EOFs) [1, 12, 21, 23], zonal averaging in space [5], low-pass time filtering [3, 18, 25], or a combination of these procedures. Decomposing the dynamic equation in (2.1) by projecting on the $\vec{x}$- and $\vec{y}$-variables yields the equations

$$\begin{aligned} \frac{d\vec{x}}{dt} &= \vec{f}_1(t) + L_{11}\vec{x} + L_{12}\vec{y} + B^1_{11}(\vec{x},\vec{x}) + B^1_{12}(\vec{x},\vec{y}) + B^1_{22}(\vec{y},\vec{y}) \,, \\ \frac{d\vec{y}}{dt} &= \vec{f}_2(t) + L_{21}\vec{x} + L_{22}\vec{y} + B^2_{11}(\vec{x},\vec{x}) + B^2_{12}(\vec{x},\vec{y}) + B^2_{22}(\vec{y},\vec{y}) \,. \end{aligned} \tag{2.3}$$

Generally, stochastic climate modeling amounts to simplifying the dynamic equations in (2.3) by representing some of the terms involving the variables $\vec{y}$, which are not resolved in detail, by a linear stochastic model. This procedure is applied implicitly or explicitly in most of the works in the literature [1, 3, 5, 8, 10, 12, 15, 18, 21, 22, 23, 25]. In this paper, we systematically discuss this strategy so we shall assume that the explicit nonlinear self-interaction through $B^2_{22}(\vec{y},\vec{y})$ of the variables $\vec{y}$ can be represented by a linear stochastic operator. More precisely, we use the following:

Working assumption of stochastic modeling:

$$B^2_{22}(\vec{y},\vec{y})dt \approx -\frac{\Gamma}{\varepsilon}\vec{y}\,dt + \frac{\sigma}{\sqrt{\varepsilon}}\,d\vec{W}(t)\,, \quad 0 < \varepsilon \ll 1\,. \tag{2.4}$$

Here Γ, σ are positive definite matrices, and $\vec{W}(t)$ is a vector-valued Wiener process. The parameter ε measures the ratio of the correlation time of the unresolved variables $\vec{y}$ to the climate variables $\vec{x}$, and the requirement $\varepsilon \ll 1$ is very natural for stochastic climate models where the climate variables should change more slowly.

By (2.4), we assume that the nonlinear self-interactions can be modeled by an Ornstein-Uhlenbeck process. The choice of this particular process is not essential for the theory but is convenient for the calculations because of the full computability of the Ornstein-Uhlenbeck process (see Appendix A). We also note that the process defined through (2.4) has zero mean; there is no loss of generality in this assumption, since it can always be enforced by appropriate definition of the variables $\vec{y}$ and the various operators entering the equations in (2.3).

If we coarse-grain the equations in (2.3) with the approximation from (2.4) on a longer time scale, $t \to \varepsilon t$, to measure the slowly evolving climate variables, we obtain

$$
\begin{aligned}
d\vec{x} &= \frac{1}{\varepsilon}\Bigg(\vec{f}_1\left(\frac{t}{\varepsilon}\right) + L_{11}\vec{x} + L_{12}\vec{y} \\
&\qquad + B^1_{11}(\vec{x},\vec{x}) + B^1_{12}(\vec{x},\vec{y}) + B^1_{22}(\vec{y},\vec{y})\Bigg)dt\,, \\
d\vec{y} &= \frac{1}{\varepsilon}\Bigg(\vec{f}_2\left(\frac{t}{\varepsilon}\right) + L_{21}\vec{x} + L_{22}\vec{y} + B^2_{11}(\vec{x},\vec{x}) + B^2_{12}(\vec{x},\vec{y})\Bigg)dt \\
&\qquad - \frac{\Gamma}{\varepsilon^2}\vec{y}dt + \frac{\sigma}{\varepsilon}d\vec{W}(t)\,.
\end{aligned}
\tag{2.5}
$$

In fact, with a few additional assumptions that are well suited for climate modeling, which will be explained below, we derive from the equations in (2.5) the following:

Stochastic climate model:

$$
\begin{aligned}
d\vec{x} &= \vec{F}_1(t)dt + \frac{1}{\varepsilon}\vec{f}_1\left(\frac{t}{\varepsilon}\right)dt + D\vec{x}\,dt + \frac{1}{\varepsilon}\left(L_{11}\vec{x} + L_{12}\vec{y}\right)dt \\
&\qquad + B^1_{11}(\vec{x},\vec{x})dt + \frac{1}{\varepsilon}\left(B^1_{12}(\vec{x},\vec{y}) + B^1_{22}(\vec{y},\vec{y})\right)dt, \\
d\vec{y} &= \frac{1}{\varepsilon}\vec{f}_2\left(\frac{t}{\varepsilon}\right)dt + \frac{1}{\varepsilon}\left(L_{21}\vec{x} + L_{22}\vec{y} + B^2_{11}(\vec{x},\vec{x}) + B^2_{12}(\vec{x},\vec{y})\right)dt \\
&\qquad - \frac{\Gamma}{\varepsilon^2}\vec{y}dt + \frac{\sigma}{\varepsilon}d\vec{W}(t)\,.
\end{aligned}
\tag{2.6}
$$

To obtain the equations in (2.6), we first made the following modification to account more appropriately for various climate effects:

A0. We have included damping and forcing terms acting on the slow time scale in the equations for the climate variables. Thus, in (2.6) we have added a term $D\vec{x}$ and we have set

$$
\vec{f}_1\left(\frac{t}{\varepsilon}\right) \to \varepsilon\vec{F}_1(t) + \vec{f}_1\left(\frac{t}{\varepsilon}\right). \tag{2.7}
$$

We have also made the following additional assumptions:

A1. We assume that the forcing terms $\vec{f}_1$, $\vec{f}_2$ in the equations in (2.6) have zero mean with respect to time average

$$
\lim_{T\to\infty}\frac{1}{T}\int_0^T \vec{f}_1(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T \vec{f}_2(t)dt = 0\,. \tag{2.8}
$$

A2. In the equations in (2.6) for the climate variables, we assume that the linear operator $L_{11}\vec{x}$ accounts for fast-wave effects only; i.e., L_{11} is skew-symmetric.

A3. We assume that the nonlinear self-interaction of the climate variables is a slow-time-scale driving effect, i.e., in (2.6) we have set

$$
B^1_{11}(\vec{x},\vec{x}) \to \varepsilon B^1_{11}(\vec{x},\vec{x})\,. \tag{2.9}
$$

A4. We assume that the nonlinear interaction in $\vec{y}$ in the equations in (2.6) for the climate variables has zero expectation with respect to the invariant measure of the Ornstein-Uhlenbeck process in (2.4), i.e.,

$$\mathbf{P}B^1_{22}(\vec{y},\vec{y}) = 0\,. \tag{2.10}$$

The modification in A0 and assumption A1 for the climate-damping and the climate-forcing functions are very natural since external solar effects provide a systematic forcing on the annual cycle and many interesting climate-modeling problems involve accumulated effects over many decades. In the specific applications to the barotropic equations in Sections 5.1, 6, and 7, assumption A4 in (2.10) is trivially satisfied. This is the typical situation for many applications to geophysical flows.

On the other hand, as will be shown below, it follows from assumptions A1 through A4 that a stochastic model for the climate variables alone can be derived for $\varepsilon \ll 1$. More precisely, these assumptions ensure the existence of the limit as $\varepsilon \to 0$ of the equations for the climate variables in (2.6) because they imply that, asymptotically, there are no effects of order ε^{-1} on the climate variables induced by the various driving terms in (2.6) and, in particular, by the terms involving the unresolved variables. As such, assumptions A1 through A4 may be regarded as the very definition for the distinction between climate and unresolved variables, and the mathematical framework developed in this work is the effective tool that will allow us to explicitly derive the stochastic model for the climate variables alone for $\varepsilon \ll 1$.

3 Stochastic Modeling for the Truncated Barotropic Equations

In this section we demonstrate the feasibility of the general strategy for stochastic modeling introduced in Section 2 on the idealized climate model equations in (2.2) for a barotropic flow on a beta plane with topography and mean flow introduced by Leith [15]. These are especially attractive climate models because they are highly inhomogeneous yet involve both a well-defined mean climate state as well as an energy spectrum. In spherical geometry such models capture a number of large-scale features of the atmosphere [7].

We proceed in two steps. We first introduce a finite-dimensional truncation of the barotropic equations in (2.2), which we call the truncated barotropic equations and are given in (3.6). These equations are well-suited for numerical simulations (see Section 6.3) and are readily shown to belong to the class of the abstract model in (2.1). Next, we introduce a stochastic model approximation for the truncated barotropic equations by appropriate identification of climate and unresolved variables and stochastic modeling of the nonlinear self-interaction of the unresolved variables. The stochastic model for the truncated barotropic equations is given in (3.17) and belongs to the class of the abstract model in (2.6).

The finite-dimensional truncation of the barotropic equations in (2.2) is obtained by making a Galerkin approximation where the equations are projected into a finite-dimensional subspace. Consistent with the two-dimensional periodic boundary conditions used in the numerical simulations, the truncation is readily accomplished with the standard Fourier basis. More precisely, we introduce the Fourier series expansion of the truncated small-scale stream function ψ_Λ, the truncated vorticity ω_Λ, and the truncated topography h_Λ in the term of the truncated basis $B_\Lambda = \{e^{i\vec{k}\cdot\vec{x}} : \vec{k} \in \sigma_\Lambda\}$, where $\sigma_\Lambda = \{\vec{k} : 1 \le |\vec{k}| \le \Lambda\}$,

$$\begin{gathered} \psi_\Lambda(\vec{x},t) = \sum_{1\le|\vec{k}|^2\le\Lambda} \hat{\psi}_k(t) e^{i\vec{k}\cdot\vec{x}}\,, \qquad h_\Lambda(\vec{x}) = \sum_{1\le|\vec{k}|^2\le\Lambda} \hat{h}_k e^{i\vec{k}\cdot\vec{x}}\,, \\ q_\Lambda(\vec{x},t) = \sum_{1\le|\vec{k}|^2\le\Lambda} \hat{q}_k(t) e^{i\vec{k}\cdot\vec{x}}\,. \end{gathered} \tag{3.1}$$

For simplicity of notation we omit the arrow for the subscripts: $k \equiv \vec{k}$. We have also assumed that the topography has zero mean with respect to spatial average; i.e., we have taken $\hat{h}_{(0,0)} = 0$. This condition ensures that the solvability condition for the steady-state equations associated with the equations in (2.2) is automatically satisfied. The amplitudes $\hat{\psi}_k$, $\hat{\omega}_k$, and $\hat{h}_k$ satisfy the reality conditions

$$\hat{\psi}_k^* = \hat{\psi}_{-k}\,, \quad \hat{\omega}_k^* = \hat{\omega}_{-k}\,, \quad \hat{h}_k^* = \hat{h}_{-k}\,. \tag{3.2}$$

Denote by P_Λ the orthogonal projector onto the finite-dimensional space V_Λ spanned by the basis B_Λ. The truncated barotropic equations are obtained by projecting the original barotropic equations in (2.2) on V_Λ:

$$\begin{gathered} \frac{\partial q_\Lambda}{\partial t} + P_\Lambda(\nabla^\perp\psi_\Lambda \cdot \nabla q_\Lambda) + U\frac{\partial q_\Lambda}{\partial x} + \beta\frac{\partial \psi_\Lambda}{\partial x} = 0\,, \\ q_\Lambda = \omega_\Lambda + h_\Lambda\,, \quad \omega_\Lambda = \Delta\psi_\Lambda\,, \quad \frac{dU}{dt} = ⨍ h_\Lambda \frac{\partial \psi_\Lambda}{\partial x}\,. \end{gathered} \tag{3.3}$$

For the remainder of this section, it will be convenient to work with the amplitude associated with velocity rather than with $\hat{\psi}_k(t)$, and we define

$$u_k(t) = |\vec{k}|\hat{\psi}_k(t)\,, \tag{3.4}$$

where u_k satisfies $u_k^* = u_{-k}$. The equation for the potential vorticity q_Λ in (3.3) is readily solved in terms of u_k as

$$\hat{q}_k(t) = -|\vec{k}|u_k(t) + \hat{h}_k\,. \tag{3.5}$$

Substituting (3.4) and (3.5) into the equations in (3.3) for ψ_Λ and U, we obtain a finite-dimensional system of ordinary differential equations for the Fourier coefficients with $\vec{k} \in \sigma_\Lambda$. We will refer to the equations in this system as the truncated barotropic equations.

Truncated barotropic equations:

$$\frac{dU}{dt} = \mathrm{Im} \sum_{\vec{k} \in \sigma_\Lambda} H_k u_k^*,$$

$$\frac{du_k}{dt} = i H_k U - i(k_x U - \Omega_k) u_k + \sum_{\vec{l} \in \sigma_\Lambda} L_{kl} u_l + \frac{1}{2} \sum_{\substack{\vec{l}, \vec{m} \in \sigma_\Lambda \\ \vec{k}+\vec{l}+\vec{m}=0}} B_{klm} u_l^* u_m^* , \tag{3.6}$$

where

$$L_{kl} = \frac{(k_x l_y - k_y l_x)}{|\vec{k}||\vec{l}|} h_{k-l} , \qquad B_{klm} = (l_y m_x - l_x m_y) \frac{|\vec{l}|^2 - |\vec{m}|^2}{|\vec{k}||\vec{l}||\vec{m}|} , \tag{3.7}$$

and

$$\Omega_k = \frac{k_x \beta}{|\vec{k}|^2} , \qquad H_k = \frac{k_x \hat{h}_k}{|\vec{k}|} . \tag{3.8}$$

Notice that L_{kl} is skew-symmetric,

$$L_{kl} = -L_{lk}^* . \tag{3.9}$$

Since the terms accounting for linear coupling between U and u_k are also skew-symmetric, it follows that assumption A2 is automatically satisfied for the truncated barotropic equations.

The equations in (3.6) can be written as a system of ordinary differential equations with real coefficients of the type of the equations in (2.1) upon defining

$$\vec{z} = (U, a_{k_1}, b_{k_1}, a_{k_2}, b_{k_2}, \dots) , \tag{3.10}$$

where $a_k = \mathrm{Re}\, u_k$, $b_k = \mathrm{Im}\, u_k$. The $\vec{k}_j$'s span the set $\bar{\sigma}_\Lambda \subset \sigma_\Lambda$, where $\bar{\sigma}_\Lambda$ is an arbitrary subset of σ_Λ such that the set of equations for $\vec{z}$ are complete using the reality condition $a_k = a_{-k}$, $b_k = -b_{-k}$ (in other words, if $\vec{k}_j \in \bar{\sigma}$, then $-\vec{k}_j \notin \bar{\sigma}$). With this notation, the equations in (3.6) can be written in more compact form as

$$\frac{d\vec{z}}{dt} = L\vec{z} + B(\vec{z}, \vec{z}) . \tag{3.11}$$

We now derive a stochastic model for the truncated barotropic equations in (3.6). Following our general strategy, we assume that the variables U, u_k in the truncated barotropic equations can be separated into climate variables and unresolved variables, depending on the time scale on which they evolve. For simplicity of presentation, we also assume that the climate variables are the mean flow U and the u_k's with $|\vec{k}| < \bar{\Lambda}$, corresponding to the large scales. Numerical support for these assumptions in some regimes of parameters is given in Section 6.3. For simplicity of identification, we will denote by v_k the u_k declared climate variables, i.e.,

$$v_k \equiv u_k \quad \text{for } \vec{k} \in \sigma_1 = \{\vec{k} : 1 \le |\vec{k}| < \bar{\Lambda}\} , \tag{3.12}$$

and by w_k the u_k declared unresolved variables, i.e.,

$$w_k \equiv u_k \quad \text{for } \vec{k} \in \sigma_2 = \{\vec{k} : \bar{\Lambda} \le |\vec{k}| \le \Lambda\} . \tag{3.13}$$

The next step is to make a stochastic model assumption as in (2.4) for the nonlinear self-interaction of the unresolved variables w_k. For the truncated barotropic equations for the unresolved variables w_k in (3.6), we will use the following stochastic model assumption, which generalizes to the complex case the model in (2.4):

$$\sum_{\vec{l}\in\sigma_2} L_{kl} w_l \, dt + \frac{1}{2} \sum_{\substack{\vec{l},\vec{m}\in\sigma_2 \\ \vec{k}+\vec{l}+\vec{m}=0}} B_{\vec{k}\vec{l}\vec{m}} w_l^* w_m^* \, dt \approx -\frac{1}{\varepsilon}\gamma_k (w_k - \bar{w}_k) dt + \frac{\sigma_k}{\sqrt{2\varepsilon}} (dW_k(t) + dW_{-k}^*(t)) , \tag{3.14}$$

where γ_k, σ_k are positive real parameters satisfying

$$\gamma_k = \gamma_{-k} , \qquad \sigma_k = \sigma_{-k} , \tag{3.15}$$

and the W_k's are independent Wiener processes satisfying

$$\mathbf{E} W_k(t) W_l^*(s) = 2\delta_{k,l} \min(t, s) , \quad \mathbf{E} W_k(t) W_l(s) = \mathbf{E} W_k^*(t) W_l^*(s) = 0 , \tag{3.16}$$

where $\mathbf{E}$ denotes the expectation over the statistics of the W_k's. (We assume that γ_k is real for the simplicity of presentation only; the present formalism generalizes easily to the situation with γ_k complex satisfying $\gamma_k^* = \gamma_{-k}$, as will be shown elsewhere by the authors.) The structure of the approximation in (3.14) together with the conditions in (3.15), (3.16) guarantee that the reality condition, $u_k^* = u_{-k}$, is automatically satisfied. Notice that in (3.14) we model both the nonlinear self-interaction of the unresolved variables w_k and the interaction between the w_k and the small-scale topography: The latter is modeled by the term $\sum_{\vec{l}\in\sigma_2} L_{kl} w_l$. This is consistent with our general strategy for stochastic modeling since both terms on the right-hand side of (3.14) account for the nonlinear self-interaction of the unresolved variables in terms of the original variables $\hat{\psi}_k$, $\hat{q}_k$, as can be seen from the first equation in (3.3).

We use assumption (3.14) in the truncated barotropic equation in (3.6) and coarse-grained time, $t \to \varepsilon t$. This gives the following:

Stochastic model for the barotropic equations:

$$dU = \frac{1}{\varepsilon} \operatorname{Im} \sum_{\vec{k}\in\sigma_1} H_k^* v_k \, dt + \frac{1}{\varepsilon} \operatorname{Im} \sum_{\vec{k}\in\sigma_1} H_k^* w_k \, dt \tag{3.17}$$

$$
\begin{aligned}
dv_k &= \frac{i}{\varepsilon} H_k U dt - i k_x \left(U - \frac{1}{\varepsilon}\Omega_k \right) v_k \, dt + \frac{1}{\varepsilon} \sum_{\vec{l} \in \sigma_1} L_{kl} v_l \, dt \\
&\quad + \frac{1}{\varepsilon} \sum_{\vec{l} \in \sigma_2} L_{kl} w_l \, dt + \frac{1}{2} \sum_{\substack{\vec{l}, \vec{m} \in \sigma_1 \\ \vec{k} + \vec{l} + \vec{m} = 0}} B_{klm} v_l^* v_m^* \, dt \\
&\quad + \frac{1}{\varepsilon} \sum_{\substack{\vec{l} \in \sigma_1, \vec{m} \in \sigma_2 \\ \vec{k} + \vec{l} + \vec{m} = 0}} B_{klm} v_l^* w_m^* \, dt + \frac{1}{2\varepsilon} \sum_{\substack{\vec{l}, \vec{m} \in \sigma_2 \\ \vec{k} + \vec{l} + \vec{m} = 0}} B_{klm} w_l^* w_m^* \, dt \, , \\
dw_k &= \frac{i}{\varepsilon} H_k U dt - \frac{i}{\varepsilon} (k_x U - \Omega_k) w_k \, dt + \frac{1}{\varepsilon} \sum_{\vec{l} \in \sigma_1} L_{kl} v_l \, dt \\
&\quad + \frac{1}{2\varepsilon} \sum_{\substack{\vec{l}, \vec{m} \in \sigma_1 \\ \vec{k} + \vec{l} + \vec{m} = 0}} B_{klm} v_l^* v_m^* \, dt + \frac{1}{\varepsilon} \sum_{\substack{\vec{l} \in \sigma_1, \vec{m} \in \sigma_2 \\ \vec{k} + \vec{l} + \vec{m} = 0}} B_{klm} v_l^* w_m^* \, dt \\
&\quad - \frac{1}{\varepsilon^2} \gamma_k (w_k - \bar{w}_k) dt + \frac{\sigma_k}{\sqrt{2\varepsilon}} (dW_k(t) + dW_{-k}^*(t)) \, ,
\end{aligned}
$$

where consistent with the assumption in A3, we have treated as slow effects the nonlinear self-interactions of the climate variables by setting

$$
\begin{aligned}
(3.18) \qquad -\frac{i}{\varepsilon} k_x U v_k dt + \frac{1}{2\varepsilon} \sum_{\substack{\vec{l}, \vec{m} \in \sigma_1 \\ \vec{k} + \vec{l} + \vec{m} = 0}} B_{klm} v_l^* v_m^* &\to \\
&- i k_x U v_k \, dt + \frac{1}{2} \sum_{\substack{\vec{l}, \vec{m} \in \sigma_1 \\ \vec{k} + \vec{l} + \vec{m} = 0}} B_{klm} v_l^* v_m^* \, .
\end{aligned}
$$

Notice that we can use a compact notation in (3.10) and identify the climate variables $\vec{x}$ as

$$
\vec{x} = (U, \mathrm{Re}\, v_{k_1}, \mathrm{Im}\, v_{k_1}, \mathrm{Re}\, v_{k_2}, \mathrm{Im}\, v_{k_2}, \dots) \, , \tag{3.19}
$$

where the $\vec{k}_j$'s span the set $\bar{\sigma}_1 = \{\vec{k} : \vec{k} \in \bar{\sigma} \text{ and } 1 \le |\vec{k}| < \bar{\Lambda}\}$, and the unresolved variables $\vec{y}$ as

$$
\vec{y} = (\mathrm{Re}\, \tilde{w}_{l_1}, \mathrm{Im}\, \tilde{w}_{l_1}, \mathrm{Re}\, \tilde{w}_{l_2}, \mathrm{Im}\, \tilde{w}_{l_2}, \dots) \, , \tag{3.20}
$$

where $\tilde{w}_l = w_l - \bar{w}_l$ and the $\vec{l}_j$'s span the set $\bar{\sigma}_2 = \{\vec{l} : \vec{l} \in \bar{\sigma} \text{ and } \bar{\Lambda} \le |\vec{l}| \le \Lambda\}$. In terms of $\vec{x}$, $\vec{y}$, the stochastic model for barotropic equations in (3.17) fits into the generic stochastic climate model in (2.6).

In Section 6, we illustrate an important variant of this model that is constrained through systematic principles to be automatically consistent with equilibrium statistical mechanics.

To conclude this section, it is worth pointing out that a rigorous derivation of the stochastic model for the barotropic equations in (3.17) from the original equations in (3.6) is beyond the scope of the mathematical framework developed in the present paper and involves the difficult issues of ergodicity and mixing in dynamical systems. The truncated barotropic equations in (3.6) and their associated stochastic model equations in (3.17) will be studied in more detail in Section 6. The relevance of the stochastic model for the barotropic equations in (3.17) will be verified numerically in Section 6.3 where we check the working assumption in (2.4) or (3.14) for several regimes of parameters. In Section 7, we will also study the stochastic model for the barotropic equations in (3.17) in some simple settings that demonstrate the appearance of new phenomena.

4 Consistent Reduced Stochastic Equations for the Climate Variables Alone

In this section we summarize our main results for stochastic model equations for the climate variables alone that are derived from the stochastic model equations in (2.6). In particular, we demonstrate the appearance of new phenomena relevant to stochastic climate modeling. In Section 4.1, we consider a generic model for wave–mean flow interaction, and we show that the effect of the unresolved wave variables on the climate mean flow variables is accounted for by linear Langevin terms that can be both stabilizing or destabilizing in contrast to what is usually assumed in the literature. Explicit criteria for instability are given. Furthermore, we show that the unresolved wave variables can modify the mean of the climate variables. In Section 4.2, we consider a generic model for climate scattering interaction. We show that, generally, the unresolved variables induce nonlinear corrections in the dynamics for the climate variables, as well as multiplicative noises, and the structure of these terms is deduced systematically from the theory. In Section 4.3, we consider the general stochastic model equations in (2.6) without fast-wave and fast-forcing effects, and we show that, generally, all kinds of effects as described in Sections 4.1 and 4.2 interact in the stochastic climate model. Section 4.4 contains the details about the systematic asymptotic strategy for elimination of the fast, unresolved variables in the cases where there are no fast-wave effects in the climate variables. Finally, in Section 4.5, we give an alternative, direct method for eliminating the unresolved variables in the special case where the equations for the unresolved variables are linear and diagonal in $\vec{y}$. Fast-wave effects will be considered in Section 5.

In the developments below, it will be convenient to have a more explicit representation for the equations in (2.6). To this end we represent the variable $\vec{z}$ with an index notation j running in a set σ, i.e., $\vec{z}(t) = \{z_j(t) : j \in \sigma\}$. The decomposition of $\vec{z}$ into climate and unresolved variables then amounts to splitting σ into two subsets σ_1 and σ_2 such that $\sigma = \sigma_1 \cup \sigma_2$ and the climate variables $\vec{x}(t)$ are those $\vec{z}(t)$ for which $j \in \sigma_1$, i.e., $\vec{x}(t) = \{x_j(t) : j \in \sigma_1\}$, whereas the unresolved variables $\vec{y}(t)$ are those $\vec{z}(t)$ for which $j \in \sigma_2$, i.e., $\vec{y}(t) = \{y_j(t) : j \in \sigma_2\}$. Usually the context makes it clear to which set an index belongs in any expression, and we only specify it explicitly in case of ambiguity. Thus we represent (2.6) as

$$
\begin{aligned}
dx_j &= F_j^1(t)dt + \frac{1}{\varepsilon} f_j^1\left(\frac{t}{\varepsilon}\right)dt + \sum_k \left(-D_{jk}x_k + \frac{1}{\varepsilon}L_{jk}^{11}x_k + \frac{1}{\varepsilon}L_{jk}^{12}y_k\right)dt \\
&\quad + \frac{1}{2}\sum_{k,l}\left(B_{jkl}^{111}x_kx_l + \frac{2}{\varepsilon}B_{jkl}^{112}x_ky_l + \frac{1}{\varepsilon}B_{jkl}^{122}y_ky_l\right)dt\,, \\
dy_j &= \frac{1}{\varepsilon} f_j^2\left(\frac{t}{\varepsilon}\right)dt + \sum_k \left(\frac{1}{\varepsilon}L_{jk}^{21}x_k + \frac{1}{\varepsilon}L_{jk}^{22}y_k\right)dt \\
&\quad + \frac{1}{2}\sum_{k,l}\left(\frac{1}{\varepsilon}B_{jkl}^{211}x_kx_l + \frac{2}{\varepsilon}B_{jkl}^{221}y_kx_l\right)dt - \frac{\gamma_j}{\varepsilon^2}y_j\,dt + \frac{\sigma_j}{\varepsilon}\,dW_j(t)\,,
\end{aligned}
\tag{4.1}
$$

where the W_j's are statistically independent Wiener processes satisfying

$$
\mathbf{E}W_j(t)W_k(s) = \delta_{j,k}\min(t,s)\,. \tag{4.2}
$$

To simplify the presentation, we have assumed that the stochastic-model term in the equation for the unresolved variables in (2.6) is diagonal in the representation for $\vec{y}$. The stochastic model for the barotropic equation in (3.17) can be mapped onto the equations in (4.1). In fact, with appropriate identification, the stochastic model equations in (4.1) can describe a wide variety of situations relevant to climate modeling, as we demonstrate now.

4.1 Wave–Mean Flow Interaction

We consider the following special setting of the stochastic model equations in equation (4.1):

$$
\begin{aligned}
dx_j &= F_j^1(t)dt - \sum_k D_{jk}x_k\,dt + \frac{1}{2\varepsilon}\sum_{k,l} B_{jkl}^{122}y_ky_l\,dt\,, \\
dy_j &= \frac{1}{\varepsilon}\sum_k L_{jk}^{22}y_k\,dt + \frac{1}{\varepsilon}\sum_{k,l}B_{jkl}^{221}y_kx_l\,dt - \frac{\gamma_j}{\varepsilon^2}y_j\,dt + \frac{\sigma_j}{\varepsilon}\,dW_j(t)\,.
\end{aligned}
\tag{4.3}
$$

The equations in (4.3) can be regarded as a generic model for wave–mean flow interaction. The mean flow is the declared climate variable, hence represented by the x_j's, whereas the waves are the unresolved variables y_j. Consistent with this identification, the mean flow responds to nonlinear driving by the waves through the terms

$$
\frac{1}{2\varepsilon}\sum_{k,l} B_{jkl}^{122}y_ky_l\,dt
$$

in the equation for x_j in (4.3), and slow forcing and damping through the terms

$$
F_j^1(t)dt - \sum_k D_{jk}x_k\,dt\,.
$$

The waves experience back reaction from the mean flow through frequency shift, as is apparent from the terms

$$\frac{1}{\varepsilon}\sum_{k} L^{22}_{jk} y_k\,dt + \frac{1}{\varepsilon}\sum_{k,l} B^{221}_{jkl} y_k x_l\,dt$$

in the equation for y_j in (4.3), while all nonlinear self-interactions between the waves are modeled stochastically as

$$-\frac{\gamma_j}{\varepsilon^2} y_j\,dt + \frac{\sigma_j}{\varepsilon} dW_j(t)\,,$$

consistent with our general strategy.

To illustrate wave–mean flow interaction in a simple situation, we utilize the stochastic model for the barotropic equations in (3.17) with no beta effect and mean flow, $\beta = U = 0$. The climate variables in this example are the subspace of functions with Fourier coefficients $\vec{k} = (0, q)$, with $q \neq 0$—these are the zonal mean flows—and the unresolved variables to be modeled stochastically are the projection on all the remaining variables. With the standard stochastic approximation for the nonlinear self-interaction of the unresolved variables, equations with the structure in (4.3) emerge. A simple example of this sort is presented explicitly in Section 7.1. Another important case of the system in (4.3) occurs when the climate variables $\vec{x}$ are determined by zonal averaging in baroclinic flows. An illustrative important example of stochastic modeling in geophysical flows that has the structure of the system in (4.3) can be found in [5]. The general theory developed below will be applied to those concrete examples by the authors in the near future.

We consider the asymptotic behavior of the climate variables for small ε. We have the following:

THEOREM 4.1 *Denote by $x^\varepsilon_j(t)$ the solution of the first equation in* (4.3). *In the limit as $\varepsilon \to 0$, $x^\varepsilon_j(t)$ converges to $x_j(t)$ where the $x_j(t)$ satisfy*

$$dx_j = F^1_j(t)dt + a_j\,dt - \sum_{k\in\sigma_1}(D_{jk} + \gamma_{jk})x_k\,dt + \sum_{k,l\in\sigma_2}\sigma_{jkl}\,dW_{kl}(t)\,, \tag{4.4}$$

where $W_{kl}(t)$ are independent Wiener processes satisfying

$$\mathbf{E}W_{kl}(t)W_{\bar{k}\bar{l}}(s) = \delta_{k,\bar{k}}\delta_{l,\bar{l}}\min(t,s)\,, \tag{4.5}$$

and a_j, γ_{jk}, σ_{jkl} are given by

$$a_j = \frac{1}{2}\sum_{k,l\in\sigma_2}\frac{\sigma_l^2 B^{122}_{jkl} L^{22}_{kl}}{\gamma_l(\gamma_k+\gamma_l)}\,, \qquad \gamma_{jk} = -\frac{1}{2}\sum_{l,m\in\sigma_2}\frac{\sigma_l^2 B^{122}_{jlm} B^{221}_{mlk}}{\gamma_l(\gamma_l+\gamma_m)}\,,$$
$$\sigma_{jkl} = \frac{B^{122}_{jkl}\sigma_k\sigma_l}{2\sqrt{(\gamma_k+\gamma_l)\gamma_k\gamma_l}}\,. \tag{4.6}$$

Remark. It follows immediately from the definition of W_{kl} that we have for the noise term in (4.6)

$$\sum_{k,l\in\sigma_2} \sigma_{jkl}\, dW_{kl}(t) \stackrel{D}{=} \sum_{k\in\sigma_1} \bar{\sigma}_{jk}\, dW_k(t)\,, \tag{4.7}$$

where $\stackrel{D}{=}$ denotes equality in law, the $W_j(t)$'s are independent Wiener processes, and the matrix $\bar{\sigma}_{jk}$ satisfies

$$\sum_{l,m\in\sigma_2} \sigma_{jlm}\sigma_{klm} = \sum_{l\in\sigma_1} \bar{\sigma}_{jl}\bar{\sigma}_{kl}\,. \tag{4.8}$$

In other words, at the price of evaluating the square root of a matrix once and for all, the noise in (4.4) can be represented by n independent Wiener processes, with n being the cardinal of the set σ_1 corresponding to the climate variables.

