

Problem 1:

(a) For the polynomial $p(x) = x^5 - x^4 + x^3 - x^2$ find the matrix of a linear map T on $\mathbb{R}^5$ such that $p(x)$ is the minimal polynomial for T .

Answer The linear map will have with respect to the five unit vectors the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

(b) Find the components of the primary decomposition for T .

Answer We need to find the prime factorization of $p(x)$. We first see that $p(x) = x^5 - x^4 + x^3 - x^2 = x^2(x^3 - x^2 + x - 1)$. The polynomial $(x^3 - x^2 + x - 1)$ has root 1 and $(x^3 - x^2 + x - 1) = (x - 1)(x^2 + 1)$. Thus $p(x) = x^2(x - 1)(x^2 + 1)$. This is $\mathbb{R}^5 = \{X | A^2 X = 0\} \oplus \{X | (A - I)X = 0\} \oplus \{X | (A^2 + I)X = 0\}$

Notice: The three direct summands are cyclic and they have dimensions 2, 1, 2 respectively. This is because T is cyclic, $\mathbb{R}^5 = \langle e_1 \rangle_T$, and invariant subspaces of a cyclic space are cyclic.

(c) Find bases for these primary components.

Answer We need to solve three homogeneous systems

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}^2 \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

This leads to the linear system:

$x = -u - v, y = u, z = -u$. There are two base solutions

$$f_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, f_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } \{X | A^2 X = 0\} = \langle f_1, f_2 \rangle$$

$$A - I = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ has a unique solution } f_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{check: } \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Finally, we need to solve $(A^2 + I)X = 0$, We have that $(A^2 + I) =$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

and therefore we need to solve:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ This is equivalent to}$$

$x = 0, y = 0, z = -u - v$. This gives us two basic solutions:

$$f_4 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, f_5 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

(d) What is the matrix B of T with respect to the basis $f_1, \dots, f_5$?

Notice: Our three subspaces are invariant. Thus we will get B as a block of three matrices.

$$f_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, f_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$Af_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = - \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = -f_1 + f_2,$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = -f_1 + f_2$$

$$Af_3 = f_3$$

$$\text{Finally, } f_4 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, f_5 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} = -f_4 + f_5$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{pmatrix} = -2f_4 + f_5$$

$$\mathbf{Mat}(T; f_1, \dots, f_5) = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \text{ minimum polynomial: } X^5 - X^4 + X^3 - X^2$$

Problem 2 Let $T : V \rightarrow V$ be a cyclic map on the finite dimensional vector space V . Prove that every invariant subspace U of V is cyclic.

Hint: Let $V = \langle e \rangle_T$ where e is a cyclic generator. Then let

$I = \{p(x) \mid p(T)(e) \in U\}$. Show that I is an ideal. Thus $I = (r(x))$. What does the cyclic space $\langle r(T)(e) \rangle_T$ look like?

Problem 3 Prove that a map T on the finite dimensional vector space has a trivial kernel, that is $\ker(T) = \{0\}$, iff the minimal polynomial has a non-zero constant term.