Theorem 4.1 can be proven using the asymptotic procedure outlined in Section 4.4, and it tells us two things. First, the effect of the unresolved wave variables on the climate mean flow variables can be either stabilizing or destabilizing. Stabilization is observed if γ_{jk} is a positive definite matrix, i.e., if for all ξ_j, $j\in\sigma_1$, such that $\sum_j \xi_j^2 \neq 0$, one has

$$\sum_{j,k\in\sigma_1} \gamma_{jk}\xi_j\xi_k > 0\,. \tag{4.9}$$

If this criterion fails to be satisfied, the unresolved wave variables destabilize the climate mean flow variables, and overall stability of the climate stochastic model equation in (4.4) requires

$$\sum_{j,k\in\sigma_1} (\gamma_{jk} + D_{jk})\xi_j\xi_k > 0\,. \tag{4.10}$$

In particular, the predictability of the climate variables can be diminished through interaction with the unresolved variables provided the explicit matrix γ_{jk} is not positive definite. Explicit examples of these phenomena are presented in Section 7.1.

The other important consequence of Theorem 4.1 is that the linear part of wave-wave interaction modifies the climate mean. This is apparent from the term $a_j\,dt$ in (4.4), with a_j proportional to L^{22}_{jk} as given by the first equation in (4.6).

4.2 Climate Scattering Interaction

As a second illustration consistent with the model equations in (4.1), we consider

$$\begin{aligned} dx_j &= F_j^1(t)dt - \sum_k D_{jk}x_k\,dt + \frac{1}{2\varepsilon}\sum_{k,l} B^{112}_{jkl}x_k y_l\,dt\,, \\ dy_j &= \frac{1}{\varepsilon}\sum_{k,l} B^{211}_{jkl}x_k x_l\,dt - \frac{\gamma_j}{\varepsilon^2}y_j\,dt + \frac{\sigma_j}{\varepsilon}\,dW_j(t)\,. \end{aligned} \tag{4.11}$$

The equations in (4.11) are a simple model where the climate variables x_j evolve through nonlinear scattering interaction with the unresolved variables, as is apparent from the term

$$\frac{1}{\varepsilon}\sum_{k,l} B^{112}_{jkl} x_k y_l \, dt$$

in the equation for x_j in (4.11). Here the amplitudes of some climate variables scatter energy into other climate variables through interaction with the unresolved variables. For illustration, we have assumed that the unresolved variables respond only to nonlinear driving by the climate variables alone through the term

$$\frac{1}{2\varepsilon}\sum_{k,l} B^{211}_{jkl} x_k x_l \, dt$$

in the equation for y_j in (4.11), and we have modeled the nonlinear self-interaction by the stochastic term

$$-\frac{\gamma_j}{\varepsilon^2} y_j \, dt + \frac{\sigma_j}{\varepsilon} \, dW_j(t) \, ,$$

as dictated by our general strategy.

The asymptotic behavior of the climate variables for small ε is specified by the following:

THEOREM 4.2 *Denote by $x^\varepsilon_j(t)$ the solution of the first equation in* (4.11). *In the limit as $\varepsilon \to 0$, $x^\varepsilon_j(t)$ tends to $x_j(t)$ where the $x_j(t)$ satisfy*

$$\begin{aligned} dx_j = {} & F^1_j(t)dt - \sum_k D_{jk} x_k \, dt \\ & + \frac{1}{2}\sum_{k,l\in\sigma_1}\sum_{m\in\sigma_2} \frac{\sigma^2_m}{\gamma^2_m} B^{112}_{jkm} B^{112}_{klm} x_l \, dt \\ & + \frac{1}{2}\sum_{k,l,m\in\sigma_1}\sum_{n\in\sigma_2} \frac{1}{\gamma_n} B^{112}_{jkn} B^{211}_{nlm} x_k x_l x_m \, dt \\ & + \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \frac{\sigma_l}{\gamma_l} B^{112}_{jkl} x_k \, dW_l(t) \, , \end{aligned} \tag{4.12}$$

where the W_j's are independent Wiener processes satisfying

$$\mathbf{E} W_j(t) W_k(s) = \delta_{jk} \min(t,s) \, . \tag{4.13}$$

Remark. It follows immediately from the definition of W_k that we have for the noise term in (4.12)

$$\sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \frac{\sigma_l}{\gamma_l} B^{112}_{jkl} x_k \, dW_{kl}(t) \stackrel{D}{=} \sum_{k,l\in\sigma_1} \bar{\sigma}_{jkl} x_k \, dW_l(t) \, , \tag{4.14}$$

where $\overset{D}{=}$ denotes equality in law, the $W_j(t)$'s are independent Wiener processes, and the matrix $\bar{\sigma}_{jkl}$ satisfies

$$\sum_{n\in\sigma_2} \frac{\sigma_n^2}{\gamma_n^2} B_{jkn}^{112} B_{lmn}^{112} = \sum_{l\in\sigma_1} \bar{\sigma}_{jkn}\bar{\sigma}_{lmn} \,. \tag{4.15}$$

Thus the noise in (4.12) can be represented with n independent Wiener processes, with n being the cardinal of the set σ_1 corresponding to the climate variables.

Theorem 4.2 follows by application of the asymptotic method of averaging outlined in Section 4.4. Alternatively, as will be shown in Section 4.5, the stochastic model equations in (4.12) can be derived directly from the equation in (4.11).

Theorem 4.2 demonstrates that, generally, the effect of the unresolved variables on the climate variables has to be accounted for by nonlinear corrections in the climate variables, as well as multiplicative noises, as is apparent from the last three terms in (4.11). The effects of climate scattering interaction have been ignored thus far by researchers in attempting to model stochastically the low-frequency variability of the atmosphere [1, 3, 18, 23, 25]. These general examples indicate that other stochastic models beyond linear Langevin modeling are needed in general and can be derived systematically. A simple explicit example is discussed in Section 7.2.

4.3 General Case Without Fast Waves

We now turn to the general case where we allow all possible linear and nonlinear interactions between the climate and the unresolved variables in the stochastic model equations in (4.1), but we still neglect the fast-wave effects (i.e., the term involving L_{jk}^{11} in (4.1)). Thus we consider

$$\begin{aligned} dx_j &= F_j^1(t) + \sum_k \left(-D_{jk}x_k + \frac{1}{\varepsilon} L_{jk}^{12} y_k \right) dt \\ &\quad + \sum_{k,l} \left(\frac{1}{2} B_{jkl}^{111} x_k x_l + \frac{1}{\varepsilon} B_{jkl}^{112} x_k y_l \, dt + \frac{1}{2\varepsilon} B_{jkl}^{122} y_k y_l \right) dt \,, \\ dy_j &= \sum_k \left(\frac{1}{\varepsilon} L_{jk}^{21} x_k + \frac{1}{\varepsilon} L_{jk}^{22} y_k \right) dt \\ &\quad + \sum_{k,l} \left(\frac{1}{2\varepsilon} B_{jkl}^{211} x_k x_l + \frac{1}{\varepsilon} B_{jkl}^{221} y_k x_l \right) dt - \frac{\gamma_j}{\varepsilon^2} y_j + \frac{\sigma_j}{\varepsilon} \, dW_j(t) \,. \end{aligned} \tag{4.16}$$

The following theorem specifies the asymptotic behavior of the climate-variables solution of the equations in (4.16) for small ε:

THEOREM 4.3 *Denote by $x_j^\varepsilon(t)$ the solution of the first equation in* (4.16). *In the limit as $\varepsilon \to 0$, $x_j^\varepsilon(t)$ tends to $x_j(t)$ where the $x_j(t)$ satisfy*

$$
\begin{aligned}
dx_j = {}& F_j(t)dt - \sum_{k\in\sigma_1} D_{jk}x_k\,dt - \frac{1}{2}\sum_{k,l\in\sigma_1} B^{111}_{jkl}x_k x_l\,dt \\
&+ a_j\,dt - \sum_{k\in\sigma_1}\gamma_{jk}x_k\,dt + \sum_{k,l\in\sigma_2}\sigma_{jkl}\,dW_{kl}(t) \\
&+ \frac{1}{2}\sum_{k\in\sigma_1}\sum_{m\in\sigma_2}\frac{\sigma_m^2}{\gamma_m^2}B^{112}_{jkm}\left(L^{12}_{km} + \sum_{l\in\sigma_1}B^{112}_{klm}x_l\right)dt \\
&+ \sum_{l\in\sigma_1}\sum_{n\in\sigma_2}\frac{1}{\gamma_n}\left(L^{12}_{jn} + \sum_{k\in\sigma_1}B^{112}_{jkn}x_k\right)\left(L^{21}_{nl}x_l + \frac{1}{2}\sum_{m\in\sigma_1}B^{211}_{nlm}x_l x_m\right)dt \\
&+ \sum_{l\in\sigma_2}\frac{\sigma_l}{\gamma_l}\left(L^{12}_{jl} + \sum_{k\in\sigma_1}B^{112}_{jkl}x_k\right)dW_l(t)\,,
\end{aligned}
\tag{4.17}
$$

where W_j, W_{jk} are independent Wiener processes satisfying

$$
\mathbf{E}W_j(t)W_k(s) = \delta_{jk}\min(t,s)\,, \qquad \mathbf{E}W_{jk}(t)W_{\bar{j}\bar{k}}(s) = \delta_{j\bar{j}}\delta_{k\bar{k}}\min(t,s)\,,
\tag{4.18}
$$

and we defined

$$
\begin{gathered}
a_j = \frac{1}{2}\sum_{k,l\in\sigma_2}\frac{\sigma_l^2 B^{122}_{jkl}L^{22}_{kl}}{\gamma_l(\gamma_k+\gamma_l)}\,, \qquad \gamma_{jk} = -\frac{1}{2}\sum_{l,m\in\sigma_2}\frac{\sigma_l^2 B^{122}_{jlm}B^{221}_{mlk}}{\gamma_l(\gamma_l+\gamma_m)}\,, \\
\sigma_{jkl} = \frac{B^{122}_{jkl}\sigma_k\sigma_l}{2\sqrt{(\gamma_k+\gamma_l)\gamma_k\gamma_l}}\,.
\end{gathered}
\tag{4.19}
$$

Remark. As in the equations in (4.4) and (4.12), the noises in (4.17) can be redefined so that they involve vector- or matrix-valued Wiener processes defined on the set of climate variables alone.

The proof of Theorem 4.3 uses the asymptotic procedure for averaging outlined in Section 4.4. These calculations are presented in Appendix A. The theorem shows that, generally, all the new phenomena described in Sections 4.1 and 4.2 will interplay in the stochastic climate model equations in (4.17), with both stable or unstable Langevin terms, modification of the climate mean, nonlinear corrections of the climate-variables dynamics, and multiplicative noises. These results will be applied by the authors to a variety of geophysical applications in the near future.

4.4 Systematic Asymptotic Strategy

We illustrate the method of averaging of the unresolved variables. This is the major tool that we use to derive stochastic model equations for the climate variables alone in the limit as $\varepsilon \to 0$. To this end, we exploit the property that the stochastic model equations in (4.1) define a Markov process that is singular in the limit as

$\varepsilon \to 0$. Perturbation methods for such processes were developed in the 1970s, originally for analyzing the linear Boltzmann equation in some asymptotic limit. We opt for a rather brief presentation of these methods here and refer the reader to the original papers by Kurtz [13] and Ellis and Pinsky [6] (see also [19]) for details. Here, we consider the situation with no fast-wave effects on the climate variables; i.e., we set

$$f_j^1\left(\frac{t}{\varepsilon}\right) = f_j^2\left(\frac{t}{\varepsilon}\right) = L_{jk}^{11} = 0 \tag{4.20}$$

in the stochastic model equations in (4.1). In this case the averaging method of Kurtz [13] applies. In Section 5, the situation with fast-wave effects is studied, in which case the averaging method must be modified by combining Kurtz's method [13] with a procedure of averaging over fast effects developed by Khasminski [11]; see [19].

The method of averaging exploits the Markov nature of the stochastic model equations in (4.1) and works with the backward equation associated with (4.1). To introduce the latter, suppose $f(\vec{x})$ is a suitable scalar-valued function and define

$$\begin{gathered} \varrho^\varepsilon(s, \vec{x}, \vec{y} \mid t) = \mathbf{E}\, f(\vec{x}^\varepsilon(t))\,, \qquad s \le t\,, \\ \text{where } \vec{x}^\varepsilon(t) \text{ solves the stochastic model equations in (4.1)} \\ \text{for the initial condition } \vec{x}^\varepsilon(s) = \vec{x},\ \vec{y}^\varepsilon(s) = \vec{y}\,. \end{gathered} \tag{4.21}$$

Here $\mathbf{E}$ denotes the expectation with respect to the statistics of the Wiener processes W_j in (4.1). We wish to determine the asymptotic behavior of ϱ^ε as $\varepsilon \to 0$; this will specify the limit of $\vec{x}^\varepsilon(t)$ as $\varepsilon \to 0$. The function ϱ^ε satisfies the backward equation associated with the equations in (4.1)

$$-\frac{\partial \varrho^\varepsilon}{\partial s} = \frac{1}{\varepsilon^2}\mathcal{L}_1\varrho^\varepsilon + \frac{1}{\varepsilon}\mathcal{L}_2\varrho^\varepsilon + \mathcal{L}_3\varrho^\varepsilon\,, \qquad \varrho^\varepsilon(t, \vec{x}, \vec{y} \mid t) = f(\vec{x})\,, \tag{4.22}$$

with the operators $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ given by

$$\begin{aligned} \mathcal{L}_1 &= \sum_j \left(-\gamma_j y_j \frac{\partial}{\partial y_j} + \frac{\sigma_j^2}{2}\frac{\partial^2}{\partial y_j^2}\right), \\ \mathcal{L}_2 &= \sum_{j,k} \left(L_{jk}^{12} y_k + \frac{1}{2}\sum_l \left(2B_{jkl}^{112} x_k y_l + B_{jkl}^{122} y_k y_l\right)\right)\frac{\partial}{\partial x_j} \\ &\quad + \sum_{j,k}\left(L_{jk}^{21} x_k + L_{jk}^{22} y_k + \frac{1}{2}\sum_l \left(B_{jkl}^{211} x_k x_l + 2B_{jkl}^{221} y_k x_l\right)\right)\frac{\partial}{\partial y_j} \\ \mathcal{L}_3 &= \sum_j \left(F_j(s) - \sum_k D_{jk} x_k + \frac{1}{2}\sum_{kl} B_{jkl}^{111} x_k x_l\right)\frac{\partial}{\partial x_j}\,. \end{aligned} \tag{4.23}$$

We now derive from the backward equation in (4.22) an equation for the limit of ϱ^ε as $\varepsilon \to 0$. To this end, let ϱ^ε be represented formally as a power series

$$\varrho^\varepsilon = \varrho_0 + \varepsilon\varrho_1 + \varepsilon^2\varrho_2 + \cdots . \tag{4.24}$$

We insert this series into (4.22) and equate the coefficients of equal power in ε. This gives the following sequence of equations:

$$\mathcal{L}_1\varrho_0 = 0\,, \tag{4.25}$$

$$\mathcal{L}_1\varrho_1 = -\mathcal{L}_2\varrho_0\,, \tag{4.26}$$

$$\mathcal{L}_1\varrho_2 = -\frac{\partial\varrho_0}{\partial s} - \mathcal{L}_3\varrho_0 - \mathcal{L}_2\varrho_1\,, \tag{4.27}$$

$$\vdots$$

From the structure of these equations we see that each requires as a solvability condition that its right-hand side belong to the range of $\mathcal{L}_1$ or, equivalently, that the right-hand sides of the equations in (4.25)–(4.27) have zero expectation with respect invariant measure of the Ornstein-Uhlenbeck process. The solvability condition is trivially satisfied for the equations in (4.25) and (4.26), but not for the one in (4.27). In fact, as we show now, the dynamic equation for ϱ_0 is determined from the solvability condition for the equation for ϱ_2 in (4.27).

The equation in (4.25) implies that ϱ_0 belongs to the null space of $\mathcal{L}_1$, i.e.,

$$\mathbf{P}\varrho_0 = \varrho_0\,, \tag{4.28}$$

where $\mathbf{P}$ denotes the expectation with respect to the invariant measure of the Ornstein-Uhlenbeck process. This, of course, is not the expected dynamic equation for ϱ_0; (4.25) essentially implies that ϱ_0 is independent of the unresolved variables $\vec{y}$. Since $\varrho_0(t \mid t) = f$, we avoid any problem near $s = t$ by assuming $\mathbf{P}f = f$. Taking next the expectation of the equation in (4.26), we obtain the solvability condition

$$\mathbf{P}\mathcal{L}_2\varrho_0 = \mathbf{P}\mathcal{L}_2\mathbf{P}\varrho_0 = 0\,. \tag{4.29}$$

It may be easily checked that this equation is trivially satisfied for our stochastic model equations in (4.1) for $f_j^1(t/\varepsilon) = f_j^2(t/\varepsilon) = L_{jk}^{11} = 0$, because of (2.10) in assumption A4. If equation (4.29) were not satisfied, the unresolved variables would induce $O(1/\varepsilon)$ effects on the climate variables, contradicting the very criterion for the distinction between these variables. Since (4.29) holds, the solution of equation (4.26) is

$$\varrho_1 = -\mathcal{L}_1^{-1}\mathcal{L}_2\mathbf{P}\varrho_0\,. \tag{4.30}$$

We insert this expression into the equation in (4.27) and take the expectation on the resulting equation to get the solvability condition for ϱ_2. The latter is the following equation for ϱ_0:

$$(4.31)\qquad -\frac{\partial \varrho_0}{\partial s} = \mathbf{P}\mathcal{L}_3\mathbf{P}\varrho_0 - \mathbf{P}\mathcal{L}_2\mathcal{L}_1^{-1}\mathcal{L}_2\mathbf{P}\varrho_0\,, \qquad \varrho_0(t,\vec{x}\mid t) = f(\vec{x})\,.$$

Despite the formal character of these manipulations, Kurtz [13] showed that in the limit as $\varepsilon \to 0$, ϱ^ε converges to ϱ_0, the solution of the equation in (4.31). In fact, we state this result as the following theorem, which potentially applies to more general equations than the stochastic model equations in (4.1):

THEOREM 4.4 (Kurtz, 1973) *Let $\varrho^\varepsilon(s,\vec{x},\vec{y}\mid t)$ satisfy*

$$(4.32)\qquad -\frac{\partial \varrho^\varepsilon}{\partial s} = \frac{1}{\varepsilon^2}\mathcal{L}_1\varrho^\varepsilon + \frac{1}{\varepsilon}\mathcal{L}_2\varrho^\varepsilon + \mathcal{L}_3\varrho^\varepsilon\,, \qquad \varrho^\varepsilon(t,\vec{x},\vec{y}\mid t) = f(\vec{x})\,,$$

where $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ are backward Fokker-Planck operators, and $\mathcal{L}_1$ generates a stationary process such that

$$(4.33)\qquad e^{\mathcal{L}_1 t}\cdot \to \mathbf{P}\,\cdot \quad \text{as } t\to\infty\,,$$

and $\mathbf{P}\mathcal{L}_2\mathbf{P} = 0$. Then, in the limit as $\varepsilon \to 0$, $\varrho^\varepsilon(s,\vec{x},\vec{y}\mid t)$ tends to $\varrho_0(s,\vec{x}\mid t)$ for $-T < s \le t$, $T < \infty$, uniformly in $\vec{x}$ and $\vec{y}$ on compact sets, where ϱ_0 satisfies $\varrho_0 = \mathbf{P}\varrho_0$ and solves the backward equation

$$(4.34)\qquad -\frac{\partial \varrho_0}{\partial s} = \bar{\mathcal{L}}\varrho_0\,, \qquad \varrho_0(t,\vec{x}\mid t) = f(\vec{x})\,,$$

with

$$(4.35)\qquad \bar{\mathcal{L}}\,\cdot = \mathbf{P}\mathcal{L}_3\mathbf{P}\,\cdot - \mathbf{P}\mathcal{L}_2\mathcal{L}_1^{-1}\mathcal{L}_2\mathbf{P}\,\cdot\,.$$

For us, the most important consequence of Theorem 4.4 is that, as applied to the stochastic model equations in (4.1) with the assumption $f_j^1(t/\varepsilon) = f_j^2(t/\varepsilon) = L_{jk}^{11} = 0$, it implies that $\bar{\mathcal{L}}$ is a Fokker-Planck operator whose actual form is effectively computable. As a direct result, a set of stochastic differential equations can be associated with $\bar{\mathcal{L}}$ (see chapter 9 of [2]): These are the stochastic climate model equations that were derived for the specific cases of wave–mean flow and climate-scattering interactions in Theorems 4.1 and 4.2, and for the general case without fast-wave effects in Theorem 4.3. The actual computation from Theorem 4.4 of the stochastic climate model equations in Theorem 4.3 is given in Appendix B. Other examples of applications of Theorem 4.4 on low-order triad models are given in Section 7.

4.5 Averaging by the Direct Method

The asymptotic strategy presented in Section 4.4 is based on a singular perturbation expansion of the partial differential equation (backward equation) associated with the stochastic model equations in (4.1). We now show that in the special case where the equations for the unresolved variables y_j are linear and diagonal in y_j, the equations for the climate variables alone that are obtained by the method of Section 4.4 can also be derived directly by working on the stochastic differential equations in (4.1). We illustrate the method on the climate scattering equations equations in (4.11) and derive the equations in (4.12).

Because the equation for y_j in (4.11) is linear in y_j, this equation with the initial condition $y_j(0) = y_j$ is equivalent to the integral equation

$$y_j(t) = e^{-\gamma_j t/\varepsilon^2} y_j + \frac{1}{\varepsilon} \sum_{k,l\in\sigma_1} \int_0^t e^{-\gamma_j(t-s)/\varepsilon^2} B^{211}_{jkl} x_k(s) x_l(s) ds + \frac{1}{\varepsilon} g_j(t)\,, \tag{4.36}$$

where

$$g_j(t) = \sigma_j \int_0^t e^{-\gamma_j(t-s)/\varepsilon^2}\, dW_j(s)\,. \tag{4.37}$$

Inserting (4.36) into (4.11) yields the following closed, non-Markovian stochastic model equations for the climate variable $x_j(t)$ valid for any ε:

$$\begin{aligned} dx_j(t) = F^1_j(t)dt - \sum_k D_{jk} x_k(t) dt + \frac{1}{2\varepsilon} \sum_{k,l} B^{112}_{jkl} x_k(t) e^{-\gamma_l t/\varepsilon^2} y_l\, dt \\ + \frac{1}{2\varepsilon^2} \sum_{k,l,m\in\sigma_1} \sum_{n\in\sigma_2} B^{112}_{jkn} B^{211}_{nlm} x_k(t) \left(\int_0^t e^{-\gamma_n(t-s)/\varepsilon^2} x_l(s) x_m(s) ds \right) dt \\ + \frac{1}{2\varepsilon^2} \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} B^{112}_{jkl} x_k(t) g_l(t) dt\,. \end{aligned} \tag{4.38}$$

We now show that, in the limit as $\varepsilon \to 0$, the equation in (4.38) reduces to the actual stochastic model given by (4.11). We consider successively the various terms involving ε at the right-hand side of (4.38). First, we have for any $t > 0$

$$\frac{1}{2\varepsilon} \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} B^{112}_{jkl} x_k(t) e^{-\gamma_l t/\varepsilon^2} y_l\, dt \to 0\,. \tag{4.39}$$

Second,

$$\begin{aligned} \frac{1}{2\varepsilon^2} \sum_{k,l,m\in\sigma_1} \sum_{n\in\sigma_2} B^{112}_{jkn} B^{211}_{nml} x_k(t) \left(\int_0^t e^{-\gamma_n(t-s)/\varepsilon^2} x_l(s) x_m(s) ds \right) dt \to \\ \frac{1}{2} \sum_{k,l,m\in\sigma_1} \sum_{n\in\sigma_2} \frac{1}{\gamma_n} B^{112}_{jkn} B^{211}_{nlm} x_k(t) x_l(t) x_m(t) ds\,. \end{aligned} \tag{4.40}$$

Finally, we use the Gaussianity of $g_j(t)$ from (4.37) combined with the following properties for any test function η:

$$\mathbf{E} \frac{1}{\varepsilon^2} \int_0^\infty \eta(t) g_j(t) dt = 0\,, \tag{4.41}$$

$$\mathbf{E} \left(\frac{1}{\varepsilon^2} \int_0^\infty \eta(t) g_j(t) dt \right) \left(\frac{1}{\varepsilon^2} \int_0^\infty \eta(t) g_k(t) dt \right) \to \frac{\sigma_j^2}{\gamma_j^2} \delta_{j,k} \int_0^\infty \eta^2(t) dt\,, \tag{4.42}$$

to deduce that the noise $g_j(t)/\varepsilon^2$ converges in mean square to a white noise, i.e.,

$$\frac{1}{\varepsilon^2} g_j(t)dt \to \frac{\sigma_j}{\gamma_j} dW_j(t)\,, \tag{4.43}$$

as $\varepsilon \to 0$, where $W_j(t)$ are independent Wiener processes. As the external limit of a process with finite correlation time, the white noise is interpreted in Stratonovich's sense (see, e.g., [2, chapter 10]), i.e.,

$$\frac{1}{2\varepsilon^2} \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} B^{112}_{jkl} x_k(t) g_l(t) dt \to \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} \frac{\sigma_l}{\gamma_l} B^{112}_{jkl} x_k(t) \circ dW_l(t)\,. \tag{4.44}$$

Collecting (4.39), (4.40), and (4.44) into (4.11), we obtain

$$\begin{aligned} dx_j = {} & F_j(t)dt - \sum_{k\in\sigma_1} D_{jk} x_k\, dt + \frac{1}{2} \sum_{k,l\in\sigma_1} B^{111}_{jkl} x_k x_l\, dt \\ & + \frac{1}{2} \sum_{k,l,m\in\sigma_1} \sum_{n\in\sigma_2} \frac{1}{\gamma_n} B^{112}_{jkn} B^{211}_{nlm} x_k x_l x_m\, dt \\ & + \frac{1}{2} \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} \frac{\sigma_l}{\gamma_l} B^{112}_{jkl} x_k \circ dW_l(t)\,. \end{aligned} \tag{4.45}$$

This equation is equivalent to Itô's equation in (4.12).

Summarizing, in the case where the equation for the unresolved variables y_j are linear and diagonal in y_j, the stochastic model for climate variables alone can be derived by a direct method alternative to the general asymptotic strategy presented in Section 4.3. It should be stressed that the direct method actually gives more than the general asymptotic strategy, since it provides us with the non-Markovian model equations in (4.38), which are valid for any ε. In particular, the equations in (4.38) can be used as a starting point for a systematic expansion in ε that goes beyond leading order. Results in this direction will be reported elsewhere by the authors. Finally, we mention that the direct method can be generalized to situations with fast-wave effects: An example of such calculation is given in the proof of Theorem 7.6.

5 The Effect of Fast-Wave Averaging in Stochastic Climate Models

In this section, we generalize the result of Section 4 by incorporating the effects of fast waves to the theory. First in Section 5.1, we consider the important example of the stochastic model for the truncated barotropic equations in (3.6) in the absence of mean flow and topography, $U = 0$, $\hat{h}_k = 0$, but with beta effect, i.e., the dispersive terms defined by Ω_k in (3.8) and associated with Rossby wave propagation; see Pedlosky [20]. These beta terms induce fast-wave effects on both the climate and the unresolved variables, and we show how to handle these effects in order to get closed equations for the climate variables alone for small ε. By comparing with the case without beta effect, we demonstrate that the beta effect induces a depletion of the effective nonlinear self-interactions in the equations for the climate variables alone, as well as a reduction of the noise in these equations. Indeed, in the situation with beta effects, additional resonance conditions between the various terms need to be satisfied in order that these terms give a nonzero contribution through the averaging procedure.

In Section 5.2, we study the complete stochastic climate equations in (4.1) when both fast waves and forcing effects are present, and we give the explicit effective equations that are obtained for the climate variables alone in this case. Finally, in Section 5.3 we give the details of the asymptotic procedure that allows us to average both on the unresolved variables and on the fast-wave effects. At the end of this section, we also give the details of the proof of Theorems 5.1 and 5.2.

5.1 Truncated Barotropic Equations with the Beta Effect

In this section, we consider the stochastic model for the truncated barotropic equations in (3.6) under the assumptions that

(1) there is no mean flow and topography. Thus, we set $U = \hat{h}_k = 0$ in the stochastic model for barotropic equations in (3.17); and
(2) the stochastic model in (3.14) has zero mean. Thus, we set $\bar{w}_k = 0$ in the stochastic model for barotropic equations in (3.17).

Under the above assumptions, the stochastic model for the truncated barotropic equations in (3.6) reduce to

$$\begin{aligned} dv_k &= \frac{i}{\varepsilon}\Omega_k v_k\,dt + \frac{1}{2}\sum_{\substack{\vec{l},\vec{m}\in\sigma_1\\ \vec{k}+\vec{l}+\vec{m}=0}} B_{klm} v_l^* v_m^*\,dt \\ &\quad + \frac{1}{\varepsilon}\sum_{\substack{\vec{l}\in\sigma_1,\vec{m}\in\sigma_2\\ \vec{k}+\vec{l}+\vec{m}=0}} B_{klm} v_l^* w_m^*\,dt + \frac{1}{2\varepsilon}\sum_{\substack{\vec{l},\vec{m}\in\sigma_2\\ \vec{k}+\vec{l}+\vec{m}=0}} B_{klm} w_l^* w_m^*\,dt\,, \\ dw_k &= \frac{i}{\varepsilon}\Omega_k w_k\,dt + \frac{1}{2\varepsilon}\sum_{\substack{\vec{l},\vec{m}\in\sigma_1\\ \vec{k}+\vec{l}+\vec{m}=0}} B_{klm} v_l^* v_m^*\,dt + \frac{1}{\varepsilon}\sum_{\substack{\vec{l}\in\sigma_1,\vec{m}\in\sigma_2\\ \vec{k}+\vec{l}+\vec{m}=0}} B_{klm} v_l^* w_m^*\,dt \\ &\quad - \frac{1}{\varepsilon^2}\gamma_k w_k\,dt + \frac{\sigma_k}{\sqrt{2}\varepsilon}\big(dW_k(t) + dW_{-k}^*(t)\big)\,. \end{aligned} \tag{5.1}$$

In these equations, the beta effects are respectively accounted for by the terms $i\Omega_k v_k\,dt/\varepsilon$ and $i\Omega_k w_k\,dt/\varepsilon$ which obviously induce fast-wave rotation effects on both the unresolved and the climate variables.

We ask about the asymptotic behavior of the climate variables v_k for small ε. We have

THEOREM 5.1 *Denote by* $v_k^\varepsilon(t)$ *the solution of* (5.1). *In the limit as* $\varepsilon \to 0$, $e^{i\Omega_k t/\varepsilon} v_k^\varepsilon(t)$ *converges to* $v_k(t)$ *where the* $v_k(t)$ *satisfy*

(5.2)
$$
\begin{aligned}
dv_k &= \Bigg\{ \frac{1}{2} \sum_{\substack{\vec{l},\vec{m}\in\sigma_1 \\ \vec{k}+\vec{l}+\vec{m}=0}} B_{klm}\theta_{klm} v_l^* v_m^*\, dt + \frac{1}{2} \sum_{\substack{\vec{l},\vec{m},\vec{n}\in\sigma_1,\, \vec{p}\in\sigma_2 \\ \vec{k}+\vec{l}+\vec{p}=0 \\ \vec{m}+\vec{n}+\vec{p}=0}} \frac{B_{klp}B_{pmn}}{\gamma_p} \theta_{k,l|m,n} v_l^* v_m v_n\, dt \Bigg\}_1 \\
&\quad + \Bigg\{ \sum_{\substack{\vec{l},\vec{m}\in\sigma_2 \\ \vec{k}+\vec{l}+\vec{m}=0}} \frac{\sigma_m^2 B_{klm} B_{lkm}}{\gamma_m(\gamma_l+\gamma_m)} v_k\, dt \\
&\qquad + \sum_{\substack{\vec{l},\vec{m}\in\sigma_2 \\ \vec{k}+\vec{l}+\vec{m}=0}} \frac{\sigma_l\sigma_m B_{klm}}{2(\gamma_l+\gamma_m)} \left(\frac{(\gamma_l+\gamma_m)}{\gamma_l\gamma_m} \right)^{1/2} (dW_{l,m}(t) + dW_{-l,-m}^*(t)) \Bigg\}_2 \\
&\quad + \Bigg\{ \sum_{\substack{\vec{l}\in\sigma_1 \vec{m}\in\sigma_2 \\ \vec{k}+\vec{l}+\vec{m}=0}} \frac{\sigma_m^2 B_{klm} B_{lkm}}{\gamma_m^2} v_k\, dt \\
&\qquad + \sum_{\substack{\vec{l}\in\sigma_1,\vec{m}\in\sigma_2 \\ \vec{k}+\vec{l}+\vec{m}=0}} \frac{\sigma_m B_{klm}}{\sqrt{2}\gamma_m} \tilde{\theta}_{klm} v_l^* (dW_m(t) + dW_{-m}^*(t)) \Bigg\}_3 ,
\end{aligned}
$$

where $W_k(t)$, $W_{k,l}(t)$ *are independent complex Wiener processes satisfying*

(5.3)
$$
\begin{aligned}
\mathbf{E} W_k(t) W_l^*(s) &= 2\delta_{k,l} \min(t,s)\,, \\
\mathbf{E} W_k(t) W_l(s) &= \mathbf{E} W_k^*(t) W_l^*(s) = 0\,, \\
\mathbf{E} W_{k,l}(t) W_{m,n}^*(s) &= 2\delta_{k,l}\delta_{m,n} \min(t,s)\,, \\
\mathbf{E} W_{k,l}(t) W_{m,n}(s) &= \mathbf{E} W_{k,l}^*(t) W_{m,n}^*(s) = 0\,,
\end{aligned}
$$

the coupling parameter $\tilde{\theta}_{klm}$ *satisfies*

(5.4)
$$
\sum_{\vec{m}\in\sigma_2} \tilde{\theta}_{klm} \tilde{\theta}_{k'l'm} = \theta_{k,l|k',l'}\,,
$$

and we have defined

(5.5)
$$
\begin{aligned}
\theta_{k,l,m} &= \begin{cases} 1 & \textit{if } \Omega_k + \Omega_l + \Omega_m = 0 \\ 0 & \textit{otherwise}, \end{cases} \\
\theta_{k,l|m,n} &= \begin{cases} 1 & \textit{if } \Omega_k + \Omega_l - \Omega_m - \Omega_n = 0 \\ 0 & \textit{otherwise}. \end{cases}
\end{aligned}
$$

The proof of Theorem 5.1 uses the asymptotic method presented in Section 5.3 and is given at the end of this section.

The conditions in (5.5) are the three-wave and four-wave resonance conditions for nonlinear wave theory, which might be familiar to the reader. The first terms in braces, $\{\cdot\}_1$, in (5.2) represent the depleted and augmented nonlinearities, the terms in $\{\cdot\}_2$ are additive noises with related damping, while the terms in $\{\cdot\}_3$ are the multiplicative noise contributions in Itô form. It is apparent from the equation in (5.2) that the effect of the unresolved variables on the climate variables must be accounted for by nonlinear self-interaction terms in the climate variables, as well as multiplicative noise. On the other hand, the effects of the fast waves in the original equations in (5.1) give rise to depletion of nonlinearity and noise weakening in (5.2). Indeed, the resonance conditions in (5.5) that involve three or four modes must be satisfied in order for the corresponding terms in (5.1) to give nonzero contribution in the equation in (5.2). This point can be emphasized even more upon comparing the results in Theorem 5.1 with those in the following theorem, which is obtained if the beta effects in the equations in (5.2) are set to zero:

THEOREM 5.2 *Set* $\Omega_k = 0$ *in the equations in* (5.1), *and denote by* $v_k^\varepsilon(t)$ *the solution of these equations in this case. In the limit as* $\varepsilon \to 0$, $v_k^\varepsilon(t)$ *converges to* $v_k(t)$ *where the* $v_k(t)$ *satisfy*

(5.6)

$$
\begin{aligned}
dv_k = {} & \Bigg\{ \frac{1}{2} \sum_{\substack{\vec{l},\vec{m}\in\sigma_1 \\ \vec{k}+\vec{l}+\vec{m}=0}} B_{klm} v_l^* v_m^* \, dt + \frac{1}{2} \sum_{\substack{\vec{l},\vec{m},\vec{n}\in\sigma_1,\, \vec{p}\in\sigma_2 \\ \vec{k}+\vec{l}+\vec{p}=0 \\ \vec{m}+\vec{n}+\vec{p}=0}} \frac{B_{klp} B_{pmn}}{\gamma_p} v_l^* v_m v_n dt \Bigg\}_1 \\
& + \Bigg\{ \sum_{\substack{\vec{l},\vec{m}\in\sigma_2 \\ \vec{k}+\vec{l}+\vec{m}=0}} \frac{\sigma_m^2 B_{klm} B_{lkm}}{\gamma_m(\gamma_l+\gamma_m)} v_k \, dt \\
& \quad + \sum_{\substack{\vec{l},\vec{m}\in\sigma_2 \\ \vec{k}+\vec{l}+\vec{m}=0}} \frac{\sigma_l \sigma_m B_{klm}}{2(\gamma_l+\gamma_m)} \left(\frac{(\gamma_l+\gamma_m)}{\gamma_l \gamma_m} \right)^{1/2} (dW_{l,m}(t) + dW^*_{-l,-m}(t)) \Bigg\}_2 \\
& + \Bigg\{ \sum_{\substack{\vec{l}\in\sigma_1 \vec{m}\in\sigma_2 \\ \vec{k}+\vec{l}+\vec{m}=0}} \frac{\sigma_m^2 B_{klm} B_{lkm}}{\gamma_m^2} v_k \, dt \\
& \quad + \sum_{\substack{\vec{l}\in\sigma_1,\vec{m}\in\sigma_2 \\ \vec{k}+\vec{l}+\vec{m}=0}} \frac{\sigma_m B_{klm}}{\sqrt{2}\gamma_m} v_l^* (dW_m(t) + dW^*_{-m}(t)) \Bigg\}_3 ,
\end{aligned}
$$

where $W_k(t)$, $W_{k,l}(t)$ *are independent complex Wiener processes satisfying* (5.3).

The proof of Theorem 5.2 is a special case of the proof of Theorem 5.1 given below and is very similar to the proof of the results described in Section 4.

Clearly, equation (5.6) is similar to (5.2) except that all the resonance conditions induced by the beta effect in the latter equation are absent.

5.2 General Case with Fast Waves

We now turn to the stochastic climate equations in (4.1) and consider first the situation with fast-wave effects but no forcing. The key assumption which we utilize here is that the operator L_{11} is skew-symmetric so that there is nontrivial averaging on the climate time scales. In this case, the asymptotic behavior of the climate variables for small ε is specified by the following:

THEOREM 5.3 *Denote by* $x_j^\varepsilon(t)$ *the solution of the first equation in* (4.1) *with no fast forcing,* $f_j^1 = f_j^2 = 0$, *under the assumption in* A2 *that* L_{11} *is skew-symmetric. In the limit as* $\varepsilon \to 0$,

$$\sum_{k\in\sigma_1} \left(e^{-L_{11}t/\varepsilon}\right)_{jk} x_k^\varepsilon(t) \to x_j(t)\,, \tag{5.7}$$

where $x_j(t)$ *satisfy*

(5.8)
$$\begin{aligned}
dx_j &= \left\langle \bar F_j(t,\tau)\right\rangle dt - \sum_{k\in\sigma_1}\left\langle \bar D_{jk}(\tau) X_k(\vec x,\tau)\right\rangle dt \\
&\quad - \frac12 \sum_{k,l\in\sigma_1}\left\langle \bar B^{111}_{jkl}(\tau) X_k(\vec x,\tau) X_l(\vec x,\tau)\right\rangle + a_j\,dt \\
&\quad - \sum_{k\in\sigma_1}\left\langle \gamma_{jk}(\tau) X_k(\vec x,\tau)\right\rangle dt + \sum_{k,l\in\sigma_2}\sigma_{jkl}\,dW_{kl}(t) \\
&\quad + \frac12 \sum_{k\in\sigma_1}\sum_{m\in\sigma_2}\frac{\sigma_m^2}{\gamma_m^2}\left\langle \bar B^{112}_{jkm}(\tau)\left(\bar L^{12}_{km}(\tau) + \sum_{l\in\sigma_1}\bar B^{112}_{klm}(\tau) X_l(\vec x,\tau)\right)\right\rangle dt \\
&\quad + \sum_{l\in\sigma_1}\sum_{n\in\sigma_2}\frac{1}{\gamma_n}\left\langle \left(\bar L^{12}_{jn}(\tau) + \sum_{k\in\sigma_1}\bar B^{112}_{jkn}(\tau) X_k(\vec x,\tau)\right)\right. \\
&\qquad\qquad \left.\times\left(L^{21}_{nl} X_l(\vec x,\tau) + \frac12\sum_{m\in\sigma_1} B^{211}_{nlm} X_l(\vec x,\tau) X_m(\vec x,\tau)\right)\right\rangle dt \\
&\quad + \sum_{l\in\sigma_2}\sigma_{jl}\,dW_l(t)\,.
\end{aligned}$$

Here W_j, W_{jk} *are independent Wiener processes satisfying*

$$\mathbf{E} W_j(t) W_k(s) = \delta_{jk}\min(t,s)\,, \qquad \mathbf{E} W_{jk}(t) W_{\bar j\bar k}(s) = \delta_{j\bar j}\delta_{k\bar k}\min(t,s)\,, \tag{5.9}$$

and $\langle g(\tau)\rangle$ denotes the (τ-independent) average of any suitable function $g(\tau)$

$$\langle g(\tau)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{t-T}^{t} g(\tau)d\tau\,. \tag{5.10}$$

We also defined

$$a_j = \frac{1}{2}\sum_{k,l\in\sigma_2} \frac{\sigma_l^2\langle \bar{B}^{122}_{jkl}(\tau)\rangle L^{22}_{kl}}{\gamma_l(\gamma_k+\gamma_l)}\,, \qquad \gamma_{jk}(\tau) = -\frac{1}{2}\sum_{l,m\in\sigma_2} \frac{\sigma_l^2 \bar{B}^{122}_{jlm}(\tau) B^{221}_{mlk}}{\gamma_l(\gamma_l+\gamma_m)}\,, \tag{5.11}$$

and the matrices σ_{jk} and σ_{jkl} satisfy

$$\begin{aligned} \sum_{k,l\in\sigma_2} \sigma_{jkl}\sigma_{j'kl} &= \sum_{k,l\in\sigma_2} \frac{\langle B^{122}_{jkl}(\tau) B^{122}_{j'kl}(\tau)\rangle \sigma_k^2\sigma_l^2}{4(\gamma_k+\gamma_l)\gamma_k^2\gamma_l^2}\,,\\ \sum_{l\in\sigma_2}\sigma_{jl}\sigma_{j'l} &= \sum_{l\in\sigma_2}\frac{\sigma_l^2}{\gamma_l^2}\Bigg\langle\Bigg(\bar{L}^{12}_{jl}(\tau) + \sum_{k\in\sigma_1}\bar{B}^{112}_{jkl}(\tau)X_k(\vec{x},\tau)\Bigg)\\ &\qquad \times\Bigg(\bar{L}^{12}_{j'l}(\tau) + \sum_{k'\in\sigma_1}\bar{B}^{112}_{j'k'l}(\tau)X_{k'}(\vec{x},\tau)\Bigg)\Bigg\rangle\,. \end{aligned} \tag{5.12}$$

Finally, $X_j(\vec{x},\tau)$ is defined through the exponential of the skew-symmetric operator L_{11}

$$X_j(\vec{x},\tau) = \sum_{k\in\sigma_1}\left(e^{L_{11}\tau}\right)_{jk} x_k\,, \tag{5.13}$$

whereas the operators with a bar are defined from the original ones by action of $e^{-L_{11}\tau}$. For instance,

$$\bar{D}_{jk}(\tau) = \sum_{l\in\sigma_1}\left(e^{-L_{11}\tau}\right)_{jl} D_{lk}\,, \tag{5.14}$$

and similar relations hold for $\bar{L}^{12}_{jk}(\tau)$, $\bar{B}^{111}_{jkl}(\tau), \ldots$.

Theorem 5.3 can be proven using the asymptotic procedure of averaging modified as to account for fast averaging. In this paper we will only provide a formal derivation as outlined in Section 5.3. These calculations are similar to the ones presented in Appendix A.

The stochastic model for climate variables alone in (5.8) is necessarily complicated due to the interplay of the many phenomena associated with driving by the unresolved variables and the fast-wave effects. In particular, we observe in the equations in (5.8) both stable and unstable Langevin terms, modification of the climate mean, nonlinear corrections of the climate variables dynamics, and multiplicative noises. Besides the example in Section 5.1, a simple example illustrating these general features will be described in Section 7.3, and other more complex examples with nontrivial averaging due to topography, beta effects, and the mean U are given in the stochastic models in (6.38) and (6.42) in Section 6.

As explained in Section 5.3, the effects of fast forcing may be more complicated to account for because resonance phenomena may arise between fast-wave effects and fast forcing with the result that the time average involved in (5.10) fails to exist. If, however, we assume that no such resonance phenomena arise, then the results of Section 5.3 show that in the presence of fast forcing, the solution of the first equation in (4.1), $x_j^\varepsilon(t)$, satisfies

$$\sum_{k\in\sigma_1}\left(e^{-L_{11}t/\varepsilon}\right)_{jk}x_k^\varepsilon(t)-\int_0^t\left(e^{-L_{11}s/\varepsilon}\right)_{jk}f_k^1(s)ds\to x_j(t) \tag{5.15}$$

as $\varepsilon\to 0$, where $x_j(t)$ obeys an equation similar to the one in (5.8) with $X_j(\vec{x},\tau)$ replaced by

$$\hat{X}_j(\vec{x},\tau)g=\sum_{k\in\sigma_1}\left(e^{L_{11}\tau}\right)_{jk}x_k+\sum_{k\in\sigma_1}\int_0^\tau\left(e^{L_{11}(\tau-\tau')}\right)_{jk}f_k^1(\tau')d\tau'\,. \tag{5.16}$$

5.3 Asymptotic Procedure with Fast-Wave Averaging

We now generalize the asymptotic procedure introduced in Section 4.4 in order to deal with fast-wave effects on the climate variables. We will, however, carefully distinguish between

(1) the situation where only fast rotation wave effects are present, i.e., the situation where the term $\sum_k L_{jk}^{11}x_k/\varepsilon$ is present in (4.1), but we still assume $f_j^1(t/\varepsilon)=f_j^2(t/\varepsilon)=0$, and
(2) the complete situation where both fast rotation wave and forcing effects are present, i.e., the situation where all three terms $\sum_k L_{jk}^{11}x_k/\varepsilon$, $f_j^1(t/\varepsilon)/\varepsilon$, and $f_j^2(t/\varepsilon)$ in (4.1) are nonzero.

The reason for this distinction is that, in situation (1), the asymptotic procedure introduced in Section 4.4 carries over almost completely: We simply combine the latter method with standard multiple time-scale expansion in order to deal with the term $\sum_k L_{jk}^{11}x_k/\varepsilon$ in (4.1). On the other hand, in situation (2), the same kind of manipulations can be performed, but one has to assume the absence of resonance effects between rotation effects and forcing, which would make the averaging procedure fail. The presence or the absence of such resonance effects is hard to assess in the general case and will be discussed in specific applications in the future.

The Situation with Fast-Wave Effects But No Fast Forcing

We first consider the situation where we let the term $\sum_k L_{jk}^{11}x_k/\varepsilon$ be present in (4.1), but we assume $f_j^1(t/\varepsilon)=f_j^2(t/\varepsilon)=0$. Let $\varrho^\varepsilon(s,\vec{x},\vec{y}\mid t)$ be defined as in (4.21). The function ϱ^ε satisfies the backward equation associated with the equations in (4.1),

$$-\frac{\partial\varrho^\varepsilon}{\partial s}=\frac{1}{\varepsilon^2}\mathcal{L}_1\varrho^\varepsilon+\frac{1}{\varepsilon}\left(\mathcal{L}_2^{\mathrm{NS}}+\mathcal{L}_2^{\mathrm{S}}\right)\varrho^\varepsilon+\mathcal{L}_3\varrho^\varepsilon\,,\qquad\varrho^\varepsilon(t,\vec{x},\vec{y}\mid t)=f(\vec{x})\,. \tag{5.17}$$

The operators $\mathcal{L}_1$ and $\mathcal{L}_3$ are given as in (4.23), whereas in order to account for fast-wave effects, we have decomposed the operator $\mathcal{L}_2$ associated with the terms of order ε^{-1} in (4.1) into a non-skew-symmetric part $\mathcal{L}_2^{\mathrm{NS}}$ (which was the only one entering the backward equation in (4.22)) and a skew-symmetric part $\mathcal{L}_2^{\mathrm{S}}$ accounting for the fast-wave effect. They are given by

$$
\begin{aligned}
\mathcal{L}_2^{\mathrm{NS}} &= \sum_{j,k}\left(L^{12}_{jk}y_k + \frac{1}{2}\sum_l \left(2B^{112}_{jkl}x_k y_l + B^{122}_{jkl}y_k y_l\right)\right)\frac{\partial}{\partial x_j} \\
&\quad + \sum_{j,k}\left(L^{21}_{jk}x_k + L^{22}_{jk}y_k + \frac{1}{2}\sum_l \left(B^{211}_{jkl}x_k x_l + 2B^{221}_{jkl}y_k x_l\right)\right)\frac{\partial}{\partial y_j} \\
\mathcal{L}_2^{\mathrm{S}} &= \sum_{j,k} L^{11}_{jk}x_k\frac{\partial}{\partial x_j}\,.
\end{aligned}
\tag{5.18}
$$

We seek for a formal asymptotic solution of (5.17) with two time scales (compare equation (4.24))

$$
\varrho^{\varepsilon}(s \mid t) = \varrho_0(s,\tau \mid t) + \varepsilon\varrho_1(s,\tau \mid t) + \varepsilon^2\varrho_2(s,\tau \mid t) + \cdots\,, \qquad \tau = \frac{s}{\varepsilon}\,. \tag{5.19}
$$

Consistent with the separation of scales between s and τ, we treat these two time scales as if they were independent. Thus we set

$$
\frac{\partial}{\partial s} \to \frac{\partial}{\partial s} + \frac{1}{\varepsilon}\frac{\partial}{\partial \tau}\,. \tag{5.20}
$$

We insert (5.19) in (5.17) and use (5.20). Equating equal powers in ε gives the following sequence of equations (compare (4.25)–(4.27)):

$$
\mathcal{L}_1\varrho_0 = 0\,, \tag{5.21}
$$

$$
\mathcal{L}_1\varrho_1 = -\frac{\partial\varrho_0}{\partial\tau} - \mathcal{L}_2^{\mathrm{NS}}\varrho_0 - \mathcal{L}_2^{\mathrm{S}}\varrho_0\,, \tag{5.22}
$$

$$
\mathcal{L}_1\varrho_2 = -\frac{\partial\varrho_0}{\partial s} - \mathcal{L}_3\varrho_0 - \frac{\partial\varrho_1}{\partial\tau} - \mathcal{L}_2^{\mathrm{NS}}\varrho_1 - \mathcal{L}_2^{\mathrm{S}}\varrho_1\,, \tag{5.23}
$$

$$
\vdots
$$

Like equations (4.25)–(4.27), equations (5.21)–(5.23) require as a solvability condition that their right-hand sides have zero expectation with respect to the invariant measure of the Ornstein-Uhlenbeck process. The solvability condition is trivially satisfied for the equation in (5.21), which implies that

$$
\mathbf{P}\varrho_0 = \varrho_0\,. \tag{5.24}
$$

Thus to leading order the behavior is independent of $\vec{y}$,

$$
\varrho_0(s,\tau,\vec{x},\vec{y} \mid t) = \varrho_0(s,\tau,\vec{x} \mid t)\,.
$$

Taking next the expectation of the equation in (5.23), we obtain the solvability condition (compare (4.29))

$$0 = -\mathbf{P}\frac{\partial \varrho_0}{\partial \tau} - \mathbf{P}(\mathcal{L}_2^{\mathrm{NS}} + \mathcal{L}_2^{\mathrm{S}})\varrho_0 = -\frac{\partial \varrho_0}{\partial \tau} - \mathcal{L}_2^{\mathrm{S}}\varrho_0\,. \tag{5.25}$$

To derive (5.25) we used (5.24) combined with the property that the expectation $\mathbf{P}$ commutes with $\partial/\partial\tau$ and $\mathcal{L}_2^{\mathrm{S}}\mathbf{P}$. We have also set $\mathbf{P}\mathcal{L}_2^{\mathrm{NS}}\varrho_0 = \mathbf{P}\mathcal{L}_2^{\mathrm{NS}}\mathbf{P}\varrho_0 = 0$ consistent with (2.10) in assumption A4 in Section 2. The solution of (5.25) can be expressed as

$$\varrho_0(s, \tau, \vec{x} \mid t) = e^{\mathcal{L}_2^{\mathrm{S}}(\tau_0 - \tau)}\bar{\varrho}_0(s, \vec{x} \mid t)\,, \qquad \tau_0 = \frac{t}{\varepsilon}\,. \tag{5.26}$$

The action of the operator $e^{\mathcal{L}_2^{\mathrm{S}}(\tau_0-\tau)}$ on a suitable scalar-valued function $g(\vec{x})$ is given by

$$e^{\mathcal{L}_2^{\mathrm{S}}(\tau_0-\tau)}g(\vec{x}) = g\left(\left(e^{L_{11}(\tau_0-\tau)}\right)\vec{x}\right). \tag{5.27}$$

Since (5.25) holds, the solution of (5.22) is

$$\varrho_1 = -\mathcal{L}_1^{-1}\mathcal{L}_2^{\mathrm{NS}}e^{\mathcal{L}_2^{\mathrm{S}}(\tau_0-\tau)}\mathbf{P}\bar{\varrho}_0\,. \tag{5.28}$$

Inserting this expression in the equation in (5.23) and taking the expectation on the resulting equation to get the solvability condition for ϱ_2 yields

$$-e^{\mathcal{L}_2^{\mathrm{S}}(\tau_0-\tau)}\frac{\partial\bar{\varrho}_0}{\partial s} = \mathbf{P}\mathcal{L}_3 e^{\mathcal{L}_2^{\mathrm{S}}(\tau_0-\tau)}\mathbf{P}\bar{\varrho}_0 - \mathbf{P}\mathcal{L}_2^{\mathrm{NS}}\mathcal{L}_1^{-1}\mathcal{L}_2^{\mathrm{NS}}e^{\mathcal{L}_2^{\mathrm{S}}(\tau_0-\tau)}\mathbf{P}\bar{\varrho}_0\,. \tag{5.29}$$

The backward equation for $\bar{\varrho}_0$ given in (5.47) is obtained from (5.29) by applying $e^{-\mathcal{L}_2^{\mathrm{S}}(\tau_0-\tau)}$ on both sides and averaging with respect to time the resulting equation. This gives

$$-\frac{\partial\bar{\varrho}_0}{\partial s} = \hat{\mathcal{L}}\bar{\varrho}_0\,, \qquad \bar{\varrho}_0(t, \vec{x} \mid t) = f(\vec{x})\,, \tag{5.30}$$

where

$$\hat{\mathcal{L}}\,\cdot = \mathbf{P}\left(\lim_{T\to\infty}\frac{1}{T}\int_{t-T}^{t} e^{-\mathcal{L}_2^{\mathrm{S}}(\tau_0-\tau)}\left(\mathcal{L}_3 - \mathcal{L}_2^{\mathrm{NS}}\mathcal{L}_1^{-1}\mathcal{L}_2^{\mathrm{NS}}\right)e^{\mathcal{L}_2^{\mathrm{S}}(\tau_0-\tau)}\,d\tau\right)\mathbf{P}\,\cdot\,. \tag{5.31}$$

Notice that the time average in (5.31) exists because of the skew-symmetric nature of $\mathcal{L}_2$.

We now show that the operator $\hat{\mathcal{L}}$ in (5.31) is a Fokker-Planck operator that is effectively computable. This essentially amounts to evaluating the action of the operators $e^{-\mathcal{L}_2^{\mathrm{S}}(\tau_0-\tau)}$ in (5.31). We do so by taking into account the fast-wave effects in the equations in (4.1) from the very beginning by an appropriate change of variables in these equations. The key step is to introduce the function

$$\bar{\varrho}^{\varepsilon}(s, \vec{x}, \vec{y} \mid t) = e^{-\mathcal{L}_2^{\mathrm{S}}(t-s)/\varepsilon}\varrho^{\varepsilon}(s, \vec{x}, \vec{y} \mid t)\,. \tag{5.32}$$

From (5.26), it follows that the function $\bar{\varrho}^{\varepsilon}$ is such that, in the limit as $\varepsilon \to 0$,

$$\bar{\varrho}^{\varepsilon} \to \bar{\varrho}_0\,, \tag{5.33}$$

where $\bar{\varrho}_0$ is the function defined in (5.26) that satisfies the equation in (5.30). On the other hand, as we now show, working with $\bar{\varrho}^\varepsilon$ instead of ϱ^ε allows us to account for the fast-wave rotation effects from the very beginning because it amounts to making an appropriate change of dependent variables in (4.1). More precisely, $\bar{\varrho}^\varepsilon$ satisfies the backward equation associated with the equations that are obtained from (4.1) upon changing dependent variables consistent with the equation of the operator $e^{-\mathcal{L}_2^{\mathrm{S}}(t-s)/\varepsilon}$, namely, upon defining $\bar{\vec{x}}$ from

$$\bar{\vec{x}}(t) = \left(e^{-L_{11}t/\varepsilon}\right)\vec{x}(t)\,, \tag{5.34}$$

or, with index notation,

$$\bar{x}_j(t) = \sum_{k\in\sigma_1}\left(e^{-L_{11}t/\varepsilon}\right)_{jk} x_k(t)\,. \tag{5.35}$$

In terms of $(\bar{\vec{x}}, \vec{y})$ the equations in (4.1) with $f^1 = f^2 = 0$ become

$$\begin{aligned} d\bar{x}_j &= \bar{F}_j^1(t)dt + \sum_k\left(-\bar{D}_{jk}(t)X_k + \frac{1}{\varepsilon}\bar{L}_{jk}^{12}(t)y_k\right)dt \\ &\quad + \frac{1}{2}\sum_{k,l}\left(\bar{B}_{jkl}^{111}(t)X_kX_l + \frac{2}{\varepsilon}\bar{B}_{jkl}^{112}(t)X_ky_l + \frac{1}{\varepsilon}\bar{B}_{jkl}^{122}(t)y_ky_l\right)dt\,, \\ dy_j &= \sum_k\left(\frac{1}{\varepsilon}L_{jk}^{21}X_k + \frac{1}{\varepsilon}L_{jk}^{22}y_k\right)dt \\ &\quad + \frac{1}{2}\sum_{k,l}\left(\frac{1}{\varepsilon}B_{jkl}^{211}X_kX_l + \frac{2}{\varepsilon}B_{jkl}^{221}y_kX_l\right)dt - \frac{\gamma_j}{\varepsilon^2}y_j\,dt + \frac{\sigma_j}{\varepsilon}\,dW_j(t)\,, \end{aligned} \tag{5.36}$$

where we defined

$$X_j = \sum_{k\in\sigma_1}\left(e^{L_{11}t/\varepsilon}\right)_{jk}\bar{x}_k\,, \tag{5.37}$$

and the operators with a bar are defined from the original ones by action of $e^{-L_{11}t/\varepsilon}$. For instance,

$$\bar{D}_{jk}(t) = \sum_{l\in\sigma_1}\left(e^{-L_{11}t/\varepsilon}\right)_{jl} D_{lk}\,, \tag{5.38}$$

and similar relations hold for $\bar{L}_{jk}^{12}$, $\bar{B}_{jkl}^{111}$, ... Thus, the backward equation for $\bar{\varrho}^\varepsilon$ is given by (omitting now the bar on $\bar{\vec{x}}$, i.e., setting $\bar{\vec{x}} \to \vec{x}$)

$$-\frac{\partial\bar{\varrho}^\varepsilon}{\partial s} = \frac{1}{\varepsilon^2}\mathcal{L}_1\varrho^\varepsilon + \frac{1}{\varepsilon}\bar{\mathcal{L}}_2^{\mathrm{NS}}(\tau)\bar{\varrho}^\varepsilon + \bar{\mathcal{L}}_3(s,\tau)\bar{\varrho}^\varepsilon\,, \qquad \bar{\varrho}^\varepsilon(t,\vec{x},\vec{y}\mid t) = f(\vec{x})\,, \tag{5.39}$$

where $\tau = s/\varepsilon$, and we have explicitly distinguished the dependence in slow, s, and fast, $\tau = s/\varepsilon$, time scales in defining the operators $\bar{\mathcal{L}}_2^{\mathrm{NS}}(\tau)$ and $\bar{\mathcal{L}}_3^{\mathrm{NS}}(s,\tau)$.

They are given by

$$
\begin{aligned}
\bar{\mathcal{L}}_2^{\mathrm{NS}}(\tau) &= \sum_{j,k} \bar{L}^{12}_{jk}(\tau) y_k \frac{\partial}{\partial x_j} \\
&\quad + \frac{1}{2} \sum_{j,k,l} \left(2\bar{B}^{112}_{jkl}(\tau) X_k(x,\tau) y_l + \bar{B}^{122}_{jkl}(\tau) y_k y_l \right) \frac{\partial}{\partial x_j} \\
&\quad + \sum_{j,k} \left(L^{21}_{jk} X_k(\vec{\bar{x}},\tau) + L^{22}_{jk} y_k \right) \frac{\partial}{\partial y_j} \\
&\quad + \frac{1}{2} \sum_{l} \left(B^{211}_{jkl} X_k(\vec{x},\tau) X_l(\vec{x},\tau) + 2B^{221}_{jkl} y_k X_l(\vec{\bar{x}},\tau) \right) \frac{\partial}{\partial y_j} \\
\bar{\mathcal{L}}_3(s,\tau) &= \sum_j \left(\bar{F}^1_j(s,\tau) - \sum_k \bar{D}_{jk}(\tau) X_k(\vec{x},\tau) \right) \frac{\partial}{\partial x_j} \\
&\quad + \frac{1}{2} \sum_{j,k,l} \bar{B}^{111}_{jkl}(\tau) X_k(\vec{x},\tau) X_l(\vec{x},\tau) \frac{\partial}{\partial x_j} ,
\end{aligned} \tag{5.40}
$$

where we defined

$$X_j(\vec{x},\tau) = \sum_{k\in\sigma_1} \left(e^{L_{11}\tau} \right)_{jk} x_k , \tag{5.41}$$

and, similarly to (5.38), the operators with a bar are defined from the original ones by action of $e^{-L_{11}\tau}$. For instance,

$$\bar{D}_{jk}(\tau) = \sum_{l\in\sigma_1} \left(e^{-L_{11}\tau} \right)_{jl} D_{lk} . \tag{5.42}$$

Manipulations similar to the one that led to (5.30) can now be performed for the backward equation in (5.39). Of course, there are no rotation effects to treat in the backward equation in (5.39) since these effects were taken into account from the very beginning by replacing the equations in (4.1) by the equations in (5.36). This means that by manipulating the backward equation given in (5.39), we obtain an alternative but, of course, equivalent expression for $\bar{\mathcal{L}}$, or, in short, we have succeeded in evaluating the action of the operators $e^{-\mathcal{L}_2^{\mathrm{S}}(\tau_0-\tau)}$ in (5.31). The expression for $\hat{\mathcal{L}}$ equivalent to the one in (5.31) is

$$\hat{\mathcal{L}} \cdot = \mathbf{P} \left(\lim_{T\to\infty} \frac{1}{T} \int_{t-T}^{t} \left(\bar{\mathcal{L}}_3(s,\tau) - \bar{\mathcal{L}}_2^{\mathrm{NS}}(\tau) \mathcal{L}_1^{-1} \bar{\mathcal{L}}_2^{\mathrm{NS}}(\tau) \right) d\tau \right) \mathbf{P} \cdot , \tag{5.43}$$

where the integration on τ is performed with s kept fixed.

These manipulations can be summarized into the following formal:

THEOREM 5.4 *Let $\varrho^\varepsilon(s,\vec{x},\vec{y} \mid t)$ satisfy*

$$-\frac{\partial \varrho^\varepsilon}{\partial s} = \frac{1}{\varepsilon^2} \mathcal{L}_1 \varrho^\varepsilon + \frac{1}{\varepsilon} \left(\mathcal{L}_2^{NS} + \mathcal{L}_2^{S} \right) \varrho^\varepsilon + \mathcal{L}_3 \varrho^\varepsilon , \qquad \varrho^\varepsilon(t,\vec{x},\vec{y} \mid t) = f(\vec{x}) , \tag{5.44}$$

where $\mathcal{L}_1$, $\mathcal{L}_2^{NS}$, $\mathcal{L}_2^{S}$, and $\mathcal{L}_3$ are backward Fokker-Planck operators given in (4.23) *and* (5.18). *$\mathcal{L}_2^S$ is a first-order skew-symmetric transport operator, whereas $\mathcal{L}_1$ generates a stationary process such that*

$$e^{\mathcal{L}_1 t}\cdot \to \mathbf{P}\cdot \quad \text{as } t\to\infty\,. \tag{5.45}$$

Then,

$$e^{\mathcal{L}_2^S (t-s)/\varepsilon}\varrho^\varepsilon(s,\vec{x},\vec{y}\mid t) \to \bar{\varrho}_0(s,\vec{x}\mid t)\,, \tag{5.46}$$

in the limit as $\varepsilon\to 0$, for $-T<s\le t$, $T<\infty$, uniformly in $\vec{x}$ and $\vec{y}$ on compact sets, where $\bar{\varrho}_0$ satisfies $\bar{\varrho}_0 = \mathbf{P}\bar{\varrho}_0$ and solves the backward equation

$$-\frac{\partial\bar{\varrho}_0}{\partial s} = \hat{\mathcal{L}}\bar{\varrho}_0\,, \qquad \bar{\varrho}_0(t,\vec{x}\mid t) = f(\vec{x})\,, \tag{5.47}$$

where

$$\begin{aligned}\hat{\mathcal{L}}\,\cdot &= \mathbf{P}\biggl(\lim_{T\to\infty}\frac{1}{T}\int_{t-T}^{t} e^{-\mathcal{L}_2^S(\tau_0-\tau)}\bigl(\mathcal{L}_3 - \mathcal{L}_2^{NS}\mathcal{L}_1^{-1}\mathcal{L}_2^{NS}\bigr)e^{\mathcal{L}_2^S(\tau_0-\tau)}\,d\tau\biggr)\mathbf{P}\,\cdot \\ &= \mathbf{P}\biggl(\lim_{T\to\infty}\frac{1}{T}\int_{t-T}^{t}\bigl(\bar{\mathcal{L}}_3(s,\tau) - \bar{\mathcal{L}}_2^{NS}(\tau)\mathcal{L}_1^{-1}\bar{\mathcal{L}}_2^{NS}(\tau)\bigr)d\tau\biggr)\mathbf{P}\,\cdot\,.\end{aligned} \tag{5.48}$$

Here $\bar{\mathcal{L}}_3(s,\tau)$, $\bar{\mathcal{L}}_2^{NS}(\tau)$ are the operators given in (5.40), *and the integration on τ is performed with s kept fixed.*

The Situation with Fast-Wave Effects and Fast Forcing

We briefly comment on the complete situation where both fast-wave and forcing effects are present and all three terms $\sum_k L_{jk}^{11}x_k/\varepsilon$, $f_j^1(t/\varepsilon)/\varepsilon$, and $f_j^2(t/\varepsilon)$ in (4.1) are nonzero. A backward equation similar to the equation in (5.17) can be associated with the complete equations in (4.1), but an additional difficulty arises because the skew-symmetric operator $\mathcal{L}_2^{\mathrm{S}}$ in (5.18) is replaced by

$$\mathcal{L}_2^{\text{“S”}} = \sum_j f_j^1\left(\frac{s}{\varepsilon}\right)\frac{\partial}{\partial x_j} + \sum_j f_j^2\left(\frac{s}{\varepsilon}\right)\frac{\partial}{\partial y_j} + \sum_{j,k} L_{jk}^{11}x_k\frac{\partial}{\partial x_j}\,. \tag{5.49}$$

Due to the terms involving forcing, this operator is skew-symmetric at fixed argument but has time dependence. This implies that all the manipulations we did in situation (i) can be formally performed in the present situation, and an effective equation similar to the one in (5.47) with $\hat{\mathcal{L}}$ given as in (5.48) but with $\mathcal{L}_2^{\text{“S”}}$ given by (5.49) instead of $\mathcal{L}_2^{\mathrm{S}}$ given by (5.18) can be associated when both fast-rotation-wave and -forcing effects are present in (4.1). However, these manipulations may be formal because the time average involved in (5.48) may fail to exist with $\mathcal{L}_2^{\text{“S”}}$. This will typically be the case if there are resonance effects between rotation waves and forcing effects. Assuming that no resonance effects arise and the time average involved exists, the operator $\hat{\mathcal{L}}$ entering the backward equation in (5.47) associated with the complete system of equations in (4.1) is given by

$$\hat{\mathcal{L}}\,\cdot = \mathbf{P}\left(\lim_{T\to\infty}\frac{1}{T}\int_{t-T}^{t}\bigl(\bar{\mathcal{L}}_3(s,\tau) - \bar{\mathcal{L}}_2^{\mathrm{NS}}(\tau)\mathcal{L}_1^{-1}\bar{\mathcal{L}}_2^{\mathrm{NS}}(\tau)\bigr)d\tau\right)\mathbf{P}\,\cdot\,, \tag{5.50}$$

where

$$
\begin{aligned}
\bar{\mathcal{L}}_2^{\mathrm{NS}}(\tau) &= \sum_{j,k} \bar{L}_{jk}^{12}(\tau) y_k \frac{\partial}{\partial x_j} \\
&\quad + \frac{1}{2}\sum_{j,k,l} \big(2\bar{B}_{jkl}^{112}(\tau)\hat{X}_k(x,\tau) y_l + \bar{B}_{jkl}^{122}(\tau) y_k y_l\big)\frac{\partial}{\partial x_j} \\
&\quad + \sum_j f_j^2(\tau)\frac{\partial}{\partial y_j} + \sum_{j,k}\big(L_{jk}^{21}\hat{X}_k(\vec{\bar{x}},\tau) + L_{jk}^{22} y_k\big)\frac{\partial}{\partial y_j} \\
&\quad + \frac{1}{2}\sum_l \big(B_{jkl}^{211}\hat{X}_k(\vec{x},\tau)\hat{X}_l(\vec{x},\tau) + 2B_{jkl}^{221} y_k \hat{X}_l(\vec{\bar{x}},\tau)\big)\frac{\partial}{\partial y_j} \\
\bar{\mathcal{L}}_3(s,\tau) &= \sum_j \left(\bar{F}_j^1(s,\tau) - \sum_k \bar{D}_{jk}(\tau)\hat{X}_k(\vec{x},\tau)\right)\frac{\partial}{\partial x_j} \\
&\quad + \frac{1}{2}\sum_{j,k,l}\bar{B}_{jkl}^{111}(\tau)\hat{X}_k(\vec{x},\tau)\hat{X}_l(\vec{x},\tau)\frac{\partial}{\partial x_j}\,.
\end{aligned}
\tag{5.51}
$$

Here we defined

$$
\hat{X}_j(\vec{x},\tau) = \sum_{k\in\sigma_1}\big(e^{L_{11}\tau}\big)_{jk} x_k + \sum_{k\in\sigma_1}\int_0^\tau \big(e^{L_{11}(\tau-\tau')}\big)_{jk} f_k^1(\tau')\,d\tau'\,, \tag{5.52}
$$

and, as in (5.42), operators with a bar are defined from the original ones by action of $e^{-L_{11}\tau}$. For instance,

$$
\bar{D}_{jk}(\tau) = \sum_{l\in\sigma_1}\big(e^{L_{11}\tau}\big)_{jl} D_{lk}\,. \tag{5.53}
$$

The set of stochastic differential equations associated with the operator $\hat{\mathcal{L}}$ given in (5.50) is the stochastic climate model given in Theorem 5.3. Examples with potential resonance due to climate forcing will be discussed elsewhere by the authors.

PROOF OF THEOREM 5.1: The proof generalizes to the complex case the procedure given in Section 5.3, and then uses the results of Appendix A to compute the operator $\hat{\mathcal{L}}$ given in (5.48) in a way similar to what is done in Appendix B. Let

$$
\varrho^\varepsilon\big(s, v_k, v_k^*, w_k, w_k^* \mid t\big) = \mathbf{E} f\big(v_k^\varepsilon(t), w_k^\varepsilon(t)\big)\,, \tag{5.54}
$$

where f is a suitable scalar-valued function and $(v_k^\varepsilon(t), w_k^\varepsilon(t))$ solves the equations in (5.1) for the initial condition

$$
(v_k^\varepsilon(s), w_k^\varepsilon(s)) = (v_k, w_k)\,. \tag{5.55}
$$

As a function of the independent variables (v_k, v_k^*, w_k, w_k^*), ϱ^ε satisfies the backward equation analog to the equation in (5.44), i.e.,

$$
-\frac{\partial \varrho^\varepsilon}{\partial s} = \frac{1}{\varepsilon^2}\mathcal{L}_1\varrho^\varepsilon + \frac{1}{\varepsilon}\big(\mathcal{L}_2^{\mathrm{NS}} + \mathcal{L}_2^{\mathrm{S}}\big)\varrho^\varepsilon\,, \tag{5.56}
$$

where the operator $\mathcal{L}_1$, $\mathcal{L}_2^{\mathrm{NS}}$, $\mathcal{L}_2^{\mathrm{S}}$ are identified from equations (5.1) as explained in Section 5.2: $\mathcal{L}_1$ is the Fokker-Planck operator associated with the model in (3.14), and $\mathcal{L}_2^{\mathrm{NS}}$ and $\mathcal{L}_2^{\mathrm{S}}$ are the non-skew-symmetric and skew-symmetric operators associated with the terms of order ε^{-1} in (5.1). These operators are given explicitly by

$$
\begin{aligned}
\mathcal{L}_1 &= -\sum_{\vec{k}} \gamma_k \frac{\partial}{\partial w_k} - \sum_{\vec{k}} \gamma_k \frac{\partial}{\partial w_k^*} \\
&\quad + \sum_{\vec{k}} \sigma_k^2 \frac{\partial^2}{\partial w_k \partial w_k^*} + \sum_{\vec{k}} \sigma_k^2 \frac{\partial^2}{\partial w_k \partial w_{-k}} + \sum_{\vec{k}} \sigma_k^2 \frac{\partial^2}{\partial w_k^* \partial w_{-k}^*}\,, \\
\mathcal{L}_2^{\mathrm{NS}} &= \frac{1}{2} \sum_{\vec{k},\vec{l},\vec{m}} \left(B_{klm} v_l^* v_m^* + 2B_{klm} v_l^* w_m^* + B_{klm} w_l^* w_m^*\right) \frac{\partial}{\partial v_k} \\
&\quad + \frac{1}{2} \sum_{\vec{k},\vec{l},\vec{m}} \left(B_{klm} v_l v_m + 2B_{klm} v_l w_m + B_{klm} w_l w_m\right) \frac{\partial}{\partial v_k^*} \\
&\quad + \frac{1}{2} \sum_{\vec{k},\vec{l},\vec{m}} \left(B_{klm} v_l^* v_m^* + 2B_{klm} v_l^* w_m^*\right) \frac{\partial}{\partial w_k} \\
&\quad + \frac{1}{2} \sum_{\vec{k},\vec{l},\vec{m}} \left(B_{klm} v_l v_m + 2B_{klm} v_l w_m\right) \frac{\partial}{\partial w_k^*}\,, \\
\mathcal{L}_2^{\mathrm{S}} &= i \sum_{\vec{k}} \Omega_k \frac{\partial}{\partial v_k} - i \sum_{\vec{k}} \Omega_k \frac{\partial}{\partial v_k^*} + i \sum_{\vec{k}} \Omega_k \frac{\partial}{\partial w_k} - i \sum_{\vec{k}} \Omega_k \frac{\partial}{\partial w_k^*}\,,
\end{aligned}
\tag{5.57}
$$

where for simplicity of notation we have omitted the explicit summation sets. To compute the operator $\hat{\mathcal{L}}$ obtained from the equation in (5.48), we need to derive $\bar{\mathcal{L}}_2^{\mathrm{NS}}$ as given in (5.40). This is particularly simple in the present case because the operator $\mathcal{L}_2^{\mathrm{S}}$ is diagonal. Hence the rotation induced by this operator is easily accounted for and amounts to setting

$$
v_k \to v_k e^{i\Omega_k \tau}\,, \qquad w_k \to w_k e^{i\Omega_k \tau}\,. \tag{5.58}
$$

It follows that

$$
\begin{aligned}
\bar{\mathcal{L}}_2^{\mathrm{NS}}(\tau) &= \frac{1}{2} \sum_{\vec{k},\vec{l},\vec{m}} \left(B_{klm}(\tau) v_l^* v_m^* + 2B_{klm}(\tau) v_l^* w_m^* + B_{klm}(\tau) w_l^* w_m^*\right) \frac{\partial}{\partial v_k} \\
&\quad + \frac{1}{2} \sum_{\vec{k},\vec{l},\vec{m}} \left(B_{klm}^*(\tau) v_l v_m + 2B_{klm}^*(\tau) v_l w_m + B_{klm}^*(\tau) w_l w_m\right) \frac{\partial}{\partial v_k^*} \\
&\quad + \frac{1}{2} \sum_{\vec{k},\vec{l},\vec{m}} \left(B_{klm}(\tau) v_l^* v_m^* + 2B_{klm}(\tau) v_l^* w_m^*\right) \frac{\partial}{\partial w_k} \\
&\quad + \frac{1}{2} \sum_{\vec{k},\vec{l},\vec{m}} \left(B_{klm}^*(\tau) v_l v_m + 2B_{klm}^*(\tau) v_l w_m\right) \frac{\partial}{\partial w_k^*}\,,
\end{aligned}
\tag{5.59}
$$

in which we defined

$$B_{klm}(\tau) = B_{klm} e^{-i(\Omega_k+\Omega_l+\Omega_m)\tau} \,. \tag{5.60}$$

The equation in (5.48) can be written as

$$\hat{\mathcal{L}} \cdot = -\mathbf{P}\left(\lim_{T\to\infty} \frac{1}{T}\int_{t-T}^{t} \bar{\mathcal{L}}_2^{\mathrm{NS}}(\tau)\mathcal{L}_1^{-1}\bar{\mathcal{L}}_2^{\mathrm{NS}}(\tau)d\tau\right)\mathbf{P}\cdot\,, \tag{5.61}$$

and Theorem 5.1 follows after explicit evaluation of this operator. We skip these calculations since they are a straightforward generalization of what is done in Appendix B. □

6 Idealized Climate Models from Equilibrium Statistical Mechanics

In this section we study in more detail the systematic design of stochastic climate models for the truncated barotropic equations in (3.3), which we recall for the reader's convenience:

$$\begin{gathered} \frac{\partial q_\Lambda}{\partial t} + P_\Lambda\left(\nabla^\perp\psi_\Lambda\cdot\nabla q_\Lambda\right) + U\frac{\partial q_\Lambda}{\partial x} + \beta\frac{\partial\psi_\Lambda}{\partial x} = 0\,, \\ q_\Lambda = \omega_\Lambda + h_\Lambda\,, \quad \omega_\Lambda = \Delta\psi_\Lambda, \quad \frac{dU}{dt} = \fint h_\Lambda\frac{\partial\psi_\Lambda}{\partial x}\,, \end{gathered} \tag{3.3}$$

where P_Λ is the projection operator associated with a defining set $\bar{\sigma}_\Lambda$ (see the discussion below (3.19)) defined for any suitable function f as

$$f_\Lambda(\vec{x}) = P_\Lambda f(\vec{x}) = \sum_{k\in\bar{\sigma}_\Lambda}\left(\hat{f}_k e^{i\vec{k}\cdot\vec{x}} + \hat{f}_k^* e^{-i\vec{k}\cdot\vec{x}}\right). \tag{6.1}$$

The equations in (3.3) are an important first test case for stochastic climate modeling since they include large- and small-scale inhomogeneity and anisotropy through the interaction of the geophysical effects from U, β, and the topography, h. The systematic approach we develop below can be extended to a number of important climate models directly such as two-layer models or barotropic flow on the sphere. These applications will be developed elsewhere.

In this section, we incorporate in the theory the important fact that an equilibrium statistical theory can be developed for the truncated barotropic equations in (3.3). The equilibrium statistical theory is based on the existence of two conserved quantities—energy and enstrophy—and is presented in Section 6.1. In Section 6.2, we show that the stochastic model for the truncated barotropic equations can be made fully consistent with equilibrium statistical theory by appropriate constraints on the parameters in the stochastic model. In other words, the stochastic model for the truncated barotropic equations shares the same Gaussian invariant measure with density in (6.16) as the original truncated barotropic equations. In addition, we demonstrate that the stochastic model for the climate variables alone that is derived from the truncated barotropic equations also satisfies an equilibrium statistical theory. Furthermore, the invariant measure for the effective climate model is the projection on the climate variables alone of the invariant measure for the original system of truncated barotropic equations.

In Section 6.3, we report on numerical simulations of the truncated barotropic equations in several different parameter regimes that demonstrate effective stochasticity and separation of time scales for the evolution of specific variables. In other words, we justify numerically the possibility of distinguishing unambiguously between climate and unresolved variables for the truncated barotropic equations. We also show that the equilibrium statistical theory is supported by the numerical simulations.

Finally, in Section 6.4, we show how the numerical simulations can be used to identify the parameters entering the stochastic model equations. We also indicate how the numerical results for the climate variables can be compared with the solutions of the equations for these variables alone that are derived by the asymptotic strategy of Section 4.

6.1 Equilibrium Statistical Theory for Geophysical Models

It can be shown by direct calculation that the dynamics in the truncated barotropic equations in (3.3) conserves the truncated energy E_Λ and the truncated enstrophy $\mathcal{E}_\Lambda$

$$\begin{aligned} E_\Lambda &= \frac{1}{2}U^2 + \frac{1}{2}\fint |\nabla\psi_\Lambda|^2 = \frac{1}{2}U^2 - \frac{1}{2}\fint \psi_\Lambda\omega_\Lambda\,, \\ \mathcal{E}_\Lambda &= \beta U + \frac{1}{2}\fint q_\Lambda^2\,. \end{aligned} \tag{6.2}$$

Based on these two conserved quantities, it is possible to construct an equilibrium statistical theory for the truncated barotropic equations in (3.3), as we show now. We sketch the argument in a rather heuristic way here; more details can be found in [4, 17]. We proceed in two steps.

Step 1. Consider the truncated barotropic equations written in the compact notation introduced in (3.10) as

$$\frac{d\vec{z}}{dt} = \vec{F}(\vec{z}) \equiv L\vec{z} + B(\vec{z},\vec{z})\,. \tag{6.3}$$

The vector field $\vec{F}(\vec{z})$ is divergence free, or incompressible, in the phase space $\Omega = \{\vec{z}\}$, i.e.,

$$\operatorname{div}\vec{F} = 0 \quad \text{or} \quad \nabla_{\vec{z}}\cdot\vec{F} = 0\,. \tag{6.4}$$

It follows that the flow map $\{\vec{\varphi}_t(\vec{z})\}$ associated with the equations in (3.11) defined by

$$\frac{d}{dt}\vec{\varphi}_t(\vec{z}) = \vec{F}\big(\vec{\varphi}_t(\vec{z})\big)\,, \quad \vec{\varphi}_0(\vec{z}) = \vec{z}\,, \tag{6.5}$$

is volume (or measure) preserving on the phase space, i.e.,

$$\det\big(\nabla_{\vec{z}}\cdot\vec{\varphi}_t(\vec{z})\big) = 1\,. \tag{6.6}$$

Utilizing the measure-preserving property of the flow map $\{\vec{\varphi}_t(\vec{z})\}$, it is possible to define probability measures on the phase space Ω. Specifically, for any suitable function $A(\vec{z})$ defined on Ω, we can associate an observable as the image value of A under the map $\{\vec{\varphi}_t(\vec{z})\}$

$$A\big(\vec{\varphi}_t(\vec{z})\big)\,. \tag{6.7}$$

Let $P_0(\vec{z})$ be a probability density on the phase space Ω, and define the average value of A with respect to P_0 at initial time $t = 0$ as

$$\mathbf{P}_0 A(\vec{z}) = \int_\Omega A(\vec{z}) P_0(\vec{z}) d\vec{z}\,. \tag{6.8}$$

Then, using the measure-preserving property of the flow map $\{\vec{\varphi}_t(\vec{z})\}$, it follows that the average value of the observable $A(\vec{\varphi}_t(\vec{z}))$ is given at time $t > 0$ by

$$\mathbf{P}_0 A\big(\vec{\varphi}_t(\vec{z})\big) = \int_\Omega A\big(\vec{\varphi}_t(\vec{z})\big) P_0(\vec{z}) d\vec{z} = \int_\Omega A(\vec{z}) P_0\big(\vec{\varphi}_t^{-1}(\vec{z})\big) d\vec{z}\,, \tag{6.9}$$

where $\{\vec{\varphi}_t^{-1}(\vec{z})\}$ is the flow map inverse to $\{\vec{\varphi}_t(\vec{z})\}$. It follows from the equation in (6.9) that

$$P(\vec{z}, t) = P_0(\vec{\varphi}_t^{-1}(\vec{z})) \tag{6.10}$$

is a probability density on Ω that allows us to compute the average of all observables on Ω at time $t > 0$. Notice also that, from (6.5) and (6.10), the probability density $P(\vec{z}, t)$ satisfies the Liouville (or forward) equation

$$\frac{\partial P}{\partial t} + \vec{F}(\vec{z}) \cdot \nabla_{\vec{z}} P = 0\,. \tag{6.11}$$

Step 2. By definition, the invariant measures on Ω are those measures whose associated densities are preserved by the map $\{\vec{\varphi}_t^{-1}(\vec{z})\}$, i.e., such that for all $t > 0$

$$P^\star(\vec{z}) = P^\star\big(\vec{\varphi}_t^{-1}(\vec{z})\big)\,. \tag{6.12}$$

Equivalently, the probability densities of the invariant measures are steady-state solutions of the Liouville equation in (6.11)

$$\vec{F}(\vec{z}) \cdot \nabla_{\vec{z}} P^\star = 0\,. \tag{6.13}$$

Invariant measures are readily obtained for the truncated barotropic equations in (3.3). Indeed, by definition of the conserved quantities in (6.2), any function $G(E_\Lambda, \mathcal{E}_\Lambda)$ of the energy E_Λ and enstrophy $\mathcal{E}_\Lambda$ is preserved by the map $\{\vec{\varphi}_t^{-1}(\vec{z})\}$ or, equivalently, satisfies

$$\vec{F}(\vec{z}) \cdot \nabla_{\vec{z}} G = 0\,. \tag{6.14}$$

In equilibrium statistical theory, given some conserved quantities, it is postulated that the actual invariant measure is the canonical measure whose density is given by

$$P_C^\star = C e^{-\theta E_\Lambda - \alpha \mathcal{E}_\Lambda}\,. \tag{6.15}$$

In fact, the measure with density in (6.16) arises from a maximum entropy principle as the Gibbs probability measure with the least bias given the information in the two conserved quantities E_Λ, $\mathcal{E}_\Lambda$ [17]. Here C is a normalization constant and θ, α are parameters playing the role of the inverse temperature in the usual equilibrium statistical mechanics theory. The values of θ, α depend on the actual values of E_Λ, $\mathcal{E}_\Lambda$, and it is customary to introduce $\mu = \theta/\alpha$ and write the density in (6.15) as

$$P_C^\star = Ce^{-\alpha(\mu E_\Lambda + \mathcal{E}_\Lambda)}\,. \tag{6.16}$$

Equilibrium statistical theory applies under the assumption of ergodicity with respect to time average of the dynamics defined by the truncated barotropic equations in (3.3). Ergodicity implies that time averaging and ensemble averaging are identical, i.e.,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T A\big(\vec{\varphi}_t(\vec{z})\big)dt = \int_\Omega A(\vec{z})P_C^*(\vec{z})d\vec{z} \tag{6.17}$$

for any suitable function $A(\vec{z})$. The numerical simulations presented in Section 6.3 support the ergodicity assumption in (6.17) for suitable functions $A(\vec{z})$ involving low-order moments.

The density in (6.16) is a Gaussian density that is completely characterized by its mean and variance in each Fourier component

$$\begin{aligned} \bar{U} &= \operatorname{mean} U = -\frac{\beta}{\mu}\,, \qquad && \operatorname{var} U = \frac{1}{\alpha\mu}\,, \\ \bar{u}_k &= \operatorname{mean} u_k = \frac{|\vec{k}|\hat{h}_k}{\mu + |\vec{k}|^2}\,, \qquad && \operatorname{var} u_k = \frac{1}{\alpha(\mu + |\vec{k}|^2)}\,, \end{aligned} \tag{6.18}$$

with $u_k = |\vec{k}|\hat{\psi}_k$. Thus, a nontrivial mean exists that is the idealized climate mean for nonzero topography, h_Λ. The Gaussian measures with density in (6.16) are finite and realizable for $\mu > 0$ for general h_Λ with β, U nonzero; if $\beta = 0$ and $U = 0$, the measures with density in (6.16) are realizable in the regime $\mu > -1$ including a "negative temperature" regime [4, 17]. The combination $\mu E_\Lambda + \mathcal{E}_\Lambda$ is called the *pseudoenergy* associated with the mean state, $\bar{U}$, $\bar{u}_k$.

6.2 Stochastic Climate Models Consistent with Equilibrium Statistical Theory

We now develop the stochastic model for the truncated barotropic equations in (3.3) consistent with the equilibrium statistical theory developed in Section 6.1. We leave a detailed analysis of the stochastic climate model equations associated with the stochastic model for a future publication and, in the present section, we content ourselves with the analysis of some general properties of the stochastic climate model, which can be summarized in the following statement:

(S) *There is a simple explicit procedure to construct a stochastic model from the truncated barotropic equations in* (3.3) *that satisfies the same equilibrium statistical theory as the original system. Furthermore, equilibrium statistical theory for the stochastic model projects into equilibrium statistical theory for the stochastic model for the climate variables alone. In particular, the reduced stochastic climate model automatically has the same mean and energy spectrum as the projection of the nonlinear dynamical model.*

A more precise phrasing of the statement in (S) is given in Propositions 6.5 and 6.6 below. The key idea in these systematic developments involves detailed conservation of pseudoenergy separately for appropriate parts of the equations in (3.3).

Detailed Balance for Pseudoenergy with a Climate Mean

From (6.18), for a fixed realizable value of μ, the climate mean state is given by

$$\bar{q}_\Lambda = \Delta\bar{\psi}_\Lambda + h_\Lambda = \mu\bar{\psi}_\Lambda\,, \quad \bar{\omega}_\Lambda = \Delta\bar{\psi}_\Lambda\,, \quad \bar{U} = -\frac{\beta}{\mu}\,. \tag{6.19}$$

The considerations involving detailed balance are properties of general solutions of the truncated equations centered on the climate mean in (3.3) and related partitions of these equations. Once these are developed below, it will be straightforward to build stochastic models consistent with the predictions from equilibrium statistical mechanics in Section 6.1. Thus, it is natural to center the variables in (3.3) about the climate mean in (6.19) through

$$q_\Lambda = \bar{q}_\Lambda + \tilde{\omega}_\Lambda\,, \quad \Delta\tilde{\psi}_\Lambda = \tilde{\omega}_\Lambda\,, \quad U = \bar{U} + \tilde{U}\,. \tag{6.20}$$

Within an irrelevant constant that can be absorbed in the normalization constant C, the argument in the Gibbs measure in (6.16) is a positive multiple of the pseudoenergy.

Pseudoenergy:

$$\mu E_\Lambda + \mathcal{E}_\Lambda = \frac{1}{2}\mu\tilde{U}^2 + \frac{1}{2}⨏ \left(-\mu\tilde{\psi}_\Lambda + \tilde{\omega}_\Lambda\right)\tilde{\omega}_\Lambda\,. \tag{6.21}$$

The pseudoenergy is conserved by the truncated dynamics in (3.3) since it is a linear combination of two conserved quantities; it also has the important property that it is a quadratic form in perturbations about the climate mean state, $\bar{q}_\Lambda$, $\bar{U}$. Furthermore, this quadratic form is positive definite exactly when the Gibbs measure in (6.16) is realizable, i.e., for $\mu > 0$ for $\beta \neq 0$, $\tilde{U} \neq 0$, and for $\mu > -1$ for $\beta = 0$, $\tilde{U} = 0$. Notice also that the nontruncated pseudoenergy,

$$\mu E + \mathcal{E} = \frac{1}{2}\mu\tilde{U}^2 + \frac{1}{2}⨏ \left(-\mu\tilde{\psi} + \tilde{\omega}\right)\tilde{\omega} \tag{6.22}$$

is equivalent to the Sobolev H^1-norm on $\tilde{\omega}$ for general (nontruncated) functions. The conservation of this pseudoenergy implies the nonlinear stability of the climate mean state [4, 17].

By rewriting the equation in (3.3) in terms of $\tilde{\omega}_\Lambda$, $\tilde{U}$ in (6.20) and utilizing the identities in (6.19) for the mean state together with the identity $\nabla^\perp f \cdot \nabla g = -\nabla^\perp g \cdot \nabla f$ and elementary integration by parts, we obtain the following dynamical equations for $\tilde{\omega}_\Lambda$ and $\tilde{U}$:

$$
\begin{aligned}
\frac{\partial \tilde{\omega}_\Lambda}{\partial t} &= -\left\{P_\Lambda\left(\nabla^\perp \tilde{\psi}_\Lambda \cdot \nabla \tilde{\omega}_\Lambda\right)\right\}_{1\mathrm{A}} - \left\{\tilde{U}\frac{\partial \tilde{\omega}_\Lambda}{\partial x}\right\}_{1\mathrm{B}} \\
&\quad - \left\{P_\Lambda\left(\nabla^\perp \bar{\psi}_\Lambda \cdot \nabla(-\mu\tilde{\psi}_\Lambda + \tilde{\omega}_\Lambda)\right)\right\}_2 \\
&\quad + \left\{\frac{\beta}{\mu}\frac{\partial}{\partial x}(-\mu\tilde{\psi}_\Lambda + \tilde{\omega}_\Lambda) - \mu\tilde{U}\frac{\partial \bar{\psi}_\Lambda}{\partial x}\right\}_3, \\
\frac{d\tilde{U}}{dt} &= \left\{ \fint \frac{\partial \bar{\psi}_\Lambda}{\partial x}(-\mu\tilde{\psi}_\Lambda + \tilde{\omega}_\Lambda)\right\}_3, \qquad \tilde{\omega}_\Lambda = \Delta\tilde{\psi}_\Lambda\,.
\end{aligned}
\tag{6.23}
$$

The terms in $\{\cdot\}_2$ and $\{\cdot\}_3$ are linear perturbations about the climate mean, while the terms in $\{\cdot\}_{1\mathrm{A}}$ and $\{\cdot\}_{1\mathrm{B}}$ are the nonlinear contributions due to small-scale and mean advection by $\tilde{U}$.

We now show that the dynamics associated with $\{\cdot\}_{1\mathrm{A}} + \{\cdot\}_{1\mathrm{B}}$, $\{\cdot\}_2$, and $\{\cdot\}_3$ separately conserve the pseudoenergy. Considering the linear terms $\{\cdot\}_2$ and $\{\cdot\}_3$ first, this result is a consequence of the following:

LEMMA 6.1 *The two operators defined by*

$$
\begin{aligned}
L_h\begin{pmatrix} w \\ V\end{pmatrix} &= \begin{pmatrix} -\nabla^\perp\bar{\psi}_\Lambda \cdot \nabla(-\mu\phi + w) \\ 0 \end{pmatrix}, \\
L_{\bar{U}}\begin{pmatrix} w \\ V\end{pmatrix} &= \begin{pmatrix} \frac{\beta}{\mu}\frac{\partial}{\partial x}(-\mu\phi + w) - \mu V\frac{\partial\bar{\psi}_\Lambda}{\partial x} \\ \fint \frac{\partial\bar{\psi}_\Lambda}{\partial x}(-\mu\phi + w) \end{pmatrix},
\end{aligned}
\tag{6.24}
$$

with $w = \Delta\phi$ are skew-symmetric in the pseudoenergy inner product associated with (6.21). *In particular, the reduced dynamics*

$$
\frac{\partial}{\partial t}\begin{pmatrix}\tilde{\omega}_\Lambda \\ \tilde{U}\end{pmatrix} = P_\Lambda L_h P_\Lambda \begin{pmatrix}\tilde{\omega} \\ \tilde{U}\end{pmatrix}, \quad \frac{\partial}{\partial t}\begin{pmatrix}\tilde{\omega}_\Lambda \\ \tilde{U}\end{pmatrix} = P_\Lambda L_{\bar{U}} P_\Lambda \begin{pmatrix}\tilde{\omega} \\ \tilde{U}\end{pmatrix},
\tag{6.25}
$$

conserve the pseudoenergy in (6.21).

COROLLARY 6.2 *Consider the incompressible vector fields in phase space $\vec{F}_h$ and $\vec{F}_{\bar{U}}$, defined as in* (6.3) *and* (6.4) *and associated with the operators $P_\Lambda L_h P_\Lambda$ and $P_\Lambda L_{\bar{U}} P_\Lambda$, respectively. Then an arbitrary function of the pseudoenergy, $G(\mu E_\Lambda + \mathcal{E}_\Lambda)$ satisfies*

$$
\vec{F}_h \cdot \nabla G = 0\,, \quad \vec{F}_{\bar{U}} \cdot \nabla G = 0\,,
\tag{6.26}
$$

i.e., $G(\mu E_\Lambda + \mathcal{E}_\Lambda)$ is a steady-state solution of the Liouville equations associated with $\vec{F}_h$ and $\vec{F}_{\bar{U}}$. In particular, the Gibbs measure with density in (6.16) *is an invariant under both of the separate dynamics in* (6.25). *Furthermore, these results* *remain true if P_Λ is replaced by any finite-dimensional projection P that projects on $\tilde{U}$ and a finite number of Fourier components of $\tilde{w}$ and preserves the reality condition.*

The proof of Lemma 6.1 is a straightforward calculation utilizing the definitions in (6.21) and (6.24) and explicit integration by parts. Corollary 6.2 is an immediate consequence of the lemma once it is recognized that PL_hP and $PL_{\tilde{U}}P$ are also skew-symmetric operators in the pseudoenergy inner product for any finite range orthogonal projection P.

Next, we need to state a detailed energy balance condition for finite-dimensional truncation of the nonlinear operator representing two-dimensional inviscid flow involving Galerkin projection by P_Λ on an arbitrary symmetric subspace of Fourier coefficients. Consider the equations

$$\frac{\partial \tilde{\omega}_\Lambda}{\partial t} = -P_\Lambda\big(\nabla^\perp \tilde{\psi}_\Lambda \cdot \nabla \tilde{\omega}_\Lambda\big), \quad \tilde{\omega}_\Lambda = \Delta \tilde{\psi}_\Lambda . \tag{6.27}$$

The standard integration-by-parts argument utilized in proving conservation of E_Λ and $\mathcal{E}_\Lambda$ can also be used to establish that the dynamics in (6.27) *conserves the pseudoenergy in* (6.21) *for an arbitrary projection* P_Λ. In fact, the nonlinear operator at the right-hand side of (6.27) can be decomposed into triad interaction terms involving either climate or unresolved variables such that we have detailed conservation of pseudoenergy for individual triads separately, as we show now.

Let $P_{\bar{\Lambda}}$ be the projection on the variables with $\vec{k} \in \bar{\sigma}_\Lambda$ and $|\vec{k}| \le \bar{\Lambda} < \Lambda$ defining the climate variables. We decompose

$$\begin{aligned} &P_\Lambda(\nabla^\perp \tilde{\psi}_\Lambda \cdot \nabla \tilde{\omega}_\Lambda) \\ &\quad = \big\{P_{\bar{\Lambda}}\big(\nabla^\perp \tilde{P}_{\bar{\Lambda}} \psi_\Lambda \cdot \nabla \tilde{P}_{\bar{\Lambda}} \omega_\Lambda\big)\big\}_{C|CC} \\ &\qquad + \big\{P_{\bar{\Lambda}}\big(\nabla^\perp (P_\Lambda - P_{\bar{\Lambda}})\tilde{\psi}_\Lambda \cdot \nabla (P_\Lambda - P_{\bar{\Lambda}})\tilde{\omega}_\Lambda\big)\big\}_{C|UU} \\ &\qquad + \big\{(P_\Lambda - P_{\bar{\Lambda}})\big(\nabla^\perp P_{\bar{\Lambda}}\tilde{\psi}_\Lambda \cdot \nabla P_{\bar{\Lambda}}\tilde{\omega}_\Lambda\big)\big\}_{U|CC} \\ &\qquad + \big\{(P_\Lambda - P_{\bar{\Lambda}})\big(\nabla^\perp (P_\Lambda - P_{\bar{\Lambda}})\tilde{\psi}_\Lambda \cdot \nabla (P_\Lambda - P_{\bar{\Lambda}})\tilde{\omega}_\Lambda\big)\big\}_{U|UU} . \end{aligned} \tag{6.28}$$

The various terms in brackets at the left-hand side of (6.28) include different types of nonlinear triad interactions such that

$\{\cdot\}_{C|CC}$: two climate variables drive a climate variable,
$\{\cdot\}_{C|UU}$: two unresolved variables drive a climate variable,
$\{\cdot\}_{U|CC}$: two climate variables drive an unresolved variable,
$\{\cdot\}_{U|UU}$: two unresolved variables drive an unresolved variable.

The following lemma shows that the dynamics associated with $\{\cdot\}_{C|CC}$, $\{\cdot\}_{C|UU} + \{\cdot\}_{U|CC}$, and $\{\cdot\}_{U|UU}$ separately conserve pseudoenergy.

LEMMA 6.3 *The reduced dynamics in*

$$\frac{\partial\tilde\omega_\Lambda}{\partial t} = -\left\{P_{\bar\Lambda}\left(\nabla^\perp P_{\bar\Lambda}\tilde\psi_\Lambda\cdot\nabla P_{\bar\Lambda}\tilde\omega_\Lambda\right)\right\}_{C|CC} \equiv N_{\bar\Lambda}(\tilde\omega_\Lambda)\,, \tag{6.29}$$

$$\begin{aligned}\frac{\partial\tilde\omega_\Lambda}{\partial t} &= -\left\{P_{\bar\Lambda}\left(\nabla^\perp (P_\Lambda - P_{\bar\Lambda})\tilde\psi_\Lambda\cdot\nabla (P_\Lambda - P_{\bar\Lambda})\tilde\omega_\Lambda\right)\right\}_{C|UU} \\ &\quad - \left\{(P_\Lambda - P_{\bar\Lambda})\left(\nabla^\perp P_{\bar\Lambda}\tilde\psi_\Lambda\cdot\nabla P_{\bar\Lambda}\tilde\omega_\Lambda\right)\right\}_{U|CC} \equiv N_{\Lambda,\bar\Lambda}(\tilde\omega_\Lambda)\,,\end{aligned} \tag{6.30}$$

and

$$\frac{\partial\tilde\omega_\Lambda}{\partial t} = -\left\{(P_\Lambda - P_{\bar\Lambda})\left(\nabla^\perp (P_\Lambda - P_{\bar\Lambda})\tilde\psi_\Lambda\cdot\nabla (P_\Lambda - P_{\bar\Lambda})\tilde\omega_\Lambda\right)\right\}_{U|UU}\,, \tag{6.31}$$

with $\tilde\omega_\Lambda = \Delta\tilde\psi_\Lambda$, *conserve the pseudoenergy in* (6.21).

Lemma 6.3 follows by a standard integration-by-parts argument. The lemma also implies the following:

COROLLARY 6.4 *Let* $\vec F_{\bar\Lambda}$ *and* $\vec F_{\Lambda,\bar\Lambda}$ *be the incompressible vector fields associated with the nonlinear operator* $N_{\bar\Lambda}(\tilde\omega)$ *and* $N_{\Lambda,\bar\Lambda}(\tilde\omega)$ *in* (6.29) *and* (6.30), *respectively. Then any function of the pseudoenergy* $G(\mu E_\Lambda + \mathcal{E}_\Lambda)$ *satisfies*

$$\vec F_{\bar\Lambda}\cdot\nabla G = 0\,,\quad \vec F_{\Lambda,\bar\Lambda}\cdot\nabla G = 0\,. \tag{6.32}$$

In particular, the Gibbs measure with density in (6.16) *is a steady solution.*

Design of Stochastic Models Consistent with Geophysical Statistical Mechanics

Finally, with all of the detailed balance conditions for pseudoenergy in Lemmas 6.1 and 6.3 and Corollaries 6.2 and 6.4, we design stochastic models consistent with the equilibrium statistical Gibbs ensemble in (6.16) and then show that the derived stochastic models for the climate variables alone are also consistent with equilibrium statistical mechanics. Following the stochastic modeling strategy in Sections 2 and 3 above for the equations in (6.23), we need to make a stochastic model for the nonlinear interaction of the unresolved scales with themselves

$$\left\{(P_\Lambda - P_{\bar\Lambda})\left(\nabla^\perp (P_\Lambda - P_{\bar\Lambda})\tilde\psi_\Lambda\cdot\nabla (P_\Lambda - P_{\bar\Lambda})\tilde\omega_\Lambda\right)\right\}_{U|UU} \tag{6.33}$$

consistent with the energy spectrum of the Gibbs measure for the pseudoenergy in (6.16). From (6.18), the mean and variance of the Fourier coefficient $\hat{\tilde\omega}_k$ of $\tilde\omega_\Lambda$ need to be constrained to

$$\operatorname{mean}\hat{\tilde\omega}_k = 0\,,\quad \operatorname{var}\hat{\tilde\omega}_k = \frac{|\vec k|^2}{\alpha(\mu + |\vec k|^2)}\,,\quad \bar\Lambda < |\vec k| \le \Lambda\,, \tag{6.34}$$

so in each Fourier mode of the unresolved variables we approximate (6.33) by

$$-\frac{\gamma_k}{\varepsilon}\hat{\tilde\omega}_k + \frac{\sigma_k}{\sqrt{2\varepsilon}}(\dot W_k(t) + \dot W^*_{-k}(t))\,,\quad \bar\Lambda < |\vec k| \le \Lambda\,, \tag{6.35}$$

with

$$\frac{\sigma_k^2}{\gamma_k} = \frac{|\vec{k}|^2}{\alpha(\mu + |\vec{k}|^2)} . \tag{6.36}$$

Thus for the nonlinear self-interaction of the unresolved variables we use

$$\{\cdot\}_{U|UU} \approx -\frac{1}{\varepsilon}\Gamma\tilde{\omega}_\Lambda + \frac{1}{\sqrt{\varepsilon}}\sigma\dot{W}(t) , \tag{6.37}$$

where the right-hand side denotes the real-space representation of the damping and stochastic forcing in (6.35).

With the approximation in (6.37) consistent with the constraint in (6.36), we are ready to develop the stochastic climate models, which we write here in operator form to clarify the presentation. First, we consider the geophysical case with $\tilde{U} = 0$, $\beta = 0$ so that the only nonlinearity is given by the term $\{\cdot\}_{1A}$ in (6.23). Following the general strategy from Section 2 with (6.37) yields, after coarse-graining in time, the stochastic model for $\beta = 0$, $\tilde{U} = 0$.

Stochastic model for $\beta = 0$, $\tilde{U} = 0$:

$$\begin{aligned} d\tilde{\omega}_\Lambda = {} & N_{\bar{\Lambda}}(\tilde{\omega}_\Lambda)dt + \frac{1}{\varepsilon}\big(\bar{L}_h\tilde{\omega}_\Lambda + N_{\Lambda,\bar{\Lambda}}(\tilde{\omega}_\Lambda)\big)dt \\ & - \frac{1}{\varepsilon^2}\Gamma\tilde{\omega}_\Lambda\, dt + \frac{1}{\varepsilon}\sigma\, dW(t) , \end{aligned} \tag{6.38}$$

where $N_{\bar{\Lambda}}$, $N_{\Lambda,\bar{\Lambda}}$ are the operators defined above in (6.29) and (6.30), and $\bar{L}_h$ is defined as (compare (6.24))

$$\bar{L}_h\tilde{\omega}_\Lambda = -\nabla^\perp\bar{\psi}_\Lambda \cdot \nabla\big(-\mu\tilde{\psi}_\Lambda + \tilde{\omega}_\Lambda\big) , \quad \tilde{\omega}_\Lambda = \Delta\tilde{\psi}_\Lambda . \tag{6.39}$$

Note that consistent with our general strategy we have to assume that the nonlinear driving of the climate variables by nonlinear self-interaction between climate variables is weak; i.e., we have to set

$$N_{\bar{\Lambda}}(\tilde{\omega}_\Lambda) \to \varepsilon N_{\bar{\Lambda}}(\tilde{\omega}_\Lambda) . \tag{6.40}$$

The stochastic model in (6.38) satisfies all the basic assumptions in this paper, since according to Lemma 6.1, $P_{\bar{\Lambda}}\bar{L}_h P_{\bar{\Lambda}}$ is skew-symmetric in the pseudoenergy norm where all the explicit calculations for (6.38) should be developed. The terms involving $\bar{L}_h\tilde{\omega}_\Lambda$ are the *topographic beta effect* for those readers familiar with geophysical flows and yields fast-wave effects as in Section 5.

The more general stochastic model with $\tilde{U} \neq 0$, $\beta \neq 0$ can also be handled within the framework of this paper whenever $\tilde{U}$ is a climate variable by supplementing (6.40) by the assumption that the additional climate nonlinear interaction involving $\tilde{U}P_{\bar{\Lambda}}\partial\tilde{\omega}_\Lambda/\partial x$ from $\{\cdot\}_{1B}$ in (6.23) is weak of order ε. However, consistency with equilibrium statistical mechanics requires detailed conservation of pseudoenergy, which in turns requires that the *whole* additional nonlinear interaction $\tilde{U}\partial\tilde{\omega}_\Lambda/\partial x$ from $\{\cdot\}_{1B}$ in (6.23) be of order ε. Thus, in the equations for perturbations in (6.23), in addition to the assumption in (6.40), we set

$$\tilde{U}\frac{\partial\tilde{\omega}_\Lambda}{\partial x} \to \varepsilon\tilde{U}\frac{\partial\tilde{\omega}_\Lambda}{\partial x}\,. \tag{6.41}$$

With the assumptions in (6.40) and (6.41), we obtain after coarse-graining in time the following:

Stochastic model for $\beta \neq 0$, $\tilde{U} \neq 0$:

$$\begin{aligned} d\tilde{\omega}_\Lambda &= N_{\bar{\Lambda}}(\tilde{\omega}_\Lambda)dt - \tilde{U}\frac{\partial\tilde{\omega}_\Lambda}{\partial x}\,dt + \frac{1}{\varepsilon}\big(\bar{L}_h\tilde{\omega}_\Lambda + N_{\Lambda,\bar{\Lambda}}(\tilde{\omega}_\Lambda)\big)dt \\ &\quad + \frac{1}{\varepsilon}\left(\frac{\beta}{\mu}\frac{\partial}{\partial x}(-\mu\tilde{\psi}_\Lambda + \tilde{\omega}) - \mu\tilde{U}\frac{\partial\bar{\psi}_\Lambda}{\partial x}\right) \\ &\quad - \frac{1}{\varepsilon^2}\Gamma\tilde{\omega}_\Lambda dt + \frac{1}{\varepsilon}\sigma\,dW(t)\,, \\ d\tilde{U} &= \frac{1}{\varepsilon}\left(⨍\frac{\partial\bar{\psi}_\Lambda}{\partial x}(-\mu\tilde{\psi}_\Lambda + \tilde{\omega})\right)dt\,, \quad \tilde{\omega}_\Lambda = \Delta\tilde{\psi}_\Lambda\,. \end{aligned} \tag{6.42}$$

Once again this model satisfies the general hypotheses of Section 2 as a consequence of Lemmas 6.1 and 6.3. The extreme special case where $\tilde{U}$ alone is the single climate variable in the stochastic model does not need the additional assumption in (6.41). This amusing example is analyzed in detail in Section 7.4 of the present paper.

We claim that the stochastic models in (6.38) and (6.42) are consistent with geophysical equilibrium statistical mechanics. We have the following:

PROPOSITION 6.5 *The stochastic climate models in* (6.38) *and* (6.42) *have Gibbs measures from* (6.16) *involving pseudoenergy as their invariant measure for all realizable values of the parameter* μ, *i.e.,* $\mu > -1$ *for* (6.38) *and* $\mu > 0$ *for* (6.42).

PROOF: We sketch the proof for the stochastic model in (6.38). We need to check that the density $P_C^\star$ in (6.16) is a steady-state solution for the forward equation associated with (6.38), i.e.,

$$\frac{1}{\varepsilon^2}\mathcal{L}_1^\dagger P_C^\star + \frac{1}{\varepsilon}\mathcal{L}_2^\dagger P_C^\star + \mathcal{L}_3^\dagger P_C^\star = 0\,. \tag{6.43}$$

Here $\mathcal{L}_1^\dagger$ is the (forward) Ornstein-Uhlenbeck operator defined through (6.35), (6.36) so that by construction

$$\mathcal{L}_1^\dagger P_C^\star = 0\,. \tag{6.44}$$

The operators $\mathcal{L}_2^\dagger$ and $\mathcal{L}_3^\dagger$ are the Liouville operators given by

$$\begin{aligned} \mathcal{L}_2^\dagger &= -\mathcal{L}_2 = -\big(\vec{F}_{\Lambda,\bar{\Lambda}}\cdot\nabla + \vec{\bar{F}}_h\cdot\nabla\big) = \big(\mathcal{L}_2^{\mathrm{NS}^\dagger} + \mathcal{L}_2^{\mathrm{S}^\dagger}\big)\,, \\ \mathcal{L}_3^\dagger &= -\mathcal{L}_3 = -\vec{F}_{\bar{\Lambda}}\cdot\nabla\,, \end{aligned} \tag{6.45}$$

where $\vec{\bar{F}}_h$ is the incompressible vector field associated with the operator in (6.39). As a consequence of the detailed balance conditions in Lemmas 6.1 and 6.3 and

Corollaries 6.2 and 6.4, the operators in (6.45) also annihilate $P_C^\star$, so the proof for (6.38) is complete. The proof for the stochastic model in (6.42) is very similar where additionally, the property of $L_{\bar{U}}$ in (6.26) is used. The straightforward details are left to the reader. $\square$

The stochastic models in (6.38) and (6.42) satisfy all the hypotheses of the formalism developed in Sections 2, 4, and 5 above with nontrivial fast-wave averaging effects. Thus, one can apply Theorem 5.4 to get a self-consistent Fokker-Planck equation with a reduced stochastic model for the climate variables alone. Are the reduced stochastic models derived in this fashion consistent with equilibrium statistical mechanics? For this to be true, the invariant measure for the derived Fokker-Planck operator for the reduced stochastic model should coincide with the Gibbs measure with the density from (6.16) projected onto the climate variables alone. We have the following result confirming this fact:

PROPOSITION 6.6 *The projection of the density associated with the Gibbs measure from* (6.16) *on the climate variables alone,*

$$P_C^{\star,\mathrm{clim}} = Ce^{-\alpha(\mu E_{\bar{\Lambda}} + \mathcal{E}_{\bar{\Lambda}})}\,, \tag{6.46}$$

is the density associated with the invariant measure for the stochastic climate equations that are obtained from the stochastic models in (6.38) *or* (6.42) *after elimination of the unresolved variables in the limit as $\varepsilon \to 0$.*

PROOF: We give the details for mode elimination for the stochastic model in (6.38). There are fast-averaging effects from the operator $\mathcal{L}_2^{\mathrm{S}}$ in (6.45), which is associated with a skew-symmetric linear operator in the pseudoenergy metric; see Lemma 6.1. Thus, from Theorem 5.4, the Fokker-Planck operator $\hat{\mathcal{L}}$ corresponding to the stochastic climate model that is obtained from the stochastic model in (6.42) is given by

$$\hat{\mathcal{L}}\,\cdot\, = \mathbf{P}\left(\lim_{T\to\infty}\frac{1}{T}\int_{t-T}^{t} e^{-\int_\tau^{\tau_0}\mathcal{L}_2^{\mathrm{S}}}\left(\mathcal{L}_3 - \mathcal{L}_2^{\mathrm{NS}}\mathcal{L}_1^{-1}\mathcal{L}_2^{\mathrm{NS}}\right)e^{\int_\tau^{\tau_0}\mathcal{L}_2^{\mathrm{S}}}\,d\tau\right)\mathbf{P}\,\cdot\,, \tag{6.47}$$

with the operators $\mathcal{L}_2^{\mathrm{NS}}$, $\mathcal{L}_2^{\mathrm{S}}$, and $\mathcal{L}_3$ given in (6.45) and with $\mathcal{L}_1$, the Ornstein-Uhlenbeck operator defined through (6.35) and (6.36) above. The equation in (6.47) is, in fact, a slight generalization of Theorem 5.4 since the rotation accounted for by $\mathcal{L}_2^{\mathrm{S}}$ involves the unresolved variables as well. This leads to no difficulty since the developments of Section 5.3 generalize trivially to the present situation.

Let $\mathbf{P}$ denote the expectation operator with respect to the invariant measure for the Ornstein-Uhlenbeck operator $\mathcal{L}_1$. Proposition 6.6 can be rephrased as

$$\hat{\mathcal{L}}^\dagger \mathbf{P} P_C^\star = \hat{\mathcal{L}}^\dagger P_C^{\star,\mathrm{clim}} = 0\,, \tag{6.48}$$

where $\hat{\mathcal{L}}^\dagger$ is the operator adjoint to $\hat{\mathcal{L}}$. The equality in (6.48) follows immediately from the definition in (6.47) for $\hat{\mathcal{L}}$, the detailed balance properties in (6.43), Corollary 6.2 and Lemma 6.3 for $\mathcal{L}_2$ and $\mathcal{L}_3$, and the property that

$$e^{\int_\tau^{\tau_0} \mathcal{L}_2^{\mathrm{S}^\dagger}} P_C^{\star,\mathrm{clim}} = P_C^{\star,\mathrm{clim}} . \tag{6.49}$$

The latter follows from the definition of $e^{\int_\tau^{\tau_0} \mathcal{L}_2^{\mathrm{S}^\dagger}}$ and Corollary 6.2. The proof for the stochastic model in (6.42) is similar and utilizes detailed balance from Lemmas 6.1 and 6.3 and Corollaries 6.2 and 6.4. □

6.3 Numerical Evidence for Effective Stochastic Dynamics

We present numerical evidence for effective stochastic modeling of the truncated barotropic equations in (3.3) whose explicit form in the Fourier representation is given in (3.6). We study these equations in several different parameter regimes that exhibit clear separation of time scales for the evolution of appropriately selected groups of variables and hence justify the possibility of distinguishing between climate and unresolved variables in stochastic climate modeling for the truncated barotropic equations in (3.6). Depending on the parameter regimes, we also show the rich variety of possibilities for selecting climate variables for these equations.

We consider the truncated barotropic equations in (3.6) for $|\vec{k}|^2 \leq 17$ and use a pseudospectral method of integration with fourth-order Runge-Kutta time stepping. The total energy and enstrophy are conserved within 0.1% in the simulations, consistent with (6.2). The initial conditions on a given energy-enstrophy level are generated in Fourier space. We represent Fourier coefficients through their amplitudes and phases $u_{\vec{k}} = |u(\vec{k})|e^{i\theta(\vec{k})}$, and we make an additional simplifying assumption that the amplitudes of the Fourier coefficients depend only on the magnitude of $|\vec{k}|^2$, i.e., $|u(\vec{k})| \equiv f(|\vec{k}|^2)$. We sample all but two amplitudes from a uniform distribution on [0, 1] and use the remaining two amplitudes in order that the prescribed values for the energy and the enstrophy be achieved. We then sample the phases $\theta(\vec{k})$ from the uniform distribution on $[0, 2\pi)$. We use averaging with respect to time in the numerical simulations as the probability measure to compute all statistics reported below and obtain excellent agreement with the predictions of the equilibrium statistical theory. Since Monte Carlo simulation over an ensemble of initial data is not considered here but rather only a single initial datum consistent with the microcanonical ensemble, the numerical experiments below also provide strong support for the use of the canonical Gibbs ensemble a priori. This gives strong support to the assumption of ergodicity with respect to time averaging of the truncated barotropic equations in (3.6). For each $q \leq 17$ fixed we compute the energy spectrum

$$E(q) = \sum_{|\vec{k}|=q} \mathbf{E}_{\mathrm{T}} |u_k(t) - \mathbf{E}_{\mathrm{T}} u_k(t)|^2 , \tag{6.50}$$

where $\mathbf{E}_{\mathrm{T}}$ denotes time averaging. The energy spectrum $E(q)$ gives the average energy in the fluctuating part of the modes in a given shell of wave numbers.

Large-Scale Topography, $\mu = -0.76$, $\alpha = 1.9$

First, we consider the truncated barotropic equations in (3.6) without mean flow U and beta effect; i.e., we set $\beta = U = 0$. As shown below, this parameter regime provides a striking example of scale separation between the modes u_k with $|\vec{k}|^2 = 1$ and all other modes. When there is no mean flow U, the parameter μ in (6.18) is allowed to be negative, which corresponds to the negative temperature regime. The negative temperature regime is characterized by an energy spectrum sharply peaked at lower wave numbers. We performed numerical simulations with $\mu = -0.76$, $\alpha = 1.9$, which corresponds to the fluctuating energy-enstrophy level $E = 7$, $\mathcal{E} = 20$, with the following large-scale topography:

$$
\begin{aligned}
h(x, y) = H \cos(x + \theta_1) + H \cos(y + \theta_2) \\
+ H \sin(x + y + \theta_3) + H \sin(x - y + \theta_4)\,,
\end{aligned}
\tag{6.51}
$$

where the phases θ_j are selected at random. Thus, the only nonzero Fourier coefficients of the topography are the coefficients for modes with $|\vec{k}|^2 = 1$ and $|\vec{k}|^2 = 2$. For the particular choice of phases we use in the simulation described in this section, the maximum height of the topography in (6.51) is

$$
\max_{x,y} |h(x, y)| = 3.5H\,.
\tag{6.52}
$$

We present the results of the numerical simulations with $H = 0.5$, but the situation described below is generic for the parameter regime $\mu = -0.76$, $\alpha = 1.9$. We have verified that the energy spectrum and correlation functions exhibit qualitatively similar behavior for $H = 1$ and $H = 2.5$.

Figure 6.1 shows the temporal convergence of mean values of several low modes u_k. Numerical estimates for the mean values agree very well with the analytical predictions in (6.18) of equilibrium statistical theory. After the transient interval, $0 < t \lesssim 5000$, the mean values stabilize, and by the end of the simulations ($t = 100{,}000$) their relative errors do not exceed 6%.

Figure 6.2 shows numerical estimates and analytical predictions of the equilibrium statistical theory for the energy spectrum. About 60% of the energy is contained in modes with $|\vec{k}|^2 = 1$ in this parameter regime. The agreement between the numerical and analytical estimates is very good, with the largest discrepancy between them concentrated at $|\vec{k}|^2 = 1$, where the relative errors on the energy spectrum do not exceed 10%. The agreement between numerical and analytical estimates for the higher modes is even better.

Note that the simulations utilized here are an especially stringent test since we calculate both the climate mean state and the fluctuations a priori. One can also assume perfect knowledge of the climate mean and perform the numerical simulations with the perturbation equations in (6.23); in this setup, the agreement with the predicted spectrum is even better. Although the Fourier coefficients of the topography $\hat{h}_{(1,0)} = 0.164939 + i0.18787$ and $\hat{h}_{(0,1)} = -0.017259 - i0.249404$

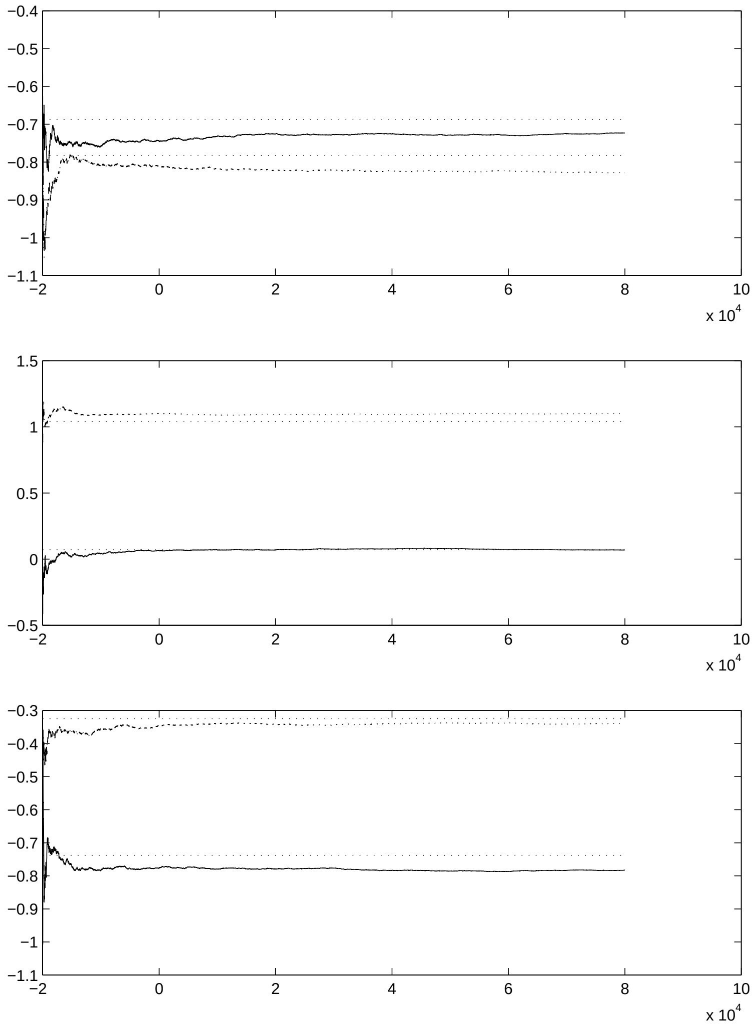

FIGURE 6.1. Temporal convergence of the mean values of the real and imaginary parts of $u_{1,0}$, $u_{0,1}$ and $u_{1,1}$. Simulations with the large-scale topography $\mu = -0.76$, $\alpha = 1.9$, $\beta = 0$.

are different, the correlation functions for $u_{(1,0)}$ and $u_{(0,1)}$ exhibit very similar behavior. The anisotropic effects due to the difference in magnitude of the Fourier coefficients of the topography are very minor in this case. The averaged (with respect to all modes with the same value of $|\vec{k}|$) correlation functions show that there is a separation of time scales between the modes with $|\vec{k}|^2 = 1$ and the rest of the modes.

To characterize the decay rate of the averaged correlation functions, we compute the correlation times τ_k^{DNS} that are proportional to the area underneath the

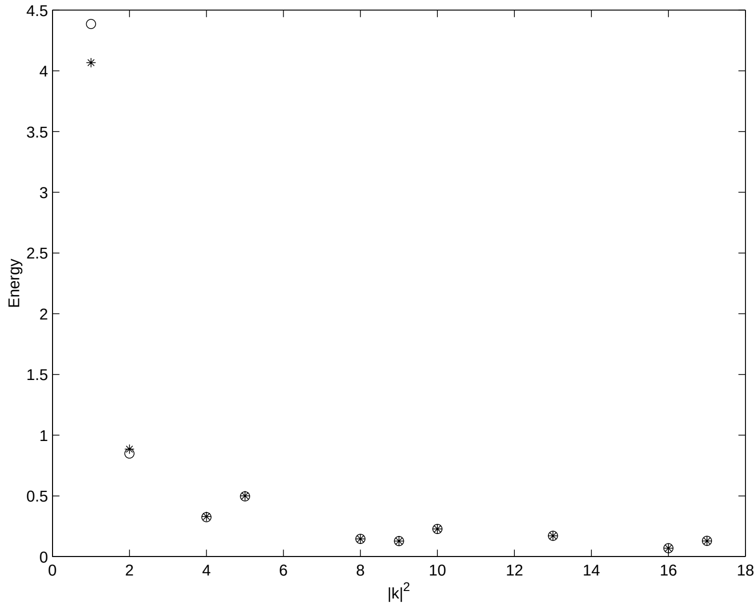

FIGURE 6.2. Analytical (circle) and numerical (star) predictions for the fluctuating part of the energy budget. Simulations with the large-scale topography $\mu = -0.76$, $\alpha = 1.9$, $\beta = 0$.

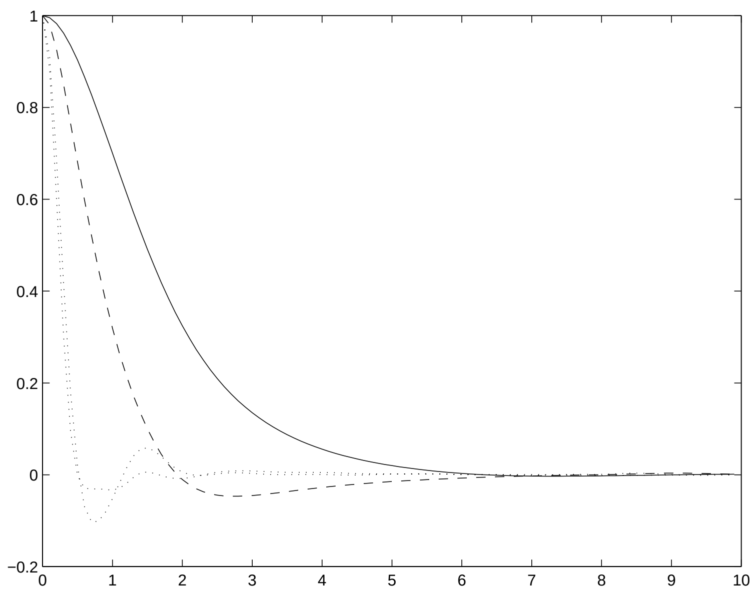

FIGURE 6.3. Averaged correlation functions for $|\vec{k}|^2 = 1$ (solid), $|\vec{k}|^2 = 2$ (dashed) and $|\vec{k}|^2 = 4, 5$ (dotted). Simulations with the large-scale topography $\mu = -0.76$, $\alpha = 1.9$, $\beta = 0$.

$\vert\vec{k}\vert^2$	τ_k^{DNS}
1	1.72
2	0.76
4	0.32
5	0.28
8	0.18
9	0.24
10	0.16
13	0.2
16	0.1
17	0.33

TABLE 6.1. Correlation times for different spectral bands, $\mu = -0.76$, $\alpha = 1.9$.

graph of the correlation functions. Figure 6.3 shows the averaged correlation functions for modes with $1 \le |\vec{k}|^2 \le 5$, and Table 6.1 summarizes the correlation times τ_k^{DNS} for different spectral bands. The correlation function for $|\vec{k}|^2 = 1$ decays more than twice as slowly as the next correlation function for $|\vec{k}|^2 = 2$, and the correlation functions for $|\vec{k}|^2 > 2$ decay even faster. Thus, the main assumption of stochastic climate modeling is clearly satisfied if we select the modes u_k with $|\vec{k}|^2 = 1$ as climate variables and those with $|\vec{k}|^2 \ge 2$ as unresolved variables.

Summarizing, in the present situation, we can replace the nonlinear system with 56 degrees of freedom by a four-dimensional stochastic model for the two complex-valued modes $u_{1,0}$ and $u_{0,1}$. Since there is no nonlinear interaction between the climate variables, this is one of the simplest possible test cases for stochastic climate modeling theory.

Large-Scale Topography, $\mu = 0.1, \alpha = 1$

Next, we describe numerical simulations in a positive temperature regime with $\mu = 0.1$, $\alpha = 1$, which corresponds to the fluctuating energy $E = 5.56$ and the fluctuating enstrophy $\mathcal{E} = 27.4$. We perform the simulations with the topography in (6.51) with $H = 1$. Even though the topography is confined to the wave numbers $|\vec{k}|^2 \le 2$, the correlation times for the modes u_k with $|\vec{k}|^2 = 4, 5$ are roughly comparable with the correlation times for the modes with $|\vec{k}|^2 = 1, 2$ in this parameter regime.

The mean values of u_k converge to the analytical predictions from the equilibrium statistical theory. Figure 6.4 shows the energy spectrum. As for the negative-temperature regime discussed earlier in this section, the numerical estimates agree very well with the predictions of equilibrium statistical theory. The relative errors for the mean values of the u_k's and the energy spectrum do not exceed 8% and 6%, respectively.

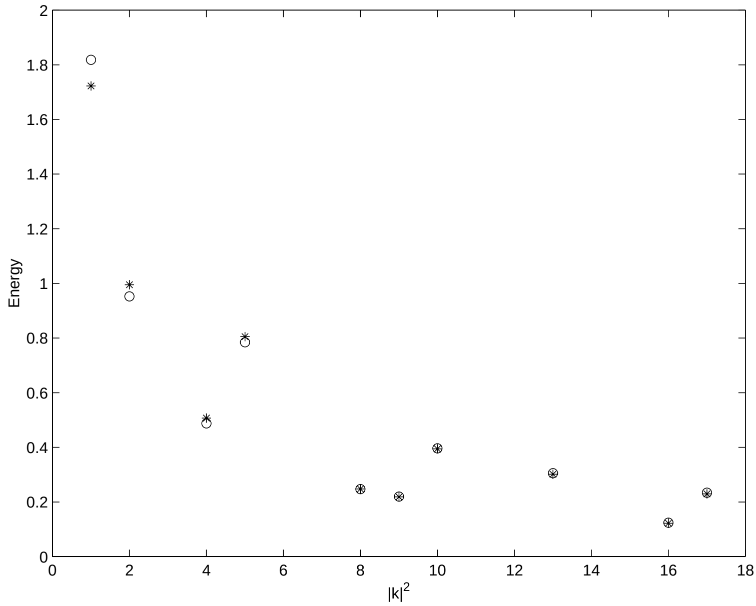

FIGURE 6.4. Analytical (circle) and numerical (star) predictions for the fluctuating part of the energy budget. Simulations with the large-scale topography $\mu = 0.1, \alpha = 1, \beta = 0$.

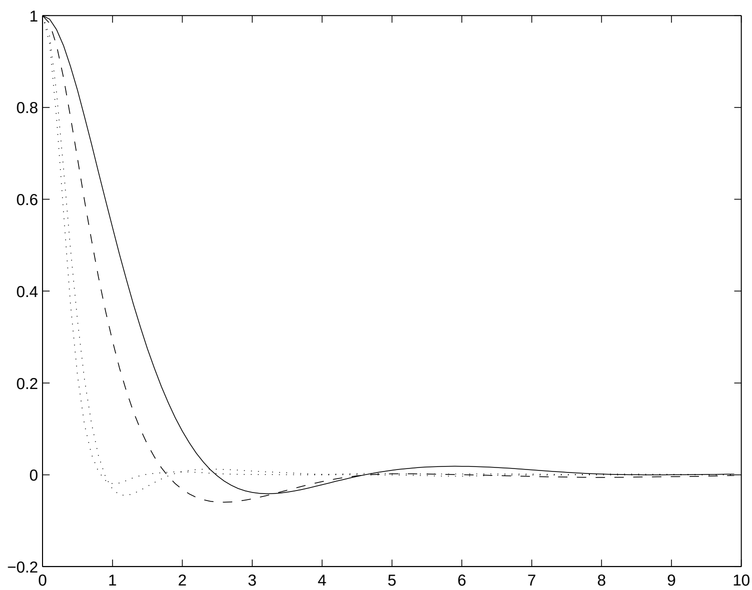

FIGURE 6.5. Averaged correlation functions for $|\vec{k}|^2 = 1$ (solid), $|\vec{k}|^2 = 2$ (dashed), and $|\vec{k}|^2 = 4, 5$ (dotted). Simulations with the large-scale topography $\mu = 0.1, \alpha = 1, \beta = 0$.

$\lvert\vec{k}\rvert^2$	τ_k^{DNS}
1	1.18
2	0.69
4	0.46
5	0.4
8	0.27
9	0.25
10	0.26
13	0.26
16	0.18
17	0.29

TABLE 6.2. Correlation times for different spectral bands, $\mu = 0.1$, $\alpha = 1$.

Figure 6.5 shows the averaged correlation functions for the modes u_k with $1 \le |\vec{k}|^2 \le 5$, and Table 6.2 summarizes the correlation times τ_k^{DNS} for averaged correlation functions. The correlation times for the modes with $|\vec{k}|^2 = 1, 2$ are roughly comparable in this case. Correlation times for spectral bands $|k|^2 = 1, 2$ indicate that that the correlation function for the modes with $|\vec{k}|^2 = 2$ decays about 1.5 times faster than the correlation function for the modes with $|\vec{k}|^2 = 1$. Thus, the modes with $|\vec{k}|^2 = 2$ should be included in the set of climate variables for this regime. The correlation times for the modes with $|\vec{k}|^2 = 4$ and $|\vec{k}|^2 = 5$ are of the same order as the correlation time for the modes with $|\vec{k}|^2 = 2$. Therefore, the time scales for the modes with $|\vec{k}|^2 = 1, 2$ and $|\vec{k}|^2 = 4, 5$ are not so well separated for this regime. This situation presents an interesting test case for stochastic climate modeling, since there are two possibilities for selecting the climate variables. The modes u_k with $|\vec{k}|^2 = 4$ and $|\vec{k}|^2 = 5$ may or may not be included in the set of climate variables, and we can compare the results of the mode elimination procedure in the two cases. The extension of the set of climate variables allows nonlinear interactions between them, and therefore, the resulting stochastic equations for the climate variables will necessarily contain nonlinear terms. Detailed results comparing theory and simulation will be reported elsewhere.

Simulations with Mean Flow and Beta Effect

Another possible test case is when the parameter β, the mean flow U, and topography effects are all present in the equations. We describe here numerical simulations with the two-mode topography

$$h(x, y) = H\left[\cos(x) + \sin(x) + \cos(2x) + \sin(2x)\right], \tag{6.53}$$

with $\beta = 0.5$ and $H = 0.36$, so that the height of the topography is equal to 1. We use the values $\mu = 2$ and $\alpha = 1$ for the canonical measure with density in

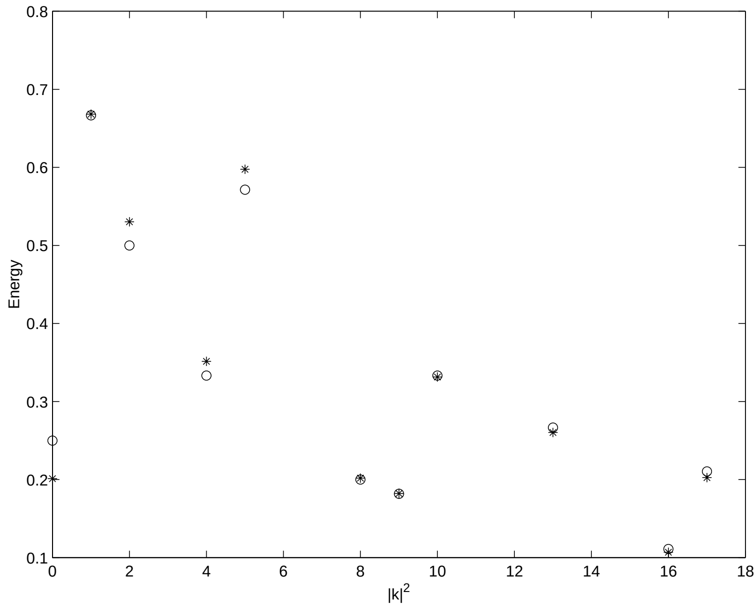

FIGURE 6.6. Analytical (circle) and numerical (star) predictions for the fluctuating part of the energy budget; estimates for $|\vec{k}|^2 = 0$ correspond to the mean flow U. Simulations with the two-mode layered topography and mean flow U with $\mu = 2$, $\alpha = 1$, $\beta = 0.5$.

(6.16) with the truncation $|\vec{k}|^2 \leq 17$ for the u_k's, so there are 57 active modes. We have verified that the situation described is similar for a variety of topographies including single-mode topography; see [16].

The numerical estimates for the mean values and variances of the mean flow U and the modes u_k agree very well with the predictions of the equilibrium statistical theory. Figure 6.6 shows the numerical and analytical estimates for the energy spectrum. The theoretical values and numerical predictions for the means and variances of the modes u_k agree within a few percent, while the relative errors for the mean value and the variance of U are about 20%.

Figure 6.7 shows the correlation function of U and the averaged correlation functions of several low modes u_k. The correlation function of the mean flow U decays much slower than correlation functions of the modes u_k.

The main assumption of the stochastic climate modeling strategy is clearly satisfied in this case, and in the truncated barotropic equations in (3.6) we can identify the mean flow U as the climate variable and the two modes u_k with $\vec{k} = (1, 0)$ and $\vec{k} = (2, 0)$ as the unresolved variables that couple to U. A detailed study of a similar example with a priori stochastic climate modeling has been reported by the authors in [16]. The general stochastic modeling procedure when U alone is declared the climate variable is the topic of Section 7.4 below.

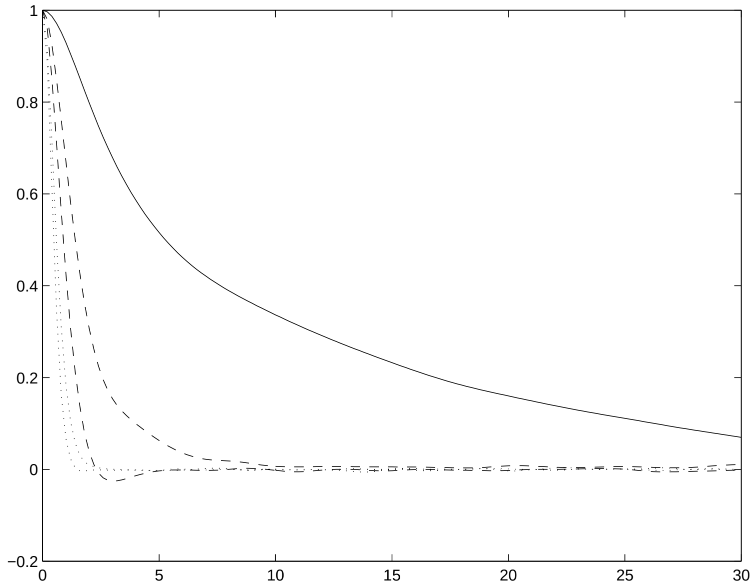

FIGURE 6.7. Averaged correlation functions for U (solid), $|\vec{k}|^2 = 1, 2$ (dashed), and $|\vec{k}|^2 = 4, 5$ (dotted). Simulations with the two-mode layered topography and mean flow U with $\mu = 2$, $\alpha = 1$, $\beta = 0.5$.

6.4 Identification of Parameters for Stochastic Modeling

The results of the numerical simulations reported above can be used to answer the following question that we have left open so far. How do we identify the parameters $\bar{w}_k$, γ_k, ω_k, and ε in the stochastic model assumption in (3.14)? We discuss this point now and also indicate how the solution of the stochastic model equations for the climate variables alone should be compared to the results of the numerical simulations for those variables.

It is important to point out first that the numerical simulations are of course performed in the original time scale t, whereas the stochastic model for barotropic equations in (3.17) is formulated in a coarse-grained time scale obtained by setting $t \to \varepsilon t$. For the present discussion it is essential to distinguish the two time scales, and we will denote by $\tau = \varepsilon t$ the coarse-grained time scale. Thus, we write the last equation in (3.17), which we will need in a moment, as

$$dw_k = \frac{i}{\varepsilon} H_k U \, d\tau - \frac{i}{\varepsilon}(k_x U - \Omega_k) w_k \, d\tau + \frac{1}{\varepsilon} \sum_{\vec{l} \in \sigma_1} L_{kl} v_l \, d\tau + \frac{1}{\varepsilon} \sum_{\vec{l} \in \sigma_2} L_{kl} w_l \, d\tau + \frac{1}{2\varepsilon} \sum_{\substack{\vec{l}, \vec{m} \in \sigma_1 \\ \vec{k}+\vec{l}+\vec{m}=0}} B_{klm} v_l^* v_m^* \, d\tau \tag{6.54}$$

$$+\frac{1}{\varepsilon}\sum_{\substack{\vec{l}\in\sigma_1,\vec{m}\in\sigma_2\\ \vec{k}+\vec{l}+\vec{m}=0}} B_{klm}v_l^* w_m^*\,d\tau - \frac{1}{\varepsilon^2}\gamma_k(w_k-\bar{w}_k)d\tau$$
$$+\frac{\sigma_k}{\sqrt{2}\varepsilon}\big(dW_k(\tau)+dW_{-k}^*(\tau)\big)\,.$$

From this equation, it follows that, to leading order in ε, the mean value of the unresolved mode w_k is given at statistical steady state by

$$\mathbf{E}w_k(t)=\bar{w}_k\,. \tag{6.55}$$

Thus, the measurement of $\mathbf{E}w_k(t)$ in the experiments provides an estimate for the parameter $\bar{w}_k$. Similarly, it follows from the equation in (6.54) that, to leading order in ε, the correlation function of the unresolved mode w_k is given at statistical steady state by

$$\mathbf{E}(w_k(\tau)-\bar{w}_k)\big(w_k^*(\tau')-\bar{w}_k^*\big)=\frac{\sigma_k^2}{\gamma_k}e^{-\gamma_k|\tau-\tau'|/\varepsilon^2}\,, \tag{6.56}$$

which, in terms of the original time scale t, reads

$$\mathbf{E}(w_k(t)-\bar{w}_k)\big(w_k^*(t')-\bar{w}_k^*\big)=\frac{\sigma_k^2}{\gamma_k}e^{-\gamma_k|t-t'|/\varepsilon}\,. \tag{6.57}$$

Thus, the ratio σ_k^2/γ_k can be estimated from $\mathbf{E}|w_k(t)-\bar{w}_k|^2$. Furthermore, the measurement of the decay rate of the correlation functions of the unresolved modes w_k, which we will denote by γ_k^{DNS}, gives the value for the ratio γ_k/ε, i.e.,

$$\gamma_k^{\mathrm{DNS}}=\frac{\gamma_k}{\varepsilon}\,. \tag{6.58}$$

Knowing the values for σ_k^2/γ_k and γ_k/ε , we immediately obtain $\sigma_k/\sqrt{\varepsilon}$. It will be convenient to define σ_k^{DNS} as

$$\sigma_k^{\mathrm{DNS}}=\frac{\sigma_k}{\sqrt{\varepsilon}}\,. \tag{6.59}$$

(Notice that if the model is taken to be consistent with equilibrium statistical theory, it follows from the equations in (6.18) that only one parameter among $\bar{w}_k$, γ_k, and ω_k is free. If we take it to be γ_k, it means that only this parameter must be estimated whereas the other two can then be obtained a priori.)

On the other hand, the parameter ε can be estimated as the ratio between the biggest measured decay rate of the climate variables and the smallest decay rate of the unresolved variables. The parameter ε is required to be small in order that the asymptotic procedure outlined in Section 4 applies. However, as we now show, the *actual value of ε is irrelevant for comparing the solutions of the equations for the climate variables alone and the results for these variables that are observed in the* (*numerical*) *experiments*. In fact, the key observation is that the equations for the climate variables alone that are provided by the asymptotic procedure of Section 4 are again given in terms of the coarse-grained time scale $\tau=\varepsilon t$ instead of the original time scale used in the experiments. Furthermore, *if we re-express the equations for the climate variables alone in terms of the original time scale t instead of the coarse-grained time scale $\tau = \varepsilon t$, the parameters entering these equations become γ_k^{DNS} and σ_k^{DNS} instead of γ_k and σ_k.* This fact can be readily proven upon verifying that the general equations for the climate variables alone in (4.17) (without fast-wave effects) or in (5.8) (with fast-wave effects) are left invariant by the transformation

$$t \to \frac{t}{\varepsilon}, \quad \gamma_j \to \frac{\gamma_j}{\varepsilon}, \quad \sigma_j \to \frac{\sigma_j}{\sqrt{\varepsilon}}, \quad j \in \sigma_2 . \tag{6.60}$$

This transformation amounts to going back to the original time scale and using γ_k^{DNS} and σ_k^{DNS} instead of γ_k and σ_k in the equations.

Here we stop comparing the analytical results with the experiments. A systematic comparison between numerical simulations of the truncated barotropic equations in (3.3) and the solutions of the stochastic climate model equations for the climate variables alone will be presented elsewhere.

7 New Phenomena in Low-Order Triad Models

In this section, we study the stochastic model for barotropic equations in (3.17) in special model cases intended to illustrate explicitly various new phenomena predicted by the theory. A wave–mean flow triad model is considered in Section 7.1. A climate scattering triad model is studied in Section 7.2. A more general triad model with fast-wave effects of the type discussed in Section 5.2 is considered in Section 7.3. Finally, the special case of the stochastic model in (3.17) where U alone is the declared climate variable is the topic of Section 7.4. The triad models discussed below arise systematically when the Galerkin truncation used to derive the barotropic equations in (3.17) involves a subspace consisting of a defining set $\bar{\sigma}_1$ involving only three modes, $(\vec{k}, \vec{l}, \vec{m})$.

In each case, we consider both situations where the triad models are constrained or not constrained by equilibrium statistical theory. The constraints in (6.36) ensuring consistency with statistical mechanics that were derived in Section 6.2 translate in terms of the variables (U, v_k, w_k) entering stochastic model in (3.17) as

$$\bar{w}_k = \frac{|\vec{k}|\hat{h}_k}{\mu + |\vec{k}|^2} \tag{7.1}$$

and

$$\frac{\sigma_k^2}{\gamma_k} = \frac{1}{\alpha(\mu + |\vec{k}|^2)} . \tag{7.2}$$

The value of $\bar{w}_k$ in (7.1) is the mean value in (6.18) for $u_k \equiv w_k$ predicted by equilibrium statistical theory, whereas the ratio in (7.2) is the variance in (6.18) for $u_k \equiv w_k$. As explained before, we study both cases where the equations in (7.1) and (7.2) are satisfied or not satisfied based on the following motivation. Equilibrium statistical models have nontrivial, well-defined mean states and energy spectra that can serve as a nontrivial test case for stochastic climate models. In this case, we check explicitly that, consistent with Theorem 6.6, in the constrained situation the stochastic climate models also satisfy equilibrium statistical theory. On the other hand, practical climate models can involve energy spectra for the unresolved modes that are not given by equilibrium statistical mechanics, so the consequences of making general assumptions are interesting.

The triad models confirm the results of Section 4. Generally, the effect of the unresolved variables on the climate variables must be accounted for by linear Langevin terms that can be both stabilizing and destabilizing, by nonlinear terms, and by multiplicative noises. The effect of the unresolved variables can also modify the climate mean. In each cases, explicit criteria for stability are derived that illuminate the mechanism for nonlinear energy transfer between the modes in triad model equations.

7.1 Wave–Mean Flow Triad Equations

We consider the equations in (3.17) under the assumption that

a1. There is no beta effect or topography. Thus, we set $\beta = \hat{h}_k = 0$ in the stochastic model for barotropic equations in (3.17).

By assumption a1 it follows from the first equation in (3.17) that there is no mean flow, $U = 0$. In addition:

a2. We identify the mode v_k with $\vec{k}$ fixed as the climate variable, and all the other modes as unresolved variables. Furthermore, we assume that there is only one pair $(\vec{l}, \vec{m})$ such that $\vec{k} + \vec{l} + \vec{m} = 0$.

It follows by assumption a2 that the climate mode v_k is coupled only to the unresolved modes w_l and w_m, and the stochastic model for barotropic equations in (3.17) reduces to the following triad model equations:

$$
\begin{aligned}
dv_k &= \frac{1}{\varepsilon} B_k w_l^* w_m^* \, dt \,, \\
dw_l &= \frac{1}{\varepsilon} B_l w_m^* v_k^* \, dt - \frac{1}{\varepsilon^2} \gamma_l w_l \, dt + \frac{\sigma_l}{\varepsilon} \, dW_l(t) \,, \\
dw_m &= \frac{1}{\varepsilon} B_m w_l^* v_k^* \, dt - \frac{1}{\varepsilon^2} \gamma_m w_m \, dt + \frac{\sigma_m}{\varepsilon} \, dW_m(t) \,,
\end{aligned}
\tag{7.3}
$$

where

$$
B_k = B_{klm} \,, \quad B_l = B_{lmk} \,, \quad B_m = B_{mkl} \,. \tag{7.4}
$$

From the equation in (3.7) and the constraint $\vec{k} + \vec{l} + \vec{m} = 0$, it follows that

$$
B_k + B_l + B_m = 0 \tag{7.5}
$$

and

$$\frac{B_k}{|\vec{l}|^2 - |\vec{m}|^2} = \frac{B_l}{|\vec{m}|^2 - |\vec{k}|^2} = \frac{B_m}{|\vec{k}|^2 - |\vec{l}|^2}\,. \tag{7.6}$$

Notice that we have set $\bar{w}_l = \bar{w}_m = 0$ in the equations in (7.3). It can be shown that this constraint is essential in order for the equation in (2.10) to be satisfied for the triad model equations in (7.3), i.e., in order for the unresolved modes w_l, w_m not to produce effects of order ε^{-1} on the climate mode v_k. Notice also that this constraint is consistent with equilibrium statistical theory since, from the equations in (7.1), we have $\bar{w}_l = \bar{w}_m = 0$ if $\hat{h}_k = \hat{h}_l = 0$. The general structure of the equations in (7.3) together with the condition in (7.5) is that for the damped and stochastically forced three-wave resonant equations [24]. The barotropic model equations are a special case.

We now ask about the asymptotic behavior of the climate mode v_k for small ε. We have the following:

THEOREM 7.1 *Denote by $v_k^\varepsilon(t)$ the solution of the first equation in* (7.3). *In the limit as $\varepsilon \to 0$, $v_k^\varepsilon(t)$ tends to $v_k(t)$ where $v_k(t)$ satisfies*

$$dv_k = -\gamma_k v_k\, dt + \sigma_k\, dW_k(t) \tag{7.7}$$

and W_k is a complex-valued Wiener process, and we define

$$\gamma_k = \frac{B_l + B_m}{2(\gamma_l + \gamma_m)}\left(\frac{\sigma_l^2 B_m}{\gamma_l} + \frac{\sigma_m^2 B_l}{\gamma_m}\right), \qquad \sigma_k = \frac{\sqrt{2}\sigma_l\sigma_m(B_l + B_m)}{2\sqrt{\gamma_l\gamma_m(\gamma_l + \gamma_m)}}\,. \tag{7.8}$$

Theorem 7.1 follows from Theorem 4.1 after mapping of the triad model equations in (7.3) onto the equations in (4.3). Alternatively, Theorem 7.1 can be proven from Theorem 4.4 by computing the operator $\bar{\mathcal{L}}$ in (4.35) associated with the equations in (7.3). Some details of this calculation are given at the end of this section.

Theorem 7.1 tells us that the unresolved modes w_l, w_m are responsible for all the driving of the climate mode v_k, since neglecting the effect of w_l, w_l in the first equation in (7.3) would have left us with the trivial result $\partial v_k/\partial t = 0$ instead of the equation in (7.7). This equation predicts that the limiting v_k is a Gaussian random process of Ornstein-Uhlenbeck type whose solution for the initial condition $v_k(t_0) = v_k$ is

$$v_k(t) = v_k e^{-\gamma_k(t-t_0)} + \sigma_k \int_{t_0}^{t} e^{-\gamma_k(t-s)}\, dW_k(s)\,. \tag{7.9}$$

It follows that a statistical steady state exists if and only if the stability criterion $\gamma_k > 0$ is satisfied, which, from the the expression for γ_k in (7.8), requires

$$B_m < -\max\left(B_l, \frac{\sigma_l^2\gamma_m B_l}{\sigma_m^2\gamma_l}\right) \quad \text{or} \quad B_m > -\min\left(B_l, \frac{\sigma_l^2\gamma_m B_l}{\sigma_m^2\gamma_l}\right). \tag{7.10}$$

In particular, a statistical steady state always exists if the nonlinear interaction coefficients satisfy $\operatorname{sgn}(B_l B_m) = 1$. On the contrary no statistical steady state exists and instability in the climate model occurs for $\operatorname{sgn}(B_l B_m) = -1$ if the conditions in (7.10) are violated. This result can be phrased in terms of the mechanism for energy transfer between the modes by nonlinear interaction. Typically, each pair (w_l, w_m), (w_m, v_k), or (v_k, w_l) transfers a positive or negative amount of energy with the corresponding mode v_k, w_l, or w_m, depending on the amplitude of the modes and the specific values of the nonlinear interaction coefficients B_k, B_l, and B_m. In the triad model equations in (7.3) the energy is fed into the unresolved modes w_l, w_l by the forcing, then transfered by nonlinear interaction to the climate mode v_k. In turn, energy can be back-scattered from v_k towards w_l and w_m. Stability or instability tells us about the balance between these two mechanisms. In particular, in the stable case, the back-scatter energy transfer from v_k to w_l and w_m is strong enough for dissipation through the damping terms in the equations for w_l and w_m to be sufficient for establishing a statistical steady state. In contrast, in the unstable case, energy piles up in the climate mode v_k, preventing the existence of a statistical steady state in the absence of additional dissipation on the mode v_k itself. This result generalizes to the random case the observation made by Smith and Waleffe for a deterministic triad model in [24]. In Section 7.2, we consider a triad model where the role of climate and unresolved variables are interchanged compared to (7.3), and we derive an analogous criterion involving the sign of the interaction coefficients that is consistent with the results obtained here.

It should also be pointed out that a statistical steady state always exists if the triad model equations in (7.3) are constrained by the results of equilibrium statistical theory, i.e., if we take the ratios σ_l^2/γ_l and σ_m^2/γ_m consistent with the energy spectrum in (7.2). In this case γ_l, γ_m can be chosen as the only free parameters, and Theorem 7.1 is changed as follows:

THEOREM 7.2 *Let the equations in* (7.3) *be constrained by* (7.2) *and denote by* $v_k^\varepsilon(t)$ *the solution of the first equation in* (7.3). *In the limit as* $\varepsilon \to 0$, $v_k^\varepsilon(t)$ *tends to* $v_k(t)$ *where* $v_k(t)$ *satisfies*

$$dv_k = -\gamma_k v_k\, dt + \sigma_k\, dW_k(t)\,, \tag{7.11}$$

where W_k *is a complex-valued Wiener process and we define*

$$\gamma_k = (\mu + |\vec{k}|^2) R_k\,, \qquad \sigma_k = \sqrt{\frac{R_k}{\alpha}}\,, \tag{7.12}$$

with

$$R_k = \frac{B_k^2}{2(\gamma_l + \gamma_m)|\vec{k}||\vec{l}||\vec{m}|(\mu + |\vec{l}|^2)(\mu + |\vec{m}|^2)}\,. \tag{7.13}$$

Clearly, $\gamma_k > 0$ from the first equation in (7.12) and, consistent with the results from equilibrium statistical theory in (6.16) and (6.18), the invariant measure for the process u_k defined through the equation in (7.11) has a Gaussian density with zero mean (since $\hat{h}_k = 0$) and variance

$$\frac{\sigma_k^2}{\gamma_k} = \frac{1}{\alpha(\mu + |\vec{k}|^2)}\,. \tag{7.14}$$

PROOF OF THEOREM 7.1: We proceed as in the proof of Theorem 5.1 and explicitly identify the operators entering $\bar{\mathcal{L}}$ in (4.35) associated with triad model equations in (7.3). The actual computation of $\bar{\mathcal{L}}$ can be done using the material in Appendix A. Consistent with the asymptotic procedure outlined in Section 4.4, we work with the backward equation associated with the Markov process defined by (7.3). Denote by $(v_k^\varepsilon(t), w_l^\varepsilon(t), w_m^\varepsilon(t))$ the solution of the equations in (7.3) for the initial condition $(v_k^\varepsilon(s), w_l^\varepsilon(s), w_m^\varepsilon(s)) = (v_k, w_l, w_m)$, and let

$$\varrho^\varepsilon\big(s, v_k, v_k^*, w_l^\varepsilon, w_l^*, w_m, w_m^* \mid t\big) = \mathbf{E}\, f\big(v_k^\varepsilon(t), w_l^\varepsilon(t), w_m^\varepsilon(t)\big)\,. \tag{7.15}$$

In this expression and the ones that follow, v_k, v_k^*, w_l, w_l^*, w_m, and w_m^* must be considered as independent variables. ϱ^ε satisfies the backward equation similar to the one in (4.22):

$$-\frac{\partial \varrho^\varepsilon}{\partial s} = \frac{1}{\varepsilon^2}\mathcal{L}_1\varrho^\varepsilon + \frac{1}{\varepsilon}\mathcal{L}_2\varrho^\varepsilon\,, \tag{7.16}$$

where we defined

$$\begin{aligned}
\mathcal{L}_1 &= -\gamma_l w_l \frac{\partial}{\partial w_l} - \gamma_l w_l^* \frac{\partial}{\partial w_l^*} - \gamma_m w_m \frac{\partial}{\partial w_m} - \gamma_m w_m^* \frac{\partial}{\partial w_m^*} \\
&\quad + 2\sigma_l^2 \frac{\partial}{\partial w_l \partial w_l^*} + 2\sigma_m^2 \frac{\partial}{\partial w_m \partial w_m^*}\,, \\
\mathcal{L}_2 &= B_k w_l w_m \frac{\partial}{\partial v_k} + B_k w_l^* w_m^* \frac{\partial}{\partial v_k^*} + B_l w_m v_k \frac{\partial}{\partial w_l} \\
&\quad + B_l w_m^* v_k^* \frac{\partial}{\partial w_l^*} + B_m w_l v_k \frac{\partial}{\partial w_m} + B_m w_l^* v_k^* \frac{\partial}{\partial w_m^*}\,.
\end{aligned} \tag{7.17}$$

The invariant measure associated with the operator $\mathcal{L}_1$ in (7.17) has the Gaussian density given by

$$P^\star = C \exp\left(-\frac{\gamma_l}{2\sigma_l^2}|w_l|^2 - \frac{\gamma_m}{2\sigma_m^2}|w_m|^2\right)\,, \tag{7.18}$$

and this expression reduces to the density (6.16) if the constraints in (7.1) and (7.2) are satisfied. The actual computation of the operator $\bar{\mathcal{L}}$ given in (4.35) can now be done using the material in Appendix A. □

7.2 Climate Scattering Triad Equations

We consider the equations in (3.17) under the assumption a1, which we recall for convenience

a1. There is no beta effect or topography. Thus, we set $\beta = \hat{h}_k = 0$ in the stochastic model for barotropic equations in (3.17).

Thus, there is no mean flow, $U = 0$. In addition, we have the following:

a3. We identify the modes v_l, v_m with $\vec{l}$, $\vec{m}$ fixed as climate variables, and all the other modes as unresolved variables. Furthermore, we assume that there is only one $\vec{k}$ such that $\vec{k} + \vec{l} + \vec{m} = 0$.

It follows by assumption a3 that the climate modes v_k, v_l are coupled to the single unresolved mode w_m, and the stochastic model for barotropic equations in (3.17) reduces to the following triad model equations

$$\begin{aligned} dv_l &= \frac{1}{\varepsilon} B_l w_k^* v_m^* \, dt \,, \qquad dv_m = \frac{1}{\varepsilon} B_m w_k^* v_l^* \, dt \,, \\ dw_k &= \frac{1}{\varepsilon} B_k v_l^* v_m^* \, dt - \frac{1}{\varepsilon^2} \gamma_k w_k \, dt + \frac{\sigma_k}{\varepsilon} \, dW_k(t) \,, \end{aligned} \tag{7.19}$$

with the condition in (7.5), $B_k + B_l + B_m = 0$. The equations in (7.19) have the structure of general three-wave interaction equations where only one of the variables is strongly damped and stochastically forced. Notice that the model equations in (7.19) can be obtained from the equations in (7.3) by interchanging the roles of climate and unresolved variables. Also, we have set $\bar{w}_k = 0$ in the last equation in (7.19) in order to avoid effects of order ε^{-1} of the unresolved mode w_k on the climate modes v_l, v_m.

The following theorem specifies the behavior of the climate modes for small ε:

THEOREM 7.3 *Denote by $v_l^\varepsilon(t)$, $v_m^\varepsilon(t)$ the solutions of the first two equations in* (7.19). *In the limit as $\varepsilon \to 0$, $(v_l^\varepsilon(t), v_m^\varepsilon(t))$ tends to $(v_l(t), v_m(t))$ where $v_l(t)$, $v_m(t)$ satisfy*

$$\begin{aligned} dv_l &= \frac{\sigma_k^2 B_l B_m}{\gamma_k^2} v_l \, dt + \frac{B_k B_l}{\gamma_k} |v_m|^2 v_l \, dt + \frac{\sigma_k B_l}{\gamma_k} v_m^* dW(t) \,, \\ dv_m &= \frac{\sigma_k^2 B_m B_l}{\gamma_k^2} v_m \, dt + \frac{B_k B_m}{\gamma_k} |v_l|^2 v_m \, dt + \frac{\sigma_k B_m}{\gamma_k} v_l^* \, dW(t) \,, \end{aligned} \tag{7.20}$$

where $W(t)$ is a complex Wiener process.

Theorem 7.3 follows from Theorem 4.2 after mapping of the triad model equations in (7.19) onto the equations in (4.11). Alternatively, Theorem 7.3 can be established by a direct method similar to the one presented in Section 4.5, as shown at the end of this section.

The equations in (7.20) clearly demonstrate that it may be necessary to account for the effect of the unresolved modes by nonlinear interaction terms and multiplicative noises in the equations for the climate modes. The exact solution of the equations in (7.20) is not available. However, since these equations are invariant under the transformation $(v_l, v_m) \to (-v_l, -v_m)$, it follows that the process $(v_l(t), v_m(t))$ predicted through these equations for the initial condition $(0, 0)$ has zero mean. Thus, we may get some insight if we linearize the equations in (7.20) around $(0, 0)$ and perform the linearized stability analysis of the resulting equations:

$$\begin{aligned} dv_l &= \frac{\sigma_k^2 B_l B_m}{\gamma_k^2} v_l \, dt + \frac{\sigma_k B_l}{\gamma_k} v_m^* \, dW(t) \,, \\ dv_m &= \frac{\sigma_k^2 B_m B_l}{\gamma_k^2} v_m \, dt + \frac{\sigma_k B_m}{\gamma_k} v_l^* \, dW(t) \,. \end{aligned} \tag{7.21}$$

An exact solution of these equations is available if we artificially force the process $(v_l(t), v_m(t))$ to be real by taking $W_k(t)$ to be a real instead of complex Wiener process. These results were reported in [16]. Here, we focus on the complex case. We have the following:

PROPOSITION 7.4 *The equations in* (7.21) *are stochastically stable around* $(0,0)$ *if and only if* B_l *and* B_m *have opposite signs. More precisely, if* $(V_l(t), V_m(t))$ *denotes the solution of the equations in* (7.21) *for the initial condition* (v_l, v_m)*, we have*

$$\lim_{v_l, v_m \to 0} \operatorname{prob}\Big(\sup_{0 \le t < \infty} \big(|V_l(t)|^2 + |V_m(t)|^2\big) \ge \varepsilon \Big) = 0 \quad \textit{for all } \varepsilon > 0 \tag{7.22}$$

if and only if $\operatorname{sgn}(B_l B_m) = -1$. *In particular, we have exponential instability if and only if* B_l *and* B_m *have the same sign.*

Notice that the criterion for stability, $\operatorname{sgn}(B_l B_m) = -1$, will always be satisfied if $|\vec{k}| > |\vec{l}|, |\vec{m}|$, i.e., if the unresolved mode has a wave number with higher amplitude than that of the climate modes. This follows from the relations in (7.6) and may also be stated as follows: One resolved mode of shorter wavelength cannot transfer energy to other two resolved modes for the barotropic equations.

PROOF: It is known [2] that, for an autonomous system of equations, stochastic stability is implied by stability in the second moment, i.e., by the property that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{0 \le t < \infty} \mathbf{E}\big(|V_l(t)|^2 + |V_m(t)|^2\big) \le \varepsilon \quad \text{if } |v_l|^2 + |v_m|^2 \le \delta\,. \tag{7.23}$$

Thus we demonstrate Proposition 7.4 by verifying the property in (7.23) for the equations in (7.21). From these equations, by standard application of the Itô calculus (see, e.g., [2]) we derive

$$\begin{aligned} \frac{d}{dt}\mathbf{E}|v_l|^2 &= \frac{2\sigma_k^2 B_l B_m}{\gamma_k^2}\mathbf{E}|v_l|^2 + \frac{2\sigma_k^2 B_l^2}{\gamma_k^2}\mathbf{E}|v_m|^2\,, \\ \frac{d}{dt}\mathbf{E}|v_m|^2 &= \frac{2\sigma_k^2 B_l B_m}{\gamma_k^2}\mathbf{E}|v_m|^2 + \frac{2\sigma_k^2 B_m^2}{\gamma_k^2}\mathbf{E}|v_l|^2\,. \end{aligned} \tag{7.24}$$

Hence

$$\begin{aligned} \mathbf{E}|V_l(t)|^2 &= \frac{B_m v_l^2 + B_l v_m^2}{2B_m} \exp\left(\frac{4\sigma_k^2 B_l B_m t}{\gamma_k^2}\right) + \frac{B_m v_l^2 - B_l v_m^2}{2B_m}\,, \\ \mathbf{E}|V_m(t)|^2 &= \frac{B_m v_l^2 + B_l v_m^2}{2B_l} \exp\left(\frac{4\sigma_k^2 B_l B_m t}{\gamma_k^2}\right) - \frac{B_m v_l^2 - B_l v_m^2}{2B_l}\,, \end{aligned} \tag{7.25}$$

and the property in (7.23) holds if $\operatorname{sgn}(B_l B_m) = -1$. Since instability of the second moments trivially implies stochastic instability, the formula in (7.25) completes the proof of Proposition 7.4. $\square$

Proposition 7.4 deserves several comments. First, this result demonstrates that it is possible to generate significant energy in two components of an interacting triad by strongly forcing and damping the third component only if these two components have the same sign of interaction, as conjectured by Smith and Waleffe [24]. Furthermore, this result is consistent with what we concluded in Section 7.1 about the mechanism of nonlinear energy transfer between the modes. Recall that the triad model equations in (7.3) and the ones in (7.26) correspond to opposite identification of the climate and the unresolved variables. Consistent with the interpretation of stability as the result of the balance between energy transfer rates between the modes, we generally observe stability of equation (7.7) when the equation in (7.20) is unstable, and vice versa. Of course, the precise criterion for stability or instability in (7.7) or (7.20) may differ slightly since the forcing and damping in the equations in (7.3) and in (7.26) are different.

Another comment about Proposition 7.4 is that, in contrast to the situations in Section 7.1, here linear instability can occur even if the parameters in (7.19) are taken that are consistent with equilibrium statistical mechanics. To discuss this point in more detail, we state the following result, which follows from Theorem 7.3 if we constrain the ratio σ_k^2/γ_k consistent with (7.2) and take γ_k to be the only free parameter:

THEOREM 7.5 *Let the last equation in* (7.19) *be constrained by* (7.2) *and denote by $v_l^\varepsilon(t)$, $v_m^\varepsilon(t)$ the solutions of the first two equations in* (7.19). *In the limit as $\varepsilon \to 0$, $(v_l^\varepsilon(t), v_m^\varepsilon(t))$ tends to $(v_l(t), v_m(t))$ where $v_l(t)$ and $v_m(t)$ satisfy*

$$\begin{aligned} dv_l &= \frac{c_k^2 B_l B_m}{\gamma_k} v_l\,dt + \frac{B_k B_l}{\gamma_k}|v_m|^2 v_l\,dt + \frac{c_k B_l}{\sqrt{\gamma_k}} v_m^*\,dW(t)\,, \\ dv_m &= \frac{c_k^2 B_m B_l}{\gamma_k} v_m\,dt + \frac{B_k B_m}{\gamma_k}|v_l|^2 v_m\,dt + \frac{c_k B_m}{\sqrt{\gamma_k}} v_l^*\,dW(t)\,, \end{aligned} \tag{7.26}$$

where $W(t)$ is a complex Wiener process and c_k is the root mean square value for w_k predicted by equilibrium statistical theory

$$c_k = \frac{1}{\sqrt{\alpha(\mu + |\vec{k}|^2)}}\,. \tag{7.27}$$

The condition for linear stability of the equations in (7.26) is the same as the one for the equations in (7.20): $\operatorname{sgn}(B_l B_m) = -1$. This result should, however, be balanced with the following remarkable property of the nonlinear equations in (7.26). Consistent with Theorem 6.6, the invariant measure for the equations in (7.26) has the following Gaussian density:

$$P_C^{\star,\mathrm{clim}} = C\exp\left(-\frac{1}{2}\alpha(\mu + |\vec{l}|^2)|v_l|^2 - \frac{1}{2}\alpha(\mu + |\vec{m}|^2)|v_m|^2\right). \tag{7.28}$$

This fact can also be verified explicitly by checking that the density in (7.28) is a statistical steady state solution for the Fokker-Planck equations associated with the equations in (7.26), i.e.,

$$
\begin{aligned}
0 = {} & \left(\frac{c_k^2 B_l B_m}{\gamma_k} + \frac{B_k B_l}{\gamma_k}|v_m|^2\right)\left(\frac{\partial}{\partial v_l}(v_l P_C^{\star,\mathrm{clim}}) + \frac{\partial}{\partial v_l^*}(v_l^* P_C^{\star,\mathrm{clim}})\right) \\
& + \left(\frac{c_k^2 B_m B_l}{\gamma_k} + \frac{B_k B_m}{\gamma_k}|v_l|^2\right)\left(\frac{\partial}{\partial v_m}(v_m P_C^{\star,\mathrm{clim}}) + \frac{\partial}{\partial v_m^*}(v_m^* P_C^{\star,\mathrm{clim}})\right) \\
& + \frac{2c_k^2 B_l^2}{\gamma_k}|v_m|^2 \frac{\partial^2 P_C^{\star,\mathrm{clim}}}{\partial v_l \partial v_l^*} + \frac{2c_k^2 B_m^2}{\gamma_k}|v_l|^2 \frac{\partial^2 P_C^{\star,\mathrm{clim}}}{\partial v_m \partial v_m^*} \\
& + \frac{2c_k^2 B_l B_m}{\gamma_k}\left(v_l^* v_m \frac{\partial^2 P_C^{\star,\mathrm{clim}}}{\partial v_l \partial v_m^*} + v_l v_m^* \frac{\partial^2 P_C^{\star,\mathrm{clim}}}{\partial v_l^* \partial v_m}\right).
\end{aligned}
\tag{7.29}
$$

The existence of an invariant measure with density (7.28) for equations in (7.26) implies that any linear instability of the equations in (7.26) will eventually be saturated by nonlinear effects in a way consistent with the density in (7.28). It would be interesting to know if a similar mechanism of nonlinear saturation exists for the equations in (7.20) in certain parameter regimes. Finally, we point out that, in the stable situation, the present analysis suggests that the actual invariant measure for the equations in (7.26) will be the point mass measure at $(0, 0)$ rather than the one with density (7.28). The existence of two invariant measures for the equations in (7.26), one stable and one unstable depending on the parameters, is a kind of pathology that is made possible by the singular nature of these equations at $(0, 0)$, but is likely to disappear for more general model equations than the ones in (7.19) because of the appearance of linear random forcing in the climate model equations as described in Sections 4 and 5 above. For instance, the introduction of fast-wave effects on the equations in (7.19), as considered in Section 7.3 below, guarantee that the invariant measure for the resulting equation for the climate variable alone is always the invariant measure with the Gaussian density in (7.28).

PROOF OF THEOREM 7.3: We use the direct method presented in Section 4.5 and proceed by direct calculation. Because the last equation in (7.19) is linear in w_k, this equation for the initial condition $w_k(0) = w_k$ is equivalent to the integral equation

$$
w_k(t) = e^{-\gamma_k t/\varepsilon^2} w_k + \frac{B_k}{\varepsilon}\int_0^t e^{-\gamma_k (t-s)/\varepsilon^2} v_l^*(s) v_m^*(s)\,ds + \frac{1}{\varepsilon} g_k(t)\,,
\tag{7.30}
$$

where

$$
g_k(t) = \sigma_k \int_0^t e^{-\gamma_k (t-s)/\varepsilon^2}\, dW_k(s)\,.
\tag{7.31}
$$

Inserting (7.30) into the first two equations in (7.19) gives closed, non-Markovian stochastic model equations for the climate modes $(v_l(t), v_m(t))$ valid for any ε.

These equations are given by

$$\begin{aligned} dv_l(t) &= \frac{B_l}{\varepsilon^2} e^{-\gamma_k t/\varepsilon^2} w_k^* v_m^*(t)dt \\ &\quad + \frac{B_l B_k}{\varepsilon^2}\left(\int_0^t e^{-\gamma_k(t-s)/\varepsilon^2} v_l(s)v_m(s)ds\right) v_m^*(t)dt \\ &\quad + \frac{B_l}{\varepsilon^2} g_k^*(t) v_m^*(t)dt \\ dv_m(t) &= \frac{B_m}{\varepsilon^2} e^{-\gamma_k t/\varepsilon^2} w_k^* v_l^*(t)dt \\ &\quad + \frac{B_m B_k}{\varepsilon^2}\left(\int_0^t e^{-\gamma_k(t-s)/\varepsilon^2} v_l(s)v_m(s)ds\right) v_l^*(t)dt \\ &\quad + \frac{B_m}{\varepsilon^2} g_k(t) v_l^*(t)dt\,. \end{aligned} \tag{7.32}$$

We now show that, in the limit as $\varepsilon \to 0$, the equations in (7.32) reduce to the actual stochastic model given in (7.20). We consider successively the various terms in (7.32) in the limit as $\varepsilon \to 0$. We have first

$$\frac{B_l}{\varepsilon} e^{-\gamma_k t/\varepsilon^2} w_k^* v_m^*(t)dt \to 0\,, \quad \frac{B_m}{\varepsilon} e^{-\gamma_k t/\varepsilon^2} w_k^* v_l^*(t)dt \to 0\,. \tag{7.33}$$

Second,

$$\begin{aligned} \frac{B_l B_k}{\varepsilon^2}\left(\int_0^t e^{-\gamma_k(t-s)/\varepsilon^2} v_l(s)v_m(s)ds\right) v_m^*(t)dt &\to \frac{B_l B_k}{\gamma_k} v_l(t)|v_m(t)|^2\,dt\,, \\ \frac{B_m B_k}{\varepsilon^2}\left(\int_0^t e^{-\gamma_k(t-s)/\varepsilon^2} v_l(s)v_m(s)ds\right) v_l^*(t)dt &\to \frac{B_l B_k}{\gamma_k} v_m(t)|v_l(t)|^2\,dt\,. \end{aligned} \tag{7.34}$$

Finally, a standard argument with test functions similar to the one given in Section 4.5 shows that

$$\begin{aligned} \frac{B_l}{\varepsilon^2} g_k^*(t) v_m^*(t)dt &\to \frac{\sigma_k B_l}{\gamma_k} v_m^*(t) \circ dW(t)\,, \\ \frac{B_m}{\varepsilon^2} g_k^*(t) v_l^*(t)dt &\to \frac{\sigma_k B_m}{\gamma_k} v_l^*(t) \circ dW(t)\,, \end{aligned} \tag{7.35}$$

where, as the external limit of a process with finite correlation time, the white noise is interpreted in Stratonovich's sense. Collecting (7.33), (7.34), and (7.35) into the first two equations in (7.19), we obtain

$$\begin{aligned} dv_l &= \frac{B_k B_l}{\gamma_k}|v_m|^2 v_l\,dt + \frac{\sigma_k B_l}{\gamma_k} v_m^* \circ dW(t)\,, \\ dv_m &= \frac{B_k B_m}{\gamma_k}|v_l|^2 v_m\,dt + \frac{\sigma_k B_m}{\gamma_k} v_l^* \circ dW(t)\,. \end{aligned} \tag{7.36}$$

These equations are equivalent to Itô's equations in (7.20), which concludes the proof. □

7.3 Triad Equations with Fast-Wave Effects

We consider the equations in (3.17) under the following assumptions:

a4. There is no beta effect or mean flow. Thus, we set $\beta = 0$, $U = 0$ in the stochastic model for barotropic equations in (3.17).
a5. We identify the modes v_l, v_m with $\vec{l}$, $\vec{m}$ fixed as climate variables, and all the other modes as unresolved variables. We also assume that there is only one pair $\vec{k}$ such that $\vec{k} + \vec{l} + \vec{m} = 0$.
a6. Finally, we assume that the only nonzero mode of topography is $\hat{h}_{l-m} \equiv h$, which we take to be real, $h = h^*$.

Assumption a6 is somewhat artificial but is meant for transparency only, and the case with all three modes of topography, $\hat{h}_k$, $\hat{h}_l$, and $\hat{h}_m$, nonzero and complex, can easily be handled.

It follows by assumptions a4 through a6 that the climate modes v_l, v_m are coupled to the single unresolved mode w_k, and the stochastic model for barotropic equations in (3.17) reduces to triad model equations that are similar to those in (7.19) except for the fast-wave effects induced by the topography

$$
\begin{aligned}
dv_l &= -\frac{\omega}{\varepsilon} v_m\, dt + \frac{1}{\varepsilon} B_l w_k^* v_m^*\, dt\,,\\
dv_m &= \frac{\omega}{\varepsilon} v_l\, dt + \frac{1}{\varepsilon} B_m w_k^* v_l^*\, dt\,,\\
dw_k &= \frac{1}{\varepsilon} B_k v_l^* v_m^*\, dt - \frac{1}{\varepsilon^2}\gamma_k w_k\, dt + \frac{\sigma_k}{\varepsilon}\, dW_k(t)\,,
\end{aligned}
\tag{7.37}
$$

where we defined

$$
\omega = \frac{h}{|\vec{k}||\vec{l}|}\,. \tag{7.38}
$$

The asymptotic behavior of the climate modes for small ε is specified by the following:

THEOREM 7.6 *Denote by $v_l^\varepsilon(t)$ and $v_m^\varepsilon(t)$ the solutions of the first two equations in* (7.37)*, and let*

$$
\begin{aligned}
\bar{v}_l^\varepsilon(t) &= \cos\left(\frac{\omega t}{\varepsilon}\right) v_l^\varepsilon(t) + \sin\left(\frac{\omega t}{\varepsilon}\right) v_m^\varepsilon(t)\,,\\
\bar{v}_m^\varepsilon(t) &= \cos\left(\frac{\omega t}{\varepsilon}\right) v_m^\varepsilon(t) - \sin\left(\frac{\omega t}{\varepsilon}\right) v_l^\varepsilon(t)\,.
\end{aligned}
\tag{7.39}
$$

In the limit as $\varepsilon \to 0$, $(\bar{v}_l^\varepsilon(t), \bar{v}_m^\varepsilon(t))$ *tends to* $(v_l(t), v_m(t))$ *where* $v_l(t)$, $v_m(t)$ *satisfy*

$$
\begin{aligned}
dv_l &= \frac{\sigma_k^2 B_l B_m}{\gamma_k^2} v_l\, dt + \frac{B_k(B_l + B_m)}{8\gamma_k}\big((2|v_m|^2 + |v_l|^2)v_l - v_m^2 v_l^*\big)dt \\
&\quad + \frac{\sigma_k}{2\gamma_k}(B_l - B_m)v_m^*\, dW_1(t) + \frac{\sigma_k}{2\sqrt{2}\gamma_k}(B_l + B_m)v_m^*\, dW_2(t) \\
&\quad + \frac{\sigma_k}{2\sqrt{2}\gamma_k}(B_l + B_m)v_l^* dW_3(t)\,, \\
dv_m &= \frac{\sigma_k^2 B_l B_m}{\gamma_k^2} v_m\, dt + \frac{B_k(B_l + B_m)}{8\gamma_k}\big((2|v_l|^2 + |v_m|^2)v_l - v_l^2 v_m^*\big)dt \\
&\quad + \frac{\sigma_k}{2\gamma_k}(B_m - B_l)v_l^*\, dW_1(t) + \frac{\sigma_k}{2\sqrt{2}\gamma_k}(B_l + B_m)v_l^*\, dW_2(t) \\
&\quad - \frac{\sigma_k}{2\sqrt{2}\gamma_k}(B_l + B_m)v_m^*\, dW_3(t)\,,
\end{aligned}
\tag{7.40}
$$

where $W_1(t)$, $W_2(t)$, *and* $W_3(t)$ *are independent complex Wiener processes.*

Theorem 7.6 follows from Theorem 5.3 after mapping of the triad model equations in (7.37) onto the equations in (4.1). Alternatively, Theorem 7.3 can be established by a direct calculation, reported at the end of this section.

The equations in (7.40) should be compared to the equations in (7.20) in Theorem 7.3. We see that the fast-wave effects in the equations in (7.37) are responsible for isotropization of the modes (v_l, v_m). If we make the original equations in (7.37) consistent with equilibrium statistical mechanics by constraining the ratio σ_k^2/γ_k as in (7.2) and taking γ_k as the only free parameters, Theorem 7.6 is changed into the following:

THEOREM 7.7 *Let the last equation in* (7.37) *be constrained by* (7.2), *denote by* $v_l^\varepsilon(t)$ *and* $v_m^\varepsilon(t)$ *the solutions of the first two equations in* (7.37), *and let*

$$
\begin{aligned}
\bar{v}_l^\varepsilon(t) &= \cos\left(\frac{\omega t}{\varepsilon}\right)v_l^\varepsilon(t) + \sin\left(\frac{\omega t}{\varepsilon}\right)v_m^\varepsilon(t)\,, \\
\bar{v}_m^\varepsilon(t) &= \cos\left(\frac{\omega t}{\varepsilon}\right)v_m^\varepsilon(t) - \sin\left(\frac{\omega t}{\varepsilon}\right)v_l^\varepsilon(t)\,.
\end{aligned}
\tag{7.41}
$$

In the limit as $\varepsilon \to 0$, $(\bar{v}_l^\varepsilon(t), \bar{v}_m^\varepsilon(t))$ *tends to* $(v_l(t), v_m(t))$ *where* $v_l(t)$ *and* $v_m(t)$ *satisfy*

$$
\begin{aligned}
dv_l &= \frac{c_k^2 B_l B_m}{\gamma_k} v_l\, dt + \frac{B_k(B_l + B_m)}{8\gamma_k}\big((2|v_m|^2 + |v_l|^2)v_l - v_m^2 v_l^*\big)dt \\
&\quad + \frac{c_k}{2\sqrt{\gamma_k}}(B_l - B_m)v_m^*\, dW_1(t) + \frac{c_k}{2\sqrt{2\gamma_k}}(B_l + B_m)v_m^*\, dW_2(t) \\
&\quad + \frac{c_k}{2\sqrt{2\gamma_k}}(B_l + B_m)v_l^*\, dW_3(t)\,,
\end{aligned}
\tag{7.42}
$$

$$
\begin{aligned}
dv_m = {} & \frac{c_k^2 B_l B_m}{\gamma_k} v_m\, dt + \frac{B_k(B_l + B_m)}{8\gamma_k}\big((2|v_l|^2 + |v_m|^2)v_l - v_l^2 v_m^*\big)dt \\
& + \frac{c_k}{2\gamma_k}(B_m - B_l)v_l^*\, dW_1(t) + \frac{c_k}{2\sqrt{2\gamma_k}}(B_l + B_m)v_l^*\, dW_2(t) \\
& - \frac{c_k}{2\sqrt{2\gamma_k}}(B_l + B_m)v_m^*\, dW_3(t)\,,
\end{aligned}
$$

where $W_1(t)$, $W_2(t)$, and $W_3(t)$ are independent complex Wiener processes and c_k is the root-mean-square value for w_k predicted by equilibrium statistical theory

$$
c_k = \frac{1}{\sqrt{\alpha(\mu + |\vec{k}|^2)}}\,. \tag{7.43}
$$

The equations in (7.42) should be compared to the equations in (7.26) in Theorem 7.5, confirming that the fast-wave effects in the equations in (7.37) are responsible for isotropization of the modes (v_l, v_m).

We now show that isotropization has a dramatic effect on the linear stability around (0, 0) of the equations in (7.40) and (7.42). In fact, in contrast to the equations in (7.20), which are linearly stable if the criterion $\operatorname{sgn}(B_l B_m) = -1$ is satisfied (see Proposition 7.8), the equations in (7.40) and (7.42) are *always* linearly unstable. More precisely, consider the equations in (7.40) linearized around (0, 0):

$$
\begin{aligned}
dv_l = {} & \frac{\sigma_k^2 B_l B_m}{\gamma_k^2} v_l\, dt + \frac{\sigma_k}{2\gamma_k}(B_l - B_m)v_m^*\, dW_1(t) \\
& + \frac{\sigma_k}{2\sqrt{2}\gamma_k}(B_l + B_m)v_m^*\, dW_2(t) + \frac{\sigma_k}{2\sqrt{2}\gamma_k}(B_l + B_m)v_l^*\, dW_3(t)\,, \\
dv_m = {} & \frac{\sigma_k^2 B_l B_m}{\gamma_k^2} v_m\, dt + \frac{\sigma_k}{2\gamma_k}(B_m - B_l)v_l^*\, dW_1(t) \\
& + \frac{\sigma_k}{2\sqrt{2}\gamma_k}(B_l + B_m)v_l^*\, dW_2(t) - \frac{\sigma_k}{2\sqrt{2}\gamma_k}(B_l + B_m)v_m^*\, dW_3(t)\,.
\end{aligned}
\tag{7.44}
$$

The equation obtained by linearizing the equations in (7.42) are similar and can be obtained from (7.44) by setting $\sigma_k/\sqrt{\gamma_k} = c_k$. We have the following:

PROPOSITION 7.8 *The equations in* (7.40) *are always stochastically unstable around* (0, 0).

PROOF: We proceed similarly as in the proof of Proposition 7.4 and verify the instability in the second moment for the equations in (7.40). From (7.40) it follows that (compare (7.24))

$$
\begin{aligned}
\frac{d}{dt}\mathbf{E}|v_l|^2 &= \frac{\sigma_k^2(B_l^2 + B_m^2 + 10B_lB_m)}{4\gamma_k^2}\mathbf{E}|v_l|^2 \\
&\quad + \frac{\sigma_k^2(3B_l^2 + 3B_m^2 - 2B_lB_m)}{4\gamma_k^2}\mathbf{E}|v_m|^2\,, \\
\frac{d}{dt}\mathbf{E}|v_m|^2 &= \frac{\sigma_k^2(B_l^2 + B_m^2 + 10B_lB_m)}{4\gamma_k^2}\mathbf{E}|v_m|^2 \\
&\quad + \frac{\sigma_k^2(3B_l^2 + 3B_m^2 - 2B_lB_m)}{4\gamma_k^2}\mathbf{E}|v_l|^2\,.
\end{aligned}
\tag{7.45}
$$

The solution of these equations is

$$
\begin{aligned}
\mathbf{E}|v_l(t)|^2 &= \frac{1}{2}(|v_l|^2 + |v_m|^2)\exp\left(\frac{\sigma_k^2(B_l + B_m)^2 t}{\gamma_k^2}\right) \\
&\quad + \frac{1}{2}(|v_l|^2 - |v_m|^2)\exp\left(-\frac{\sigma_k^2(B_l^2 + B_m^2 - 6B_lB_m)t}{2\gamma_k^2}\right), \\
\mathbf{E}|v_m(t)|^2 &= \frac{1}{2}(|v_l|^2 + |v_m|^2)\exp\left(\frac{\sigma_k^2(B_l + B_m)^2 t}{\gamma_k^2}\right) \\
&\quad - \frac{1}{2}(|v_l|^2 - |v_m|^2)\exp\left(-\frac{\sigma_k^2(B_l^2 + B_m^2 - 6B_lB_m)t}{2\gamma_k^2}\right),
\end{aligned}
\tag{7.46}
$$

which demonstrates instability. □

It is remarkable that both the equations in (7.40) and the equations in (7.42) that we obtain by using the constraint in (7.2) from equilibrium statistical mechanics are always linearly unstable. In the latter case, however, linear instability around (0, 0) should be balanced with the property that the invariant measure for the full nonlinear equation in (7.42) has the Gaussian density

$$
P_C^{\star,\mathrm{clim}} = C\exp\left(-\frac{1}{2}\alpha\big(\mu + |\vec{l}|^2\big)|v_l|^2 - \frac{1}{2}\alpha\big(\mu + |\vec{m}|^2\big)|v_m|^2\right). \tag{7.47}
$$

This follows from Theorem 6.6; alternatively, this property can be checked explicitly by a calculation similar to the one we did in Section 7.2. In other words, linear instability of the equation in (7.42) will eventually be saturated by nonlinear effects in a way consistent with the density in (7.47). In contrast, such a nonlinear saturation mechanism will generally not be present for the equation in (7.40).

PROOF OF THEOREM 7.6: The proof discussed here uses the direct method presented in Section 4.5 combined with multiple time-scale expansion to deal with fast-wave effects. We deal with the rotation first by changing dependent variables as explained in Section 5.3. Let

$$
\bar{v}_l(t) = c(t)v_l(t) + s(t)v_m(t)\,, \quad \bar{v}_m(t) = c(t)v_m(t) - s(t)v_l(t)\,, \tag{7.48}
$$

with $c(t) = \cos(\omega t/\varepsilon)$, $s(t) = \sin(\omega t/\varepsilon)$. In terms of $(w_k, \bar{v}_l, \bar{v}_m)$, the equations in (7.37) become

$$
\begin{aligned}
d\bar{v}_l &= \frac{1}{\varepsilon} B_l c V_m^* w_k^* \, dt + \frac{1}{\varepsilon} B_m s V_l^* w_k^* \, dt \,, \\
d\bar{v}_m &= \frac{1}{\varepsilon} B_m c V_l^* w_k^* \, dt - \frac{1}{\varepsilon} B_l s V_m^* w_k^* \, dt \,, \\
d w_k &= \frac{1}{\varepsilon} B_k V_l^* V_m^* \, dt - \frac{1}{\varepsilon^2} \gamma_k w_k \, dt + \frac{\sigma_k}{\varepsilon} \, dW_k(t) \,,
\end{aligned}
\tag{7.49}
$$

where we defined

$$
V_l = c\bar{v}_l - s\bar{v}_m \,, \quad V_m = c\bar{v}_m + s\bar{v}_l \,. \tag{7.50}
$$

The equation for w_k in (7.49) for the initial condition $w_k(0) = w_k$ can be solved as

$$
w_k(t) = e^{-\gamma_k t/\varepsilon^2} w_k + \frac{B_k}{\varepsilon} \int_0^t e^{-\gamma_k (t-s)/\varepsilon^2} V_l^*(s) V_m^*(s) ds + \frac{1}{\varepsilon} g_k(t) \,, \tag{7.51}
$$

where

$$
g_k(t) = \sigma_k \int_0^t e^{-\gamma_k (t-s)/\varepsilon^2} \, dW_k(s) \,. \tag{7.52}
$$

Inserting (7.51) into the first two equations in (7.49) gives the following closed, non-Markovian stochastic model equations for the climate modes $(\bar{v}_l(t), \bar{v}_m(t))$ valid for any ε:

$$
\begin{aligned}
d\bar{v}_l(t) = {} & \frac{B_l}{\varepsilon} c(t) e^{-\gamma_k t/\varepsilon^2} w_k^* V_m^*(t) dt + \frac{B_m}{\varepsilon} s(t) e^{-\gamma_k t/\varepsilon^2} w_k^* V_l^*(t) dt \\
& + \frac{B_l B_k}{\varepsilon^2} c(t) V_m^*(t) \left(\int_0^t e^{-\gamma_k (t-s)/\varepsilon^2} V_l(s) V_m(s) ds \right) dt \\
& + \frac{B_m B_k}{\varepsilon^2} s(t) V_l^*(t) \left(\int_0^t e^{-\gamma_k (t-s)/\varepsilon^2} V_l(s) V_m(s) ds \right) dt \\
& + \frac{B_l \sigma_k}{\varepsilon^2} c(t) V_m^*(t) g_k^*(t) dt + \frac{B_m \sigma_k}{\varepsilon^2} s(t) V_l^*(t) g_k^*(t) dt \,, \\
d\bar{v}_m(t) = {} & \frac{B_m}{\varepsilon} c(t) e^{-\gamma_k t/\varepsilon^2} w_k^* V_l^*(t) dt - \frac{B_l}{\varepsilon} s(t) e^{-\gamma_k t/\varepsilon^2} w_k^* V_m^*(t) dt \\
& + \frac{B_m B_k}{\varepsilon^2} c(t) V_l^*(t) \left(\int_0^t e^{-\gamma_k (t-s)/\varepsilon^2} V_l(s) V_m(s) ds \right) dt \\
& - \frac{B_l B_k}{\varepsilon^2} s(t) V_m^*(t) \left(\int_0^t e^{-\gamma_k (t-s)/\varepsilon^2} V_l(s) V_m(s) ds \right) dt \\
& + \frac{B_m \sigma_k}{\varepsilon^2} c(t) V_l^*(t) g_k^*(t) dt - \frac{B_l \sigma_k}{\varepsilon^2} s(t) V_m^*(t) g_k^*(t) dt \,.
\end{aligned}
\tag{7.53}
$$

We now show that in the limit as $\varepsilon \to 0$ the solutions of these non-Markovian model equations converge to the solutions of the actual stochastic model given in (7.40). Taking the limit as $\varepsilon \to 0$ for the equations in (7.53) is an operation more complicated than the one we did with the equations in (7.32) because of the appearance of both time scales t/ε (corresponding to rotation effects) and t/ε^2 (corresponding to the fast evolution of the nonlinear self-interaction between unresolved variables modeled stochastically in (7.37)). The right way to proceed is to exploit the separation of scale between the various time scales and use a multiple time-scale expansion argument to take the limit as $\varepsilon \to 0$ for the equations in (7.53). In effect, this amounts to doing the following three operations on these equations:

(1) Consider $c(t)$, $s(t)$ as functions of $\tau = t/\varepsilon$,
(2) let $\varepsilon \to 0$ with τ kept fixed, and
(3) time-average over τ with t kept fixed.

For simplicity, we will skip the tedious calculations involved in justifying steps (1) through (3) and simply use them as a rule. Step (1) amounts to setting in the equations in (7.53)

$$\begin{aligned} V_l(t) &\to V_l(t,\tau) = \cos(\omega\tau)\bar{v}_l(t) - \sin(\omega\tau)\bar{v}_m(t)\,, \\ V_m(t) &\to V_m(t,\tau) = \cos(\omega\tau)\bar{v}_m(t) + \sin(\omega\tau)\bar{v}_l(t)\,. \end{aligned} \tag{7.54}$$

The limit as $\varepsilon \to 0$ involved in step (2) can be obtained by a calculation very similar to the one we did in the proof of Proposition 7.3. It leads to

$$\begin{aligned} d\bar{v}_l(t) &= \frac{B_l B_k}{\gamma_k}\cos(\omega\tau)|V_m(t,\tau)|^2 V_l(t,\tau)dt \\ &\quad + \frac{B_m B_k}{\gamma_k}\sin(\omega\tau)|V_l(t,\tau)|^2 V_m(t,\tau)dt \\ &\quad + \frac{B_l\sigma_k}{\gamma_k}\cos(\omega\tau)V_m^*(t,\tau)\circ dW(t) \\ &\quad + \frac{B_m\sigma_k}{\gamma_k}\sin(\omega\tau)V_l^*(t,\tau)\circ dW(t), \\ d\bar{v}_m(t) &= \frac{B_m B_k}{\gamma_k}\cos(\omega\tau)|V_l(t,\tau)|^2 V_m(t,\tau)dt \\ &\quad - \frac{B_l B_k}{\gamma_k}\sin(\omega\tau)|V_m(t,\tau)|^2 V_l(t,\tau)dt \\ &\quad + \frac{B_m\sigma_k}{\gamma_k}\cos(\omega\tau)V_l^*(t,\tau)\circ dW(t) \\ &\quad - \frac{B_l\sigma_k}{\gamma_k}\sin(\omega\tau)V_m^*(t,\tau)\circ dW(t)\,, \end{aligned} \tag{7.55}$$

where $W(t)$ is a complex Wiener process. Step (3) now amounts to averaging these equations over τ with t kept fixed. This operation is straightforward for those terms in (7.55) that do not involve $dW(t)$. For instance, for the first term on the right-hand side of the equation for $\bar{v}_l(t)$ in (7.55), we obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{B_l B_k}{\gamma_k}\cos(\omega\tau)|V_m(t,\tau)|^2 V_l(t,\tau)d\tau = \frac{B_l B_k}{8\gamma_k}\big((2|\bar{v}_m(t)|^2 + |\bar{v}_l(t)|^2)\bar{v}_l(t) - \bar{v}_m^2(t)\bar{v}_l^*(t)\big)\,, \tag{7.56}$$

and the other terms are treated similarly. The time-averaging of the terms in (7.55) that involve $dW(t)$ is slightly more tedious since both the terms themselves and their quadratic variations must be considered. Expanding the terms involving $dW(t)$, we can express them in terms of linear combination (with factors involving $\bar{v}_l(t)$, $\bar{v}_m(t)$) of

$$\cos^2(\omega\tau)dW(t)\,, \quad \sin(\omega\tau)\cos(\omega\tau)dW(t)\,, \quad \sin^2(\omega\tau)dW(t)\,. \tag{7.57}$$

Time-averaging over τ can now be performed using a standard test function argument. For instance, we have

$$\lim_{T\to\infty} \frac{1}{T}\mathbf{E}\int_0^T \left(\int_0^\infty \eta(t)\cos^2(\omega\tau)dW(t)\right)d\tau = 0\,, \tag{7.58}$$

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T}\mathbf{E}\int_0^T \left|\int_0^\infty \eta(t)\cos^2(\omega\tau)dW(t)\right|^2 d\tau \\ = 2\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^\infty \eta^2(t)\cos^4(\omega\tau)dt\,d\tau = \frac{3}{4}\int_0^\infty \eta^2(t)dt\,,\end{aligned} \tag{7.59}$$

where η is a test function. By such manipulations, it can be concluded that time-averaging amounts to setting

$$\begin{aligned}\cos^2(\omega t)dW(t) &\to \frac{1}{2}dW_1(t) + \frac{1}{2\sqrt{2}}dW_2(t)\,,\\ \sin^2(\omega t)dW(t) &\to \frac{1}{2}dW_1(t) - \frac{1}{2\sqrt{2}}dW_2(t)\,,\\ \sin(\omega t)\cos(\omega t)dW(t) &\to \frac{1}{2\sqrt{2}}dW_3(t)\,,\end{aligned} \tag{7.60}$$

where $W_1(t)$, $W_2(t)$, and $W_3(t)$ are independent, complex white noises. Combining the results in (7.56) and (7.60), it follows that, in the limit as $\varepsilon \to 0$, the equations in (7.53) reduce to the following system of Stratonovitch's equations for $(v_l(t), v_m(t))$:

$$\begin{aligned}dv_l = {}& \frac{B_k(B_l+B_m)}{8\gamma_k}\big((2|v_m|^2 + |v_l|^2)v_m - v_m^2 v_l^*\big)dt\\ &+ \frac{\sigma_k}{2\gamma_k}(B_l - B_m)v_m^* \circ dW_1(t) + \frac{\sigma_k}{2\sqrt{2}\gamma_k}(B_l+B_m)v_m^* \circ dW_2(t)\\ &+ \frac{\sigma_k}{2\sqrt{2}\gamma_k}(B_l+B_m)v_l^* \circ dW_3(t)\,,\end{aligned} \tag{7.61}$$

$$
\begin{aligned}
dv_m = {} & \frac{B_k(B_l + B_m)}{8\gamma_k}\big((2|v_l|^2 + |v_m|^2)v_l - v_l^2 v_m^*\big)dt \\
& + \frac{\sigma_k}{2\gamma_k}(B_m - B_l)v_l^* \circ dW_1(t) + \frac{\sigma_k}{2\sqrt{2}\gamma_k}(B_l + B_m)v_l^* \circ dW_2(t) \\
& - \frac{\sigma_k}{2\sqrt{2}\gamma_k}(B_l + B_m)v_m^* \circ dW_3(t)\,.
\end{aligned}
$$

These equations are equivalent to Itô's equations in (7.40). □

7.4 The Mean U as a Single Climate Variable

As a last special example of the equations in (3.17), we consider that

a7. The mean flow U is identified as the climate variable, and the Fourier modes $u_k = w_k$ as the unresolved variables.

By assumption a7, it follows by utilizing the reality condition, $w_{-k} = w_k^*$, that the stochastic model for barotropic equations in (3.17) thus reduces to the following system:

$$
\begin{aligned}
dU &= \frac{2}{\varepsilon}\sum_{k\in\bar{\sigma}_2} k_x\,\mathrm{Im}(\hat{H}_k^* w_k)dt\,, \\
dw_k &= \frac{i}{\varepsilon}H_k U\,dt - \frac{i}{\varepsilon}(k_x U - \Omega_k)w_k\,dt \\
&\quad - \frac{1}{\varepsilon^2}\gamma_k(w_k - \bar{w}_k)dt + \frac{\sigma_k}{\varepsilon}dW_k(t)\,,
\end{aligned}
\tag{7.62}
$$

where $\bar{\sigma}_2$ is an arbitrary subset of σ_2 such that the representation of the modes w_k is complete using the reality condition, $w_{-k} = w_k^*$ (i.e., if $k \in \bar{\sigma}_2$, $-k \notin \bar{\sigma}_2$). Notice that, consistent with our general strategy, all nonlinear self-interaction terms between unresolved variables are modeled stochastically in the second equation in (7.62).

The following theorem specifies the behavior of the climate mean flow U for small ε:

THEOREM 7.9 *Denote by $U^\varepsilon(t)$ the solution of* (7.62)*, and assume that the following condition is satisfied*:

$$
\sum_{k\in\bar{\sigma}_2} \frac{\bar{w}_k}{\hat{h}_k} \in \mathbb{R}\,.
\tag{7.63}
$$

Then in the limit as $\varepsilon \to 0$, $U^\varepsilon(t)$ tends to $U(t)$ where $U(t)$ satisfies

$$
dU = -\gamma_u(U - \bar{U})dt + \sigma_u\,dW(t)\,,
\tag{7.64}
$$

where

$$\gamma_u = 2\sum_{k\in\bar\sigma_2} \frac{k_x^2|\hat h_k|^2(\hat h_k - |\vec k|^2\bar w_k)}{|\vec k|^2\gamma_k \hat h_k}, \qquad \bar U = -\frac{2\beta}{\gamma_u}\sum_{k\in\bar\sigma_2}\frac{k_x^2\bar w_k|\hat h_k|^2}{|\vec k|^2\gamma_k\hat h_k},$$
$$\sigma_u = 2\left(\sum_{k\in\bar\sigma_2}\frac{k_x^2\sigma_k^2|\hat h_k|^2}{\gamma_k^2}\right)^{1/2}. \tag{7.65}$$

Theorem 7.9 follows from Theorem 4.1 after mapping of the triad model equations in (7.62) onto the equations in (4.3). Alternatively, Theorem 7.9 can be proven from Theorem 4.4 by computing the operator $\bar{\mathcal{L}}$ in (4.35) associated with the equations in (7.62). Some details of this calculation are given below.

We now discuss the content of Theorem 7.9. First, the theorem clearly tells us that, in the present case, the unresolved modes are responsible for *all* the driving of the climate variable U, since neglecting the effect of w_k in the equation for U in (7.62) would have left us with the trivial result $dU/dt = 0$ instead of the equation in (7.64). This equation predicts that the mean flow U is a Gaussian random process of Ornstein-Uhlenbeck type whose solution for the initial condition $U(t_0) = U$ is

$$U(t) = Ue^{-\gamma_u(t-t_0)} + \bar U(1-e^{-\gamma_u(t-t_0)}) + \sigma_u\int_{t_0}^t e^{-\gamma_u(t-s)}\,dW(s)\,. \tag{7.66}$$

It follows that a statistical steady state exists if and only if the following stability criterion is satisfied:

$$\gamma_u = 2\sum_{k\in\bar\sigma_2}\frac{k_x^2|\hat h_k|^2(\hat h_k - |\vec k|^2\bar w_k)}{|\vec k|^2\gamma_k\hat h_k} > 0\,. \tag{7.67}$$

As shown below, the criterion in (7.67) is always satisfied if the $\bar w_k$ are taken consistent with equilibrium statistical theory. On the contrary, instability may occur and no statistical steady state exists if $|\hat h_k| > |\vec k|^2|\bar w_k|$.

The requirement in (7.63) also deserves comment. As shown in the sketch of the proof of Theorem 7.9 given below, this requirement is essential in order for the solution of the stochastic model equation in (7.62) to have a limit as $\varepsilon \to 0$. In other words, the requirement in (7.63) is essential for the very definition of the mean flow U as the climate variable. What (7.63) actually gives is a rather weak constraint on our stochastic model in (3.14): The mean value of modes $\vec k$ and $\bar w_k$ must be taken such that they do not introduce global phase shift with respect to the corresponding modes $\hat h_k$ of the topography. It is also important to point out that this constraint is automatically satisfied if $\bar w_k$ is taken consistent with equilibrium statistical theory, since in this case it follows from the equation in (7.1) that $\bar w_k/\hat h_k = (\mu + |\vec k|^2)^{-1} \in \mathbb{R}$.

Consider finally the question of how Theorem 7.9 is modified if we take the parameters in (7.62) consistent with equilibrium statistical theory, i.e., if we account for the constraints in (7.1) and (7.2). In this case there is only one free parameter, which we chose to be γ_k. The following is an immediate consequence of Theorem 7.9:

THEOREM 7.10 *Let the equations in* (7.62) *be constrained by* (7.1) *and* (7.2)*, and denote by* $U^\varepsilon(t)$ *the solution of the first equation in* (7.62)*. In the limit as* $\varepsilon \to 0$*,* $U^\varepsilon(t)$ *tends to* $U(t)$ *where* $U(t)$ *satisfies*

$$dU = -\gamma_u(U - \bar{U})dt + \sigma_u \, dW(t)\,. \tag{7.68}$$

Here

$$\gamma_u = 2\mu R_u\,, \quad \bar{U} = -\frac{\beta}{\mu}\,, \quad \sigma_u = 2\sqrt{\frac{R_u}{\alpha}}\,, \tag{7.69}$$

where

$$R_u = \sum_{k \in \bar{\sigma}_2} \frac{k_x^2 |\hat{h}_k|^2}{\gamma_k(\mu + |\vec{k}|^2)}\,. \tag{7.70}$$

It is a simple matter to check that the process defined by (7.68) reaches a statistical steady state because $\gamma_u > 0$ since $R_u > 0$ by the equation in (7.70) and $\mu > 0$ in the presence of a mean flow. Furthermore, consistent with Theorem 6.6, the invariant measure for (7.68) has a Gaussian density with mean and variance that agree with the values in (6.18) from equilibrium statistical theory. Indeed, the mean value of U predicted through (7.68) is simply $\bar{U}$; the variance of U is given by the ratio

$$\frac{\sigma_u^2}{2\gamma_u} = \frac{1}{\alpha\mu}\,. \tag{7.71}$$

Of course, Theorem 7.10 gives more than the results of equilibrium statistical theory since it predicts all multiple time statistics of U. In particular, it follows from the equation in (7.66) that at statistical steady state we have

$$\mathbf{E}U(t)U(s) = \frac{e^{-\gamma_u|t-s|}}{\alpha\mu}\,. \tag{7.72}$$

This expression for the correlation of U agrees well with the results of numerical simulations that were reported by the authors in [16].

PROOF OF THEOREM 7.9: We only sketch the calculation and identify explicitly the operators entering $\bar{\mathcal{L}}$ in (4.35) associated with triad model equations in (7.62). The backward equation associated with the Markov process defined by (7.62) is given by

$$-\frac{\partial \varrho^\varepsilon}{\partial s} = \frac{1}{\varepsilon^2}\mathcal{L}_1\varrho^\varepsilon + \frac{1}{\varepsilon}\mathcal{L}_2\varrho^\varepsilon\,, \tag{7.73}$$

where we defined

$$
\begin{aligned}
\mathcal{L}_1 &= -\sum_{k\in\bar\sigma_2} \gamma_k (w_k - \bar w_k)\frac{\partial}{\partial w_k} - \sum_{k\in\bar\sigma_2} \gamma_k (w_k^* - \bar w_k^*)\frac{\partial}{\partial w_k^*} \\
&\quad + 2\sum_{k\in\bar\sigma_2} \sigma_k^2 \frac{\partial}{\partial w_k \partial w_k^*}\,, \\
\mathcal{L}_2 &= i\sum_{k\in\bar\sigma_2} k_x\big(\hat H_k w_k^* - \hat H_k^* w_k\big)\frac{\partial}{\partial U} + i\sum_{k\in\bar\sigma_2}\big(\hat H_k U - (k_x U - \Omega_k) w_k\big)\frac{\partial}{\partial w_k} \\
&\quad - i\sum_{k\in\bar\sigma_2}\big(\hat H_k^* U - (k_x U - \Omega_k) w_k^*\big)\frac{\partial}{\partial w_k^*}\,.
\end{aligned}
\tag{7.74}
$$

The actual computation of the operator $\bar{\mathcal{L}}$ given in (4.35) can now be done using the material in Appendix A. We conclude the proof upon noting that from the equations in (7.74) and (7.18), it follows that

$$
\mathbf{P}\mathcal{L}_2\mathbf{P} = i\sum_{k\in\bar\sigma_2} k_x\big(\hat H_k \bar w_k^* - \hat H_k^* \bar w_k\big)\frac{\partial}{\partial U}\mathbf{P}\,. \tag{7.75}
$$

In order that the asymptotic method in Section 4.4 applies, it is required that this operator be zero; this is the content of equation (2.10) in assumption A4 that is necessary in order for the equation in (4.29) to be satisfied. On the other hand, the right-hand side of the equation in (7.75) is zero if

$$
0 = i\sum_{k\in\bar\sigma_2}(\hat H_k \bar w_k^* - \hat H_k^* \bar w_k) = 2\,\mathrm{Im}\sum_{k\in\bar\sigma_2}\hat H_k^* \bar w_k\,. \tag{7.76}
$$

This equation yields the constraints in (7.63). □

8 Concluding Discussion

We have developed a systematic mathematical strategy for stochastic climate modeling based on the following three-step procedure:

(1) Identification of two sets of variables referred to as climate and unresolved variables. The former are the variables we are ultimately interested in; the latter are the variables whose dynamics are essential to driving the climate variables but are irrelevant for a meaningful macroscopic description of the state of the system.
(2) Stochastic modeling of the nonlinear self-interaction of the unresolved variables.
(3) Elimination of the unresolved variables by averaging techniques in the limit of infinite separation between the time scales for the dynamics of the climate and the unresolved variables.

Thus, our key assumption is that the climate variables in a given nonlinear system necessarily evolve on longer time scales than the unresolved variables. This assumption justifies the approximation that the nonlinear interaction among unresolved variables can be represented stochastically in a suitably simplified fashion. Despite the relative crudeness of this assumption, we have developed an explicit and rigorous mathematical strategy for such an approach that is implicit in much of the work in stochastic climate modeling in the literature [1, 3, 5, 8, 10, 12, 15, 18, 21, 22, 23, 25].

The closed nonlinear stochastic equations are derived for the climate variables alone on longer time scales in a rigorous fashion that accounts for strong coupling between the climate variables and the unresolved variables. Furthermore, the predicted stochastic evolution equations for the climate variables are given quantitatively so the theory is effectively computable but much simpler than turbulence closure. These equations display several potentially important new phenomena not included in the previous applied efforts in climate modeling. These phenomena include systematic nonlinear corrections to the climate dynamics due to the interaction with the unresolved variables and the appearance of multiplicative stochastic noises besides additive noises for the climate variables. We also showed that stochastic equations for climate variables alone can be both linearly stable or unstable, and we gave explicit mathematical criteria and examples with unstable linear Langevin equations for the climate variables. Such examples with less stable stochastic models for the climate variables on a longer time scale indicate that interactions with the unresolved variables can diminish predictability under appropriate circumstances.

Throughout this paper we have used as an illustrative example the idealized climate model of a barotropic flow on a beta plane with topography and mean flow introduced by Leith [15]. These are especially attractive climate models because they are highly inhomogeneous yet involve both a well-defined mean climate state as well as energy spectrum. In spherical geometry such models capture a number of large-scale features of the atmosphere [7]. We have demonstrated the feasibility of our general strategy for this idealized climate model. We have also shown that the idealized climate model of a barotropic flow can be made fully consistent with equilibrium statistical theory by appropriate constraints on the parameters in the stochastic model. Furthermore, we have demonstrated that the stochastic model for the climate variables alone that is derived from the barotropic flow equations also satisfies an equilibrium statistical theory. Simpler examples illustrating the appearance of new phenomena were also given, whereas the general stochastic model equations for the climate variables alone that are derived from the barotropic flow model will be studied by the authors in the near future as well as other generalizations of geophysical interest, including baroclinic flows and coupled atmosphere/ocean systems.

Appendix A: Explicit Properties of the Ornstein-Uhlenbeck Operator

In this appendix we consider the Ornstein-Uhlenbeck operator

$$\mathcal{L}_1 = -\sum_{j\in\sigma_2} \gamma_j y_j \frac{\partial}{\partial y_j} + \frac{1}{2}\sum_{j\in\sigma_2} \sigma_j^2 \frac{\partial^2}{\partial y_j^2}\,, \tag{A.1}$$

and give some properties of this operator useful for evaluating the operator $\bar{\mathcal{L}}$ defined in (4.35).

The basic idea is to consider the properties of $\mathcal{L}_1$ in a Fourier representation defined for any suitable function $f(\vec{y})$ as

$$\hat{f}(\vec{p}) = \int_{\mathbb{R}^n} e^{i\vec{p}\cdot\vec{y}} f(\vec{y})d\vec{y}\,, \qquad f(\vec{y}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\vec{p}\cdot\vec{y}}\hat{f}(\vec{p})d\vec{p}\,, \tag{A.2}$$

where n is the cardinal of the set σ_2. In the Fourier representation (A.1) becomes

$$\hat{\mathcal{L}}_1 = \sum_{j\in\sigma_2}\gamma_j + \sum_{j\in\sigma_2}\gamma_j p_j \frac{\partial}{\partial p_j} - \frac{1}{2}\sum_{j\in\sigma_2}\sigma_j^2 p_j^2\,. \tag{A.3}$$

The following two lemmas allow for explicit evaluation of the expectation $\mathbf{P}$ and the action of $\mathcal{L}_1^{-1}$.

LEMMA A.1 *We have*

$$\mathbf{P} f(\vec{y}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{P}^\star(\vec{p})\,\hat{f}(\vec{p})d\vec{p}\,, \tag{A.4}$$

where

$$\hat{P}^\star(\vec{p}) = \exp\left(-\frac{1}{4}\sum_{j\in\sigma_2}\frac{\sigma_j^2 p_j^2}{\gamma_j}\right). \tag{A.5}$$

LEMMA A.2 *Assuming* $\mathbf{P} f = 0$, *we have*

$$\hat{\mathcal{L}}_1^{-1}\hat{f}(\vec{p}) = -\int_0^\infty \exp\left(\sum_{j\in\sigma_2}\gamma_j t - \frac{1}{4}\sum_{j\in\sigma_2}\frac{\sigma_j^2 p_j^2}{\gamma_j}\big(e^{2\gamma_j t}-1\big)\right)\hat{f}(\vec{p}(t))dt\,, \tag{A.6}$$

where

$$p_j(t) = e^{\gamma_j t}p_j\,. \tag{A.7}$$

PROOF OF LEMMA A.1: (A.4) follows from Parseval's identity upon noting that (A.5) is the Fourier representation of the invariant measure for the Ornstein-Uhlenbeck process. □

PROOF OF LEMMA A.2: Consider the following equation for $\hat{g}(\vec{p},t)$

$$\frac{\partial\hat{g}}{\partial t} = \hat{\mathcal{L}}_1\hat{g} - \hat{f}\,, \quad \hat{g}(\vec{p},0) = 0\,. \tag{A.8}$$

Then

$$\hat{g} \to \hat{\mathcal{L}}_1^{-1} \hat{f}\,, \tag{A.9}$$

as $t \to \infty$, and the limit exists since $\mathbf{P} f = 0$ by assumption. Since the equation in (A.8) is linear, this equation can be solved by the method of characteristics:

$$\hat{g}(\vec{p}, t) = -\int_0^t \exp\left(\sum_{j\in\sigma_2} \gamma_j s - \frac{1}{4} \sum_{j\in\sigma_2} \frac{\sigma_j^2 p_j^2}{\gamma_j} \left(e^{2\gamma_j s} - 1\right)\right) \hat{f}(\vec{p}(s)) ds\,. \tag{A.10}$$

(A.6) follows in the limit as $t \to \infty$. □

Appendix B: The General Derivation

In this appendix we derive the stochastic model equations in (4.17). The situation with fast-wave effects can be treated in a similar way after an appropriate change of dependent variables, as explained in Section 5.3.

The calculation amounts to evaluating the operator $\bar{\mathcal{L}}$ defined in (4.35) associated with the equations in (4.16). The corresponding operators $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ are given in (4.23). In order to use the results of Appendix A, it will be convenient to have the Fourier representations of $\mathcal{L}_2$, $\mathcal{L}_3$. They are given by

$$\begin{aligned}\hat{\mathcal{L}}_2 = &\sum_{j,k} \left(i L_{jk}^{12} \frac{\partial}{\partial p_k} + \frac{1}{2} \sum_l \left(2i B_{jkl}^{112} x_k \frac{\partial}{\partial p_l} - B_{jkl}^{122} \frac{\partial^2}{\partial p_k \partial p_l}\right)\right) \frac{\partial}{\partial x_j}\\ &+ i \sum_{j,k} \left(L_{jk}^{21} x_k + i L_{jk}^{22} \frac{\partial}{\partial p_k} + \frac{1}{2} \sum_l \left(B_{jkl}^{211} x_k x_l + 2i B_{jkl}^{221} x_l \frac{\partial}{\partial p_l}\right)\right) p_j\\ \hat{\mathcal{L}}_3 \equiv \mathcal{L}_3 = &\sum_j \left(F_j(s) - \sum_k D_{jk} x_k + \frac{1}{2} \sum_{kl} B_{jkl}^{111} x_k x_l \right) \frac{\partial}{\partial x_j}\,.\end{aligned} \tag{B.1}$$

We now evaluate the operator $\bar{\mathcal{L}}$ defined in (4.35), i.e.,

$$\bar{\mathcal{L}}\, g(\vec{x}) = \mathbf{P}\mathcal{L}_3\mathbf{P}\, g(\vec{x}) - \mathbf{P}\mathcal{L}_2\mathcal{L}_1^{-1}\mathcal{L}_2\mathbf{P} g(\vec{x})\,. \tag{B.2}$$

By definition of $\mathcal{L}_3$ we obtain

$$\mathbf{P}\mathcal{L}_3\mathbf{P} g(\vec{x}) = \mathcal{L}_3 g(\vec{x})\,, \tag{B.3}$$

whereas from the Parseval identity and using the results in Lemmas A.1 and A.2, it follows that for any suitable function $g(\vec{x})$ we have

$$\begin{aligned}&-\mathbf{P}\mathcal{L}_2\mathcal{L}_1^{-1}\mathcal{L}_2\mathbf{P} g(\vec{x}) =\\ &\quad\int_{\mathbb{R}^n} \hat{P}^\star(\vec{p})\hat{\mathcal{L}}_2 \int_0^\infty \exp\left(\sum_{j\in\sigma_2} \gamma_j t - \frac{1}{4} \sum_{j\in\sigma_2} \frac{\sigma_j^2 p_j^2}{\gamma_j} \left(e^{2\gamma_j t} - 1\right)\right)\\ &\qquad\qquad \times \left[\hat{\mathcal{L}}_2 g(\vec{x}) \delta(\vec{p})\right]_{\vec{p}=\vec{p}(t)} dt\, d\vec{p}\,.\end{aligned} \tag{B.4}$$

The action of the rightmost $\hat{\mathcal{L}}_2$ is easily accounted for. Using the property of the delta function as well as the definition for $\vec{p}(t)$ in (A.7), we obtain

$$\exp\left(\sum_{j\in\sigma_2}\gamma_j t\right)\left[\hat{\mathcal{L}}_2 g(\vec{x})\delta(\vec{p})\right]_{\vec{p}=\vec{p}(t)} = \tag{B.5}$$
$$i\sum_{j,k} L^{12}_{jk} e^{-\gamma_k t}\frac{\partial\delta(\vec{p})}{\partial p_k}\frac{\partial g}{\partial x_j}$$
$$+\frac{1}{2}\sum_{j,k,l}\left(2i B^{112}_{jkl} e^{-\gamma_l t} x_k \frac{\partial\delta(\vec{p})}{\partial p_l} - B^{122}_{jkl} e^{-(\gamma_k+\gamma_l)t}\frac{\partial^2\delta(\vec{p})}{\partial p_k\partial p_l}\right)\frac{\partial g}{\partial x_j}.$$

The sequel of the calculation is now tedious but straightforward and based on the property of the delta function that

$$\int_{\mathbb{R}^n} f(\vec{p})\frac{\partial\delta(\vec{p})}{\partial p_j}\,d\vec{p} = -\int_{\mathbb{R}^n}\frac{\partial f(\vec{p})}{\partial p_j}\delta(\vec{p})\,f\vec{p} = \left[\frac{\partial f(\vec{p})}{\partial p_j}\right]_{\vec{p}=0}. \tag{B.6}$$

A similar relation holds for higher derivatives. Thus, the calculation essentially amounts to integrating by parts in $\vec{p}$ the various terms in (B.4) as many times as necessary in order that no derivative acts on the delta functions in (B.4). Once this operation has been done, the integration on $\vec{p}$ is trivial, since it amounts to evaluating the factor of the delta function in the integrand at $\vec{p}=0$. Finally, the last integration on t can be performed. This way we obtain

(B.7)
$$\bar{\mathcal{L}} = \sum_{j\in\sigma_1}\left(F_j(t) - \sum_{k\in\sigma_1} D_{jk}x_k - \frac{1}{2}\sum_{k,l\in\sigma_1} B^{111}_{jkl}x_k x_l\right)\frac{\partial}{\partial x_j}$$
$$+\frac{1}{2}\sum_{j\in\sigma_1}\left(\sum_{k,l\in\sigma_2}\frac{\sigma_l^2 B^{122}_{jkl} L^{22}_{kl}}{\gamma_l(\gamma_k+\gamma_l)} + \sum_{k\in\sigma_1}\sum_{l,m\in\sigma_2}\frac{\sigma_l^2 B^{122}_{jlm} B^{221}_{mlk}}{\gamma_l(\gamma_l+\gamma_m)}x_k\right)\frac{\partial}{\partial x_j}$$
$$+\frac{1}{8}\sum_{j,k\in\sigma_1}\sum_{l,m\in\sigma_2}\frac{B^{122}_{jlm}B^{122}_{klm}\sigma_l^2\sigma_m^2}{(\gamma_l+\gamma_m)\gamma_l^2\gamma_m^2}\frac{\partial^2}{\partial x_j\partial x_k}$$
$$+\frac{1}{2}\sum_{j,k\in\sigma_1}\sum_{m\in\sigma_2}\frac{\sigma_m^2}{\gamma_m^2}B^{112}_{jkm}\left(L^{12}_{km} + \sum_{l\in\sigma_1} B^{112}_{klm}x_l\right)\frac{\partial}{\partial x_j}$$
$$+\sum_{j,l\in\sigma_1}\sum_{n\in\sigma_2}\frac{1}{\gamma_n}\left(L^{12}_{jn} + \sum_{k\in\sigma_1} B^{112}_{jkn}x_k\right)\left(L^{21}_{nl}x_l + \frac{1}{2}\sum_{m\in\sigma_1} B^{211}_{nlm}x_l x_m\right)\frac{\partial}{\partial x_j}$$
$$+\frac{1}{2}\sum_{j,k\in\sigma_1}\sum_{n\in\sigma_2}\frac{\sigma_n^2}{\gamma_n^2}\left(L^{12}_{jn} + \sum_{l\in\sigma_1} B^{112}_{jln}x_l\right)\left(L^{12}_{kn} + \sum_{m\in\sigma_1} B^{112}_{kmn}x_m\right)\frac{\partial^2}{\partial x_j\partial x_k}.$$

The system of stochastic differential equations associated with this operator are the equations in (4.17).

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Received May 2000